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Balanced splittings of Semi-free actions of finite groups on homotopy spheres

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Throughout this paper we work in the smooth category. In particular, the terms "manifold" and "action" mean "smooth manifold" and "smooth action" respectively.

Let Σ^{n+k} be a homotopy sphere of dimension $n+k$ and let $\rho: G\times\Sigma\to\Sigma$ be a semi-free action of the finite group G on Σ. The fixed point set of this action is an n-manifold, denoted by either F^n or Σ^G, and G acts freely on $\Sigma-F$. A *splitting* of such an action is a decomposition of Σ of the form $\Sigma=D_1\cup D_2$ where D_i $(i=1,2)$ is a closed G-invariant $(n+k)$-disk such that $\partial D_i=D_1\cap D_2$ meets F transversally. In this case, the splitting $F=F_1\cup F_2$ where $F_i=F\cap D_i$ $(i=1,2)$ is called the *induced splitting* of F. A splitting of the action is a (*strong*) *balanced* splitting if the induced splitting is a (strong) balanced splitting of F in the sense of the following definition.

Let F^n be a closed n-manifold. A *splitting* of F is a decomposition of F of the form $F=F_i\cup F_2$ where F_i $(i=1,2)$ is an n-manifold with $\partial F_i=F_1\cap F_2$. A splitting is *balanced* if $H_j(F_1)$ is isomorphic to $H_j(F_2)$ for all j. A *strong balanced splitting* is a balanced splitting $F=F_1\cup F_2$ such that A_1 is equal to A_2 where $A_i=\ker(H_{m-1}(F_0)\to H_{m-1}(F_i))$ $(i=1,2)$, $m=[n/2]$ and $F_0=F_1\cap F_2$.

In this paper, we investigate the problem(s) of whether a given semi-free action admits a (strong) balanced splitting. We show that if $\dim F=2m$, then a balanced splitting of the action always exists. When $\dim F=2m+1$, we introduce a sort of "semi-characteristic" invariant which is the main obstruction to the existence of a balanced splitting. A similar invariant is the obstruction to the existence of a strong balanced splitting of the action without regard for the parity of $\dim F$. Finally, we construct examples of semi-free actions whose semi-characteristic invariants are non-zero. Such actions, then, have no strong balanced splittings.

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One class of actions for which balanced splittings exist is obtained by the "twisted double" construction. Namely, let $\rho : G \times D^{n+k} \to D^{n+k}$ be a semi-free action of the finite group G on an $(n+k)$ disk. Let $\Sigma = D \cup_\phi D$ where $\phi : \partial D \to \partial D$ is an equivariant diffeomorphism. Our interest in the problem considered here arose from trying to understand the conditions under which a given semi-free action is a twisted double. An action that admits a balanced splitting (respectively, strong balanced splitting) resembles a twisted double (respectively, a double; i.e. ϕ is the identity), at least homologically. In that sense, such an action exhibits a rough sort of symmetry. An action with no (strong) balanced splitting is rather strongly asymmetrical.

We remark finally that the class of finite groups G that can act smoothly and semi-freely on a manifold is rather small – it consists of exactly the finite groups that admit a free linear representation. Although these groups have been classified by Wolf [22], this classification is not used in proving Theorem A and B below and it was used only to guide our search for the examples of Theorem C.

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1. Statement of results

In this section, we outline the main flow of our argument and state our main results. We begin by setting the notational convention that for any finite group G, $\mathfrak{D}(G)$ denotes the category of finite abelian groups of order prime to $|G|$, the order of G.

Let F^n be the fixed point set of a semi-free action of G on the homotopy sphere Σ^{n+k}. It follows from Smith theory that $\tilde{H}_i(F) \in \mathfrak{D}(G)$ for $i < n$. Similarly, if the splitting $F = F_1 \cup F_2$ is induced from a splitting of the action, then $\tilde{H}_i(F_j) \in \mathfrak{D}(G)$ $(j = 1, 2)$ for all i, and $\tilde{H}_i(F_0) \in \mathfrak{D}(G)$ for $i < n-1$ where $F_0 = F_1 \cap F_2$. A splitting satisfying these conditions will be called *admissible.*

We note that if F^n is any closed manifold such that $\tilde{H}_i(F^n) \in \mathfrak{D}(G)$ for $i < n$, then admissible splittings of F are abundant. For example, let $D^n \subset F^n$ be an embedded n-disk and set $F_1 = D^n$ and $F_2 = F - \operatorname{Int} D^n$. Suppose now that $F = \Sigma^G$ and that $F = F_1 \cup F_2$ is an admissible splitting of F. We wish to obtain an obstruction to extending this splitting to a splitting of the action. This is done as follows:

Let $\mathscr{C}(G)$ denote the category of finitely generated, cohomologically trivial ZG modules. If we regard the groups in $\mathfrak{D}(G)$ as trivial ZG modules, then Rim has shown that they are cohomologically trivial [14; Theorem 4.7]. Hence, there is an inclusion $k:\mathcal{D}(G)\to\mathcal{C}(G)$ and an induced homomorphism $k_*: G_0(\mathcal{D}(G))\to \tilde{G}_0(\mathcal{C}(G))$ where $G_0(\mathcal{D}(G))$ and $G_0(\mathcal{C}(G))$ are the Grothendieck groups of the categories $\mathcal{D}(G)$ and $\mathcal{C}(G)$ respectively and $\tilde{G}_0(\mathcal{C}(G))=G_0(\mathcal{C}(G))/\{[M]\mid M$ is a finitely generated free ZG module$\}$. On the other hand, every finitely generated, projective ZG module is cohomologically trivial. Hence, there is also an inclusion $l:\mathcal{P}(G)\to\mathcal{C}(G)$ where $\mathcal{P}(G)$ is the category of finitely generated projective ZG modules. The induced homomorphism $l_*:\tilde{K}_0(ZG)\to\tilde{G}_0(\mathcal{C}(G))$ is an isomorphism by another result of Rim [14; Theorem 4.12] (cf. Proposition 2.2 below) and we identify these groups via this isomorphism. Finally, let $A(G)=\operatorname{Im}(k_*: G_0(\mathcal{D}(G))\to\tilde{K}_0(ZG))$.

Suppose now that X is a finite CW complex such that $\tilde{H}_i(X)\in\mathcal{D}(G)$ for all i. Let $\tilde{\chi}(X;G)\in G_0(\mathcal{D}(G))$ and $\chi(X;G)\in A(G)$ be defined by

$$\tilde{\chi}(X,G)=\sum_{i\geq 1}(-1)^i[\tilde{H}_i(X)]$$

and

$$\chi(X,G)=k_*\tilde{\chi}(X,G).$$

We are now ready to state our first main result.

THEOREM A. *Let (Σ,ρ) be a smooth semi-free action of the finite group G on the homotopy $(n+k)$ sphere Σ with connected fixed point set F^n. If $1\leq n\leq k-2$, then the admissible splitting of F as $F_1\cup F_2$ is induced by a splitting of the action if and only if $\chi(F_1;G)=0$.*

We remark that $\chi(F_1;G)=(-1)^n\chi(F_2;G)$ so the above result does not depend on the ordering of F_1 and F_2.

The reader will note that this theorem is similar in spirit to results of Jones [6] and Oliver [9]. Indeed, it was inspired by their work.

An obvious necessary condition for the existence of a (strong) balanced splitting of the action is that there be an admissible (strong) balanced splitting of F. In section 4, we show that if $n=2m$, then F always has such splittings; but, if $n=2m+1$ and $H_1(\pi_1(F);M)=0$ for all $\pi_1(F)$ modules M then F has such splittings if and only if $|H_m(F)|$ is a square. Thus $|H_m(F)|$ is a "primary obstruction" to finding a (strong) balanced splitting of the action.

Now let F^n be a closed manifold such that $\tilde{H}_i(F^n)\in\mathcal{D}(G)$ for $i<n$ and, if $n=2m+1$, suppose $|H_m(F)|$ is a square. We define a semi-characteristic $\tilde{\chi}_{1/2}(F;G)\in A(G)$ by setting $\tilde{\chi}_{1/2}(F;G)=k_*\rho_{1/2}(F;G)$ where $\rho_{1/2}(F;G)\in$

$G_0(\mathscr{D}(G))$ is given by

$$\rho_{1/2}(F^n; G) = \begin{cases} \sum_{i=1}^{m-1} (-1)^i [H_i(F)] & \text{if} \quad n = 2m \\ \sum_{i=1}^{m-1} (-1)^i [H_i(F)] + (-1)^m [Z/q] & \text{if} \quad n = 2m+1 \end{cases}$$

where $q = |H_m(F)|^{1/2}$.

Now let $x \to \bar{x}$ be the involution on $\tilde{K}_0(ZG)$ induced by sending $[P]$ to $-[P^*]$ where $P \in \mathscr{P}(G)$ and $P^* = \mathrm{Hom}_{ZG}(P, ZG)$. In Section 2, we show that every element $x \in A(G)$ satisfies $x = \bar{x}$. In particular, $\tilde{\chi}_{1/2}(F; G) = \overline{\tilde{\chi}_{1/2}(F; G)}$ if $n = 2m$. In Section 4, we show that if $F^n = \Sigma^G$, $n = 2m+1$, and $|H_m(F)|$ is a square, then $\tilde{\chi}_{1/2}(F; G) = -\overline{\tilde{\chi}_{1/2}(F; G)}$. In either case, then $\tilde{\chi}_{1/2}(F; G)$ represents a well defined element of $H^n(Z/2; A(G))$ which we denote by $\chi_{1/2}(F; G)$.

THEOREM B. *Let (Σ, ρ) be a semi-free action of the finite group G on the homotopy $(n+k)$-sphere Σ with connected fixed point set F^n where $n \neq 3, 4$, $1 \leq n \leq k-2$.*

(i) *If $n = 2m$, then the action has a balanced splitting. It has a strong balanced splitting if and only if $\chi_{1/2}(F; G) = 0$.*

(ii) *If $n = 2m+1$ and $H_1(\pi_1(F); M) = 0$ for all $\pi_1(F)$ modules M, then the action has a balanced splitting if and only if it has a strong balanced splitting. The latter occurs if and only if $|H_m(F)|$ is a square and $\chi_{1/2}(F; G) = 0$.*

In particular, Part (ii) of Theorem B holds if $\pi_1(F) = 0$ or if $H_m(F) = 0$.

If G is cyclic of order n, then a result of Jones [6; Lemma 1.1] shows that $A(G) = 0$. Similarly, a result of Ullom [18; Proposition 2.10] shows that the exponent of $A(G)$ divides the Artin exponent of G; which, in turn, divides $|G|$. Hence, if G has odd order, so does $A(G)$ and $H^n(Z/2; A(G)) = 0$ for all n. In these two cases, then, $\chi_{1/2}(F; G)$ always vanishes.

The simplest examples of non-cyclic, even order groups that admit free linear representations are the generalized quaternion groups $Q2^l$ $(l \geq 3)$. These groups have the presentation $\langle a, b;\, a^{2^{l-1}} = 1,\ b^2 = a^{2^{l-2}},\ bab^{-1} = a^{-1} \rangle$ and have order 2^l.

THEOREM C. *Let $G = Q2^l$ and let d be the minimal dimension of a free linear representation of G. Let n and k be integers such that $6 \leq n \leq k-6$ and $k \equiv 0$ (mod d). If $n \not\equiv 1$ (mod 4) then there exists a semi-free action of G on a homotopy sphere Σ^{n+k} with fixed point set F having dimension n such that $\chi_{1/2}(F; G) \neq 0$.*

We remark that the fixed point sets of these examples are actually doubles.

We also remark that since it is easy to show that $\chi_{1/2}(F; G)$ is a cobordism invariant, these actions are not cobordant to linear actions.

2. Some algebra

In this section, we obtain the basic algebraic results about the groups $G_0(\mathscr{D}(G))$ and $A(G)$ that we will need.

PROPOSITION 2.1. *The group $G_0(\mathscr{D}(G))$ is free abelian on the generators* $\{[Z/p] \mid p$ *is a prime,* $p \nmid |G|\}$.

Proof. Any group $T \in \mathscr{D}(G)$ is isomorphic to a direct sum $C_1 \oplus \cdots \oplus C_t$ where C_i is cyclic of order $p_i^{r_i}$ for some prime p_i where $p_i \nmid |G|$. Hence, $[T] = [C_1] + \cdots + [C_t]$ in $G_0(\mathscr{D}(G))$. An easy induction argument using the exact sequence $0 \to Z/p^{r-1} \to Z/p^r \to Z/p \to 0$ shows that $[Z/p^r] = r[Z/p]$ in $G_0(\mathscr{D}(G))$. Hence, $\{[Z/p] \mid p$ is a prime, $p \nmid |G|\}$ generates $G_0(\mathscr{D}(G))$.

Now let $F(G)$ be the free abelian group with generators $\{[p] \mid p$ is a prime, $p \nmid |G|\}$. If $T \in \mathscr{D}(G)$ has order $p_1^{r_1} \cdots p_s^{r_s}$, let $\sigma(T) = r_1[p_1] + \cdots + r_s[p_s] \in F(G)$. It is easy to check that σ induces an isomorphism $\sigma_*: G_0(\mathscr{D}(G)) \to F(G)$. This completes the proof.

PROPOSITION 2.2. *The inclusion of categories $l: \mathscr{P}(G) \to \mathscr{C}(G)$ induces an isomorphism $l_*: \tilde{K}_0(ZG) \to \tilde{G}_0(\mathscr{C}(G))$.*

Proof. If $M \in \mathscr{C}(G)$, then by Rim [14; Theorem 4.12] M has a short resolution $0 \to P_1 \to P_0 \to M \to 0$ where $P_0, P_1 \in \mathscr{P}(G)$. The map $[M] \mapsto [P_0] - [P_1]$ induces a well defined homomorphism inverse to l_*.

We wish now to give an alternate description of $A(G)$. We recall that if r is a positive integer such that $(r, |G|) = 1$, then Swan [16; Proposition 7.1] has shown that the submodule $\langle r, N \rangle \subset ZG$ is in $\mathscr{P}(G)$ where $N = \sum_{g \in G} g$. Swan also shows [17; Lemma 6.1] that the map $r \to [\langle r, N \rangle]$ defines a homomorphism $\partial: (Z/|G|)^\times \to \tilde{K}_0(ZG)$ where $(Z/|G|)^\times$ denotes the group of units in $Z/|G|$.

PROPOSITION 2.3. (i) $A(G) = \operatorname{Im} \partial$.

(ii) *Every element $x \in A(G)$ is of the form $x = k_*[Z/q]$ for some integer q prime to $|G|$. Furthermore, $x = \bar{x}$.*

Proof. Define a homomorphism $\tau: G_0(\mathscr{D}(G)) \to (Z/|G|)^\times$ by sending $[Z/p]$ to p where $[Z/p]$ is one of the generators of $G_0(\mathscr{D}(G))$ given in 2.1. We note that

$\partial\tau = k_*$, for if p is a prime not dividing $|G|$, then

$$0 \to ZG \xrightarrow{\times p} \langle p, N\rangle \longrightarrow Z/p \longrightarrow 0 \qquad (*)$$

is a projective resolution of Z/p. Since τ is obviously onto, $A(G) = \operatorname{Im} k_* = \operatorname{Im} \partial$ and (i) follows.

To prove (ii), note that $x = \partial(q)$ for some $q \in (Z/|G|)^\times$. If we take q to be an integer $0 < q < |G|$, then $x = \partial(q) = \partial\tau[Z/q] = k_*[Z/q]$. To show $x = \bar{x}$ it suffices to show that $k_*[Z/p] = \overline{k_*[Z/p]}$ whenever p is an integer prime to $|G|$. To see this, take the dual of the sequence (*) above. This gives an exact sequence

$$0 \to \langle p, N\rangle^* \to (ZG)^* \to \operatorname{Ext}_{ZG}(Z/p, ZG) \to 0 \qquad (**)$$

The right hand term is isomorphic to Z/p as abelian groups. It inherits a ZG module structure from the maps "multiplication by g" ($g \in G$) on ZG. However, since the map, "multiplication by $(g-1)$" on ZG factors through $\times p : ZG \to \langle p, N\rangle$, the ZG module structure on $\operatorname{Ext}_{ZG}(Z/p, ZG)$ is trivial. Hence $k_*[Z/p] = \overline{k_*[\operatorname{Ext}_{ZG}(Z/p, ZG)]} = -[\langle p, N\rangle^*] = \overline{k_*[Z/p]}$ by the definition of $\overline{}$, and (ii) follows.

PROPOSITION 2.4. (i) *Let $C_* = \{C_i, \partial_i\}$ be a chain complex such that $C_i = 0$ for $i < 0$ and for $i > n$ where n is some positive integer. If C_i and $H_i(C_*)$ are in $C(G)$ for all i, then*

$$\sum_{i\geq 0} (-1)^i [C_i] = \sum_{i \geq 0} (-1)^i [H_i(C_*)]$$

in $\tilde{G}_0(\mathscr{C}(G)) = \tilde{K}_0(ZG)$.

(ii) *Let p and q be positive integers with $q < p$. Suppose that $0 \to H'_p \to H_p \to H''_p \to H'_{p-1} \to \cdots \to H_q \to H''_q \to 0$ is an exact sequence of modules in $\mathscr{C}(G)$. Then in $\tilde{G}_0(\mathscr{C}(G)) = \tilde{K}_0(ZG)$*

$$\sum_{i=q}^{p} (-1)^i [H_i] = \sum_{i=q}^{p} (-1)^i [H'_i] + \sum_{i=q}^{p} (-1)^i [H''_i]$$

Proof. This follows easily by standard arguments.

3. The proof of Theorem A

In this section we give the proof of Theorem A. The necessity of having $\chi(F_1; G) = 0$ for an induced splitting is contained in the following lemma.

LEMMA 3.1. *Let F^n be the fixed point set of a semi-free action of G on Σ^{n+k}. Then*

(i) $\chi(F^n - x; G) = 0$ *for any* $x \in F^n$.

(ii) *If the splitting* $F = F_1 \cup F_2$ *is induced by a splitting of the action, then* $\chi(F_1; G) = 0$.

(iii) *For any admissible splitting* $F = F_1 \cup F_2$,

$$\chi(F_1; G) = (-1)^n \chi(F_2; G).$$

Proof. We first show (ii). Let $D_1 \cup D_2$ be the splitting of the action that induces the given splitting of F. It has been shown by Illman [5] that the G-space D_1 has the structure of a finite $G-CW$ complex. It now follows from the exact sequence of the pair (D_1, F_1) and 2.4 that

$$\chi(F_1; G) = -\sum_{i \geq 0} (-1)^i [H_i(D_1, F_1)] = -\sum_{i \geq 0} (-1)^i [C_i(D_1, F_1)]$$

where $C_*(D_1, F_1)$ denotes the cellular chains on (D_1, F_1). Since G acts freely on $D_1 - F_1$, $C_i(D_1, F_1)$ is a free ZG module for all i. Hence $\chi(F_1; G) = 0$.

To prove (i), we note that if $x \in F^n$ then there exists a G-invariant disk D_1^{n+k} with center x such that $F_1 = D_1 \cap F$ is a disk. Let $D_2 = \Sigma - \operatorname{Int} D_1$ and $F_2 = D_2 \cap F$. Then by (ii) $0 = \chi(F_2; G) = \chi(F - x; G)$ and (i) follows.

To prove (iii), we note that a consideration of the exact sequence of the pair $(F - x, F_1)$ yields the exact sequence

$$\begin{aligned} 0 \to H_{n-1}(F_1) \to H_{n-1}(F - x) \to H_{n-1}(F - x, F_1) \to \cdots \\ \to H_1(F_1) \to H_1(F - x) \\ \to H_1(F - x, F_1) \to 0 \end{aligned}$$

Hence, by 2.4 and (i)

$$0 = \chi(F - x; G) = \chi(F_1; G) + \sum_{i=1}^{n-1} (-1)^i [H_i(F - x, F_1)].$$

For $i \leq n - 1$, we have the isomorphisms

$$H_i(F - x, F_1) \to H_i(F, F_1) \xleftarrow{e_*} H_i(F_2, F_0) \xleftarrow{d} H^{n-i}(F_2)$$

where $F_0 = F_1 \cap F_2 = \partial F_2$, e_* is an excision, and d is duality. Since $H_j(F_2) \in$

$\mathscr{D}(G)$ for $j \geq 1$, we also have the isomorphism

$$\operatorname{Ext}[H_{n-i-1}(F_2); Z] \to H^{n-i}(F_2)$$

It follows that $H_i(F-x, F_1) \approx H_{n-i-1}(F_2)$ for $i \leq n-2$ and that $H_{n-1}(F-x, F_1) = 0$. A similar argument shows that $H_{n-1}(F-x)$ vanishes. Hence so do $H_{n-1}(F_j)$ for $j = 1, 2$. It now follows that

$$\sum_{i=1}^{n-1} (-1)^i [H_i(F-x, F_1)] = (-1)^{n-1} \chi(F_2; G)$$

and (iii) follows from the previous equation.

The proof that $\chi(F_1; G) = 0$ implies that the splitting of F is induced by a splitting of the action is contained in the corollary to the following proposition.

PROPOSITION 3.2. *Let $m \geq 6$. Let M^m be a smooth $(m-1)$-connected m-manifold with a semi-free G-action. Let $N^n = M^G$ be the fixed point set of this action and let $N_1 \subset N$ be a connected codimension 0 submanifold with boundary such that $\tilde{H}_i(N_1) \in \mathscr{D}(G)$ for $i > 0$. If $2n+1 \leq m$, then there exists an $[m/2]-1$ connected G-invariant submanifold $M_1^m \subset M^m$ such that $M_1^G = N_1$ and $(-1)^i [H_i(M_1)] = \chi(N_1; G)$ for $i = [m/2]$.*

COROLLARY 3.3 *If $2n+2 \leq m$ and $\chi(N_1; G) = 0$, then there exists a G-invariant disk $D^m \subset M^m$ with $D^G = N_1$.*

Proof of 3.2. Let $p: E \to N$ be a closed G-tubular neighborhood of N. Let $\tilde{Q} = M - \operatorname{Int} E$, $Q = (M - \operatorname{Int} E)/G$, and $q: \tilde{Q} \to Q$ be the obvious quotient map. Using the facts that $H_i(\partial E) \to H_i(E)$ is an isomorphism for $i \leq m-n-2$ and that $2n+1 \leq m$, it is easy to show that $\tilde{Q}$ is $[m/2]-1$ connected. Let $E_1 = p^{-1}(N_1)$, $\tilde{Q}_1 = S(E_1)$ the sphere bundle of E_1, and $Q_1 = S(E_1)/G$. Then $Q_1 \subset \partial Q$. Note that since $p: \tilde{Q}_1 \to N_1$ has fiber S^{m-n-1} and $2n+1 \leq m$, $p_*: H_i(\tilde{Q}_1) \to H_i(N_1)$ is an isomorphism for $i \leq n-1$.

We shall construct a sequence of submanifolds $V_i \subset Q$, $0 \leq i \leq [m/2]-1$ such that the following conditions hold:

(a) V_i is a smooth regular neighborhood of a CW complex $X_i = Q_1 \cup$ cells of dimension $\leq i+1$.

(b) (Q, V_i) is $i+1$ connected.

(c) There exists a short exact sequence

$$0 \longrightarrow H_{i+1}(\tilde{Q}_1) \xrightarrow{k_*} H_{i+1}(\tilde{V}_i) \longrightarrow P \longrightarrow 0$$

where $\tilde{V}_i = q^{-1}(V_i)$, $k:\tilde{Q}_1 \to \tilde{V}_i$ is the inclusion, and $P \in \mathcal{P}(G)$ satisfies

$$(-1)^{i+1}[P] = \sum_{j=1}^{i} (-1)^j [H_j(N_1)]$$

(d) For $j > i+1$, $K_*: H_j(\tilde{Q}_1) \to H_j(\tilde{V}_i)$ is an isomorphism.

If we let $M_1 = E_1 \cup \tilde{V}_i$ for $i = [m/2]-1$, then M_1 is the desired submanifold of M. To see this, note that by (b), $\tilde{V}_i$ is $[m/2]-1$ connected since $\tilde{Q}$ is. Then Mayer–Victoris and Van Kampen arguments show that M_1 is $[m/2]-1$ connected and that for $i=[m/2]$, $H_i(M_i) \in \mathcal{P}(G)$ and satisfies $(-1)^i[H_i(M_1)] = \chi(N_1; G)$, where we have used (c).

The construction of the manifolds V_i starts by letting V_0 be a smooth collar neighborhood of Q_1 in Q. Clearly (a) through (d) hold. Suppose V_i satisfying (a–d) has been constructed for some i, $1 \le i < [m/2]-1$. (The construction of V_1 from V_0 is slightly different from the general case and will be described below.) Let $\partial_+ V_i = \mathrm{Cl}\,[\partial V_i - Q_1]$.

LEMMA 3.4. *The inclusion* $\pi_j(\mathrm{Cl}\,(Q - V_i), \partial_+ V_i) \to \pi_j(Q, V_i)$ *is an isomorphism for* $j \le [m/2]$.

Proof. Since V_i has the homotopy type of $Q_i \cup$ cells of dimension $\le i+1$, and the image of any map may be pushed off Q_1 by using a collar, $(V_i, \partial_+ V_i)$ is $m-(i+1)-1$ connected by general position. It follows that the inclusion maps $\pi_1(\partial_+ V_i) \to \pi_1(V_i) \to \pi_1(Q)$ are all isomorphisms. Hence, so is $\pi_1(\mathrm{Cl}\,(Q - V_i)) \to \pi_1(Q)$. By homotopy excision $\pi_j(\mathrm{Cl}\,(Q - V_i), \partial_+ V_i) \to \pi_j(Q, V_i)$ is an isomorphism for $j \le m-i-2$. Since $i < [m/2]-1$, $[m/2] \le m-i-2$ and the lemma follows.

We note that (b) implies that $\tilde{V}_i$ is i-connected since Q is. Hence we have isomorphisms, $H_{i+1}(\tilde{V}_i) \leftarrow \pi_{i+1}(\tilde{V}_i) \to \pi_{i+1}(V_i)$ since $i \ge 1$. Since $p_*: H_{i+1}(\tilde{Q}_1) \to H_{i+1}(N_1)$ is an isomorphism, $H_{i+1}(\tilde{Q}_1) \in \mathcal{D}(G)$ and we may choose a resolution

$$0 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\rho} H_{i+1}(\tilde{V}_i) \longrightarrow 0$$

with $P_k \in \mathcal{P}(G)$ $(k=0,1)$ such that P_0 is a free ZG module with basis $e_1, \dots, e_s$. Let $\rho': P_0 \to \pi_{i+2}(\mathrm{Cl}\,(Q - V_i), \partial_+ V_i)$ be a homomorphism such that the diagram below commutes

$$\begin{tikzcd}
 & & P_0 \arrow[dll, "\rho'"'] \arrow[d, "\rho"] \\
\pi_{i+2}(\mathrm{Cl}\,(Q - V_i), \partial_+ V_i) \arrow[r, "\approx"] & \pi_{i+2}(Q, V_i) \arrow[r, "\partial"] & \pi_{i+1}(V_i) = H_{i+1}(\tilde{V}_i)
\end{tikzcd}$$

We note that if $i<[m/2]-2$, then ∂ is an isomorphism and ρ' is unique, while, if $i=[m/2]-2$, ∂ is onto, but may not be an isomorphism. In this case then ρ' may not be unique.

Let $f_j:(D^{i+2}, S^{i+1})\to(\mathrm{Cl}\,(Q-V_i), \partial_+V_i)$ be a map representing $\rho'(e_j)$ $(1\leq j\leq s)$. General position, if $i<[m/2]-2$, or standard piping arguments if $i=[m/2]-2$, show that we may assume the f_j are embeddings with mutually disjoint images. We now let V_{i+1} *be a smooth regular neighborhood if* $V_i\cup\bigcup_{j=1}^{s} f_j(D^{i+2})$. It is obvious from the construction that V_{i+1} satisfies (a), while an examination of the exact sequence

$$
\begin{tikzcd}
\to \pi_{i+2}(V_{i+1}, V_i) \arrow[r] \arrow[d, equal] & \pi_{i+2}(Q, V_i) \arrow[r] \arrow[d, equal] & \pi_{i+2}(Q, V_{i+1}) \to \\
P_0 \arrow[r, "\rho'"] & H_{i+1}(\tilde{V}_i) &
\end{tikzcd}
$$

shows that (b) holds.

To prove (c) and (d), we note that since (V_{i+1}, V_i) is $(i+1)$ connected, $H_p(\tilde{V}_{i+1}, \tilde{V}_i)=0$ for $p\neq i+2$ and $H_{i+2}(\tilde{V}_{i+1}, \tilde{V}_i)=P_0$. It follows that $k_*:H_j(\tilde{Q}_1)\to H_j(\tilde{V}_{i+1})$ is an isomorphism for $j>i+2$. Furthermore, the diagram below commutes and has an exact row:

$$
\begin{tikzcd}
0 \arrow[r] & H_{i+2}(\tilde{V}_i) \arrow[r] & H_{i+2}(\tilde{V}_{i+1}) \arrow[r] & H_{i+2}(\tilde{V}_{i+1}, \tilde{V}_i) \arrow[r] & H_{i+1}(\tilde{V}_i) \arrow[r] & 0 \\
& H_{i+2}(\tilde{Q}_1) \arrow[u, "k_*", "\approx"'] \arrow[ur] & & P_0 \arrow[u, "\approx"] \arrow[ur, "\rho"] & &
\end{tikzcd}
$$

Since $\ker\rho=P_1$, we obtain the short exact sequence

$$0\to H_{i+2}(\tilde{Q}_1)\to H_{i+2}(\tilde{V}_{i+1})\to P_1\to 0$$

Since P_0 is free, we have

$$(-1)^{i+2}[P_1]=(-1)^{i+1}[H_{i+1}(\tilde{V}_i)]=$$
$$(-1)^{i+1}[H_{i+1}(\tilde{Q}_1)]+(-1)^{i+1}[P]=\sum_{j=1}^{i+1}(-1)^j[H_j(N_1)]$$

by the inductive hypothesis and since $p_*:H_{i+1}(\tilde{Q}_1)\to H_{i+1}(N_1)$ is an isomorphism. This completes the proof of (c) and (d) and of the inductive step.

The construction of V_1 from V_0 is similar to the above argument. In this case, we note that there is a commutative diagram with exact rows and the indicated isomorphism

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_2(Q, Q_1) & \longrightarrow & \pi_1(Q_1) & \longrightarrow & \pi_1(Q) & \longrightarrow & 1 \\ & & & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \approx} & & \\ 1 & \longrightarrow & \pi_1(\tilde{Q}_1) & \longrightarrow & \pi_1(Q_1) & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

since $\tilde{Q}$ is 2-connected and $q: \tilde{Q}_1 \to Q_1$ is a covering. Hence, there is an isomorphism $\pi_2(Q, Q_1) \to \pi_1(\tilde{Q}_1)$. The argument now proceeds essentially as above to kill $\pi_2(Q, V_0) = \pi_2(Q, Q_1)$. The details are left to the reader.

Proof of 3.3. If $m \geq 6$, let V_{n-1} be the manifold constructed in the proof above and notice that (c) implies that $H_n(\tilde{V}_{n-1}) \in \mathscr{P}(G)$ and that $(-1)^n[H_n(\tilde{V}_{n-1})] = \chi(N_1; G)$ since $p_*: H_n(\tilde{Q}_1) \to H_n(N_1) = 0$ is an isomorphism. Since $\chi(N_1; G) = 0$, we may attach trivial n-handles to V_{n-1} in Q to make $H_n(\tilde{V}_{n-1})$ free over ZG. Since $2n+2 \leq m$ we may then attach $(n+1)$ handles to $\tilde{V}_{n-1}$ as above to kill $H_n(\tilde{V}_{n-1})$ without introducing new homology.

If the manifold so obtained is denoted by V_n, then $W_n = \tilde{V}_n \cup E_1$ is contractible manifold with simply connected boundary. Hence it is a disk and 3.3 holds for $m \geq 6$.

If $m < 6$, then $n = 1$. In this case, N_1 must be an interval and E_1 is the desired disk.

4. The proof of Theorem B

In this section, we give the proof of Theorem B. It is based on the following lemmas whose proofs are given at the end of this section.

LEMMA 4.1. *Let $n \neq 3, 4$. Let F^n be a closed n-manifold such that $H_i(F) \in \mathscr{D}(G)$ for $i < n$. If either $n = 2m$ or $n = 2m+1$, $H_1(\pi_1(F); M) = 0$ for all $\pi_1(F)$ modules M, and $|H_m(F)|$ is a square, then there exists an admissible strong balanced splitting of F.*

ADDENDUM 4.2. *The splitting above can be chosen so that $H_i(F_j) \to H_i(F)$ is an isomorphism for $i \leq [n/2] - 1$ and $j = 1, 2$.*

LEMMA 4.3. *Let $F^n = \Sigma^G$ where G acts semi-freely on the homotopy sphere Σ. Suppose there exists an admissible balanced splitting $F_1 \cup F_2$ of F.*

(i) *If $n=2m$, let $A = \ker(H_{m-1}(F_1) \to H_{m-1}(F))$. Then*

$$\tilde{\chi}_{1/2}(F; G) = \chi(F_1; G) - 2\sum_{i=m}^{2m-1}(-1)^i[H_i(F_1)] + (-1)^m[A].$$

(ii) *If $n=2m+1$, then $|H_m(F)|$ is a square,*

$$\tilde{\chi}_{1/2}(F; G) = \chi(F_1; G), \quad \textit{and} \quad \tilde{\chi}_{1/2}(F, G) = \overline{-\tilde{\chi}_{1/2}(F; G)}.$$

LEMMA 4.4. *Let $n=2m$ $(m \geq 3)$ and q be any integer prime to $|G|$. Let $F_1 \cup F_2$ be an admissible strong balanced splitting of F.*

(i) *For any integer l $(1 \leq l < m-1)$ there exists an admissible strong balanced splitting $F_1' \cup F_2'$ of F such that*

$$\chi(F_1'; G) = \chi(F_1; G) + (-1)^l 2k_*[Z/q]$$

(ii) *There exists an admissible balanced splitting $F_1'' \cup F_2''$ of F such that*

$$\chi(F_1''; G) = \chi(F_1; G) + (-1)^{m-1}k_*[Z/q].$$

It follows from 2.3 that if $n=2m$, then $\tilde{\chi}_{1/2}(F)$ represents a cohomology class $\chi_{1/2}(F) \in H^n(Z/2; A(G))$. In this case any other representative of this class differs from $\tilde{\chi}_{1/2}(F)$ by an element of the form $2a$ for some $a \in A(G)$. If $n=2m+1$, then $\tilde{\chi}_{1/2}(F)$ represents an element of $H^n(Z/2; A(G))$ by 4.3. In this case, $H^n(Z/2; A(G)) = \{x \in A(G) \mid x = -\bar{x}\}/\{y - \bar{y} \mid y \in A(G)\} = \{x \in A(G) \mid 2x = 0\}$ since $x = \bar{x}$ for all $x \in A(G)$.

Proof of Theorem B. Suppose first that $n=2m$. By 4.1, there exists an admissible strong balanced splitting $F_1 \cup F_2$ of $F = \Sigma^G$. By 2.2, $(-1)^m\chi(F_1, G) = k_*[Z/q]$ for some integer q. By 4.4, there exists an admissible balanced splitting $F_1'' \cup F_2''$ such that $\chi(F_1''; G) = \chi(F_1; G) + (-1)^{m-1}k_*[Z/q] = 0$. By Theorem A, the splitting $F_1'' \cup F_2''$ extends to a balanced splitting of the action. This establishes the first sentence of Part (i) of Theorem B.

To establish the second sentence of Part (i), we use 4.1 and 4.2 and let $F_1 \cup F_2$ be an admissible strong balanced splitting of F such that $H_{m-1}(F_1) \to H_{m-1}(F)$ is an isomorphism. From 4.3, we then see that $\tilde{\chi}_{1/2}(F; G) = \chi(F_1; G) + 2a$ for some

$a \in A(G)$. But also if $\chi_{1/2}(F; G) = 0$, then $\tilde{\chi}_{1/2}(F, G) = 2b$ for some $b \in A(G)$. Hence $\chi(F_1; G) = 2(b - a)$. We now use 4.4(i) and proceed as above to obtain a strong balanced splitting of the action.

Suppose now that the action has a strong balanced splitting inducing the strong balanced splitting $F_1 \cup F_2$ of F with $F_0 = F_1 \cap F_2$. In this case, since $\ker(H_{m-1}(F_0) \to H_{m-1}(F_1)) = \ker(H_{m-1}(F_0) \to H_{m-1}(F_2))$ by definition, a simple chase of the diagram

$$\begin{tikzcd}
H_m(F, F_1) \arrow[r] & H_{m-1}(F_1) \arrow[r] & H_{m-1}(F) \\
H_m(F_2, F_0) \arrow[r] \arrow[u, "\approx"] & H_{m-1}(F_0) \arrow[r] \arrow[u] & H_{m-1}(F_2) \arrow[u]
\end{tikzcd}$$

shows that $A = \ker(H_{m-1}(F_1) \to H_{m-1}(F)) = 0$. But also $\chi(F_1; G) = 0$ by 3.1. Hence by 4.3, $\tilde{\chi}_{1/2}(F; G) = 2c$ from which it follows that $\chi_{1/2}(F; G) = 0$.

To prove (ii) of Theorem B, suppose the action admits a balanced splitting $D_1 \cup D_2$. Then $|H_m(F)|$ is a square and $0 = \chi(F_1; G) = \tilde{\chi}_{1/2}(F; G) = \chi_{1/2}(F; G)$ by Theorem A, 4.3, and the remarks preceeding this proof. The first of these conditions, however, implies that F has an admissible strong balanced splitting $F_1' \cup F_2'$ by 4.1; while the latter condition implies that the splitting $F_1' \cup F_2'$ extends to a strong balanced splitting of the action by 4.3 and Theorem A. Since a strong balanced splitting of the action is balanced the proof of Theorem B is completed.

We turn to the proofs of 4.1–4.4.

Proofs of 4.1 *and* 4.2. If $n = 2$, then $F^n = S^2$ obviously has an admissible strong balanced splitting. Suppose now that $n \geq 5$. We consider first the case when $n = 2m$. Fix a handlebody decomposition of F and consider K^{m-1} where K^j denotes the union of all handles of index $\leq j$. For $i < m - 1$, $H_i(K^{m-1}) \to H_i(F)$ is an isomorphism while, for $i = m - 1$, there is an exact sequence

$$0 \longrightarrow H_m(K^m) \longrightarrow H_m(K^m, K^{m-1}) \xrightarrow{\partial} H_{m-1}(K^{m-1}) \xrightarrow{i_*} H_{m-1}(K^m) \longrightarrow 0$$

where $H_{m-1}(K^m) \to H_{m-1}(F)$ is an isomorphism, $H_m(K^m, K^{m-1})$ is free abelian on the handles of index m, and $H_{m-1}(K^{m-1})$ is free abelian since K^{m-1} has the homotopy type of an $m-1$ complex. Let $T = \ker i_*$. Then T is free abelian and $\partial: H_m(K^m, K^{m-1}) \to T$ is a split epimorphism. Hence, if $x_1, \ldots, x_s$ are free generators for T, we may regard these classes as lying in $H_m(K^m, K^{m-1})$. Since (K^m, K^{m-1}) is $(m-1)$-connected, these classes may be represented by maps $f_i: (D_i^m, S^{m-1}) \to (K^m, K^{m-1})$, $i = 1, \ldots, s$. Since $n \geq 5$, by general position and standard embedding theorems we may factor $f = \cup f_i$ through an embedding

$f': \bigcup_{i=1}^{s} (D_i^m, S^{m-1}) \to (K^m - \mathrm{Int}\, K_0, \partial K_0)$, where $K_0 = K^{m-1}$ - an open boundary collar, and the inclusion $(K^m - \mathrm{Int}\, K_0, \partial K_0) \to (K^m, K^{m-1})$. Let F_1 be a thickening of $K_0 \cup \bigcup_{i=1}^{s} D_i^m$ where f' is the attaching map.

A straightforward argument shows that $H_i(F_1) \to H_i(F)$ is an isomorphism for $i \leq m-1$ and that $H_i(F_1) = 0$ for $i \geq m$. An analysis of the exact sequence of the pair (F, F_1), using techniques similar to those in the proof of 3.1, then shows that $H_i(F_2) \to H_i(F)$ is an isomorphism for $i \leq m-1$ and that $H_i(F_2) = 0$ for $i \geq m$ where $F_2 = F - \mathrm{Int}\, F_1$. This establishes 4.1 and 4.2 when $n = 2m$.

The proof of 4.1 in the case when $n = 2m+1$ requires the following lemma.

LEMMA 4.5. *Let B be a finite abelian group such that $|B|$ is a square. Then there exists a short exact sequence $0 \to A \to B \to \bar{A} \to 0$ where $\bar{A}$ is isomorphic to A.*

Proof. We divide the proof into several cases. For the first case, suppose $B = Z/p^{2s}$. Then $0 \to Z/p^s \to Z/p^{2s} \to Z/p^s \to 0$ where the first map is multiplication by p^s, is the desired short exact sequence. For the second case, suppose $B = Z/p^r \oplus Z/p^s$ where r and s are odd and $r \leq s$. In this case, set $t = (r+s)/2$, let $\rho_1 : Z/p^t \to Z/p^r$ be the obvious epimorphism, and let $\rho_2 : Z/p^s$ be multiplication by p^{s-t}. Then

$$0 \longrightarrow Z/p^t \xrightarrow{(\rho_1, \rho_2)} Z/p^r \oplus Z/p^2 \longrightarrow \mathrm{coker}\,(\rho_1, \rho_2) \longrightarrow 0$$

is the desired short exact sequence.

In the general case, write B as the direct sum of its p-primary components where p is a prime. Each such summand can be written as a direct sum of groups of the forms Z/p^{2s} and $Z/p^r \oplus Z/p^s$ where r and s are odd and $r \leq s$. The desired conclusion now follows obviously from the first two cases.

The proof of 4.1 when $n = 2m+1$ and $m \geq 2$ now proceeds as follows. Fix a handle decomposition of F. Let K^j be the union of all handles of index $\leq j$ and consider K^{m-1}. As before, we have an exact sequence

$$0 \longrightarrow H_m(K^m) \xrightarrow{i_*} H_m(K^m, K^{m-1}) \xrightarrow{\partial} H_{m-1}(K^{m-1}) \longrightarrow H_{m-1}(K^m) \longrightarrow 0$$

and an epimorphism $H_m(K^m) \to H_m(F)$. Since $|H_m(F)|$ is a square, there exists a subgroup $A \subset H_m(F)$ and a short exact sequence $0 \to A \to H_m(F) \to \bar{A} \to 0$ with $\bar{A}$ isomorphic to A. Pick elements $x_1, \ldots, x_s \in H_m(K^m)$ projecting to generators of A and a basis $y_1, \ldots, y_t$ for a direct summand of $H_m(K^m, K^{m-1})$ that maps isomorphically onto $\ker \partial$. We may proceed as in the case when $n=2m$ to add handles of index m to K^{m-1} to kill the classes $j_*x_1, \ldots, j_*x_s; y_1, \ldots, y_t$. The resulting submanifold $F_0 \subset K^m$ has the following properties

(i) F_0 is a smooth regular neighborhood of an m-complex;
(ii) (F, F_0) is $(m-1)$ connected;
(iii) $i_*: H_{m-1}(F_0) \to H_{m-1}(F)$ is an isomorphism; and
(iv) $i_*: H_m(F_0) \to H_m(F)$ maps onto A.

It follows from these properties that all the groups $\pi_1(\partial F_0)$, $\pi_1(F - \operatorname{Int} F_0)$ and $\pi_1(F)$ are isomorphic and that the pair $(F - \operatorname{Int} F_0, \partial F_0)$ is $(m-1)$ connected.

LEMMA 4.6. *If $H_1(\pi_1(F), M) = 0$ for all $\pi_1(F)$ modules, then the Hurewicz map $h: \pi_{m+1}(F - \operatorname{Int} F_0, \partial F_0) \to H_{m+1}(F - \operatorname{Int} F_0, \partial F_0)$ is onto.*

Proof. Since $\pi_1(F) = \pi_1(\partial F_0) = \pi_1(F - \operatorname{Int} F_0)$, the spectral sequence of the universal covering $p: (\widetilde{F - \operatorname{Int} F_0}, \partial\widetilde{F}_0) \to (F - \operatorname{Int} F_0, \partial F_0)$ shows that $p_*: H_{k+1}(\widetilde{F - \operatorname{Int} F_0}, \partial\widetilde{F}_0) \to H_{k+1}(F - \operatorname{Int} F_0, \partial F_0)$ is onto. But so is $h: \pi_{k+1}(\widetilde{F - \operatorname{Int} F_0}, \partial\widetilde{F}_0) \to H_{k+1}(\widetilde{F - \operatorname{Int} F_0}, \partial\widetilde{F}_0)$. The lemma now follows easily.

We remark that the proof really requires only that $H_1(\pi_1(F), M) = 0$ when $M = H_m(\widetilde{F - \operatorname{Int} F_0}, \partial\widetilde{F}_0)$. In particular the lemma holds if $\pi_1(F) = 0$ or if $H_m(F) = 0$.

It follows from 4.6 that the Hurewicz map h below is onto.

$$
\begin{tikzcd}
 & \pi_{m+1}(F - \operatorname{Int} F_0, \partial F_0) \arrow[d, "h"] & & & \\
 & H_{m+1}(F - \operatorname{Int} F_0, \partial F_0) \arrow[d, "\approx"] & & & \\
{} \arrow[r] & H_{m+1}(F, F_0) \arrow[r, "\partial"] & H_m(F_0) \arrow[r] & H_m(F) \arrow[r] & {}
\end{tikzcd}
$$

In particular, if we let $z_1, \ldots, z_r$ generate a free abelian direct summand S of $H_{m+1}(F, F_0)$ such that $\partial: S \to \operatorname{Im} \partial$ is an isomorphism, then there exist maps $f_i: (D^{m+1}, S^m) \to (F - \operatorname{Int} F_0, \partial F_0)$ such that f_i represents z_i. By standard embedding theorems, the maps f_i may be taken to be embeddings. We may now proceed essentially as before to construct a submanifold $F_1 \subset F$ such that (F, F_1) is $(m-1)$ connected, $i_*: H_{m-1}(F_1) \to H_{m-1}(F)$ is an isomorphism, and $i_*: H_m(F_1) \to H_m(F)$ is an isomorphism onto A. But then F_1, $F - \operatorname{Int} F_1$ is the desired strong balanced splitting of F.

Proof of 4.3. Suppose first that $n = 2m+1$. Let $A = \ker (H_{m-1}(F_1) \to$

$H_{m-1}(F))$ and $B = \ker(H_m(F_1) \to H_m(F))$. Then the exact sequence of (F, F_1) may be factored into the exact sequences

$$0 \to A \to H_{m-1}(F_1) \to H_{m-1}(F) \to \cdots \to H_1(F) \to H_1(F, F_1) \to 0$$

$$0 \to B \to H_m(F_1) \to H_m(F) \to H_m(F, F_1) \to A \to 0$$

$$0 \to H_{2m-1}(F_1) \to H_{2m-1}(F) \to \cdots \to H_{m+1}(F) \to H_{m+1}(F, F_1) \to B \to 0$$

where the expected terms of degree $2m$ have dropped out of the third sequence by duality (cf. the proof of 3.1). By applying the second part of 2.4 to the third sequence, we obtain the equation

$$(-1)^m[B] = \sum_{i=m+1}^{2m-1} (-1)^i[H_i(F)] - \sum_{i=m+1}^{2m-1} (-1)^i[H_i(F_1)] - \sum_{i=m+1}^{2m-1} (-1)^i[H_i(F, F_1)].$$

The proof of 3.1 shows that $H_i(F) \approx H_{2m-i}(F)$ and that $H_i(F, F_1) \approx H_{2m-i}(F_2) \approx H_{2m-i}(F_1)$ where the last isomorphism follows from the fact that $F_1 \cup F_2$ is a balanced splitting of F. Hence

$$(-1)^m[B] = \sum_{i=1}^{m-1} (-1)^i[H_i(F)] - \sum_{i=1}^{m-1} (-1)^i[H_i(F_1)] - \sum_{i=m+1}^{2m-1} (-1)^i[H_i(F_1)]$$

A similar computation starting with the first sequence gives

$$(-1)^m[A] = \sum_{i=1}^{m-1} (-1)^i[H_i(F)] - \sum_{i=1}^{m-1} (-1)^i[H_i(F_1)] - \sum_{i=m+1}^{2m-1} (-1)^i[H_i(F_1)]. \tag{1}$$

Hence $[A] = [B]$ in $G_0(\mathscr{D}(G))$.

Similar reasoning applied to the second sequence shows that

$$\begin{aligned}(-1)^m[H_m(F)] &= (-1)^m[H_m(F_1)] + (-1)^m[H_m(F, F_1)] \\ &\quad + (-1)^{m+1}[B] + (-1)^{m-1}[A] \\ &= (-1)^m 2[H_m(F_1)] + (-1)^{m+1} 2[A]\end{aligned}$$

Hence $[H_m(F)] = 2\{[H_m(F_1) - [A]\}$ in $G_0(\mathscr{D}(G))$. It now follows from 2.1 that $|H_m(F)|$ is a square and that

$$[Z/q] = [H_m(F_1)] - [A]$$

where $|H_m(F)| = q^2$. By combining this with equation (1), we seee that $\tilde{\chi}_{1/2}(F; G) = \chi(F_1; G)$.

Since $F_1 \cup F_2$ is a balanced splitting of F, and $n = 2m+1$, $\chi(F_1; G) = -\chi(F_1; G)$ by 3.1. Hence, from above, $\tilde{\chi}_{1/2}(F; G) = -\tilde{\chi}_{1/2}(F; G)$. The conclusion that $\tilde{\chi}_{1/2}(F; G) = -\overline{\tilde{\chi}_{1/2}(F; G)}$ now follows from 2.3.

The argument above establishes 4.3 in the case when $n = 2m+1$. If $n = 2m$, the formula for $\tilde{\chi}_{1/2}(F; G)$ is obtained by similar arguements starting with the first of the exact sequences above.

Before proving 4.4, we introduce some notation and prove a sublemma. Let n, l, and q be positive integers with $l < n$. Then $M_n(l, q)$ will denote any n-manifold with boundary such that

$$\tilde{H}_i(M_n(l, q)) = \begin{cases} 0 & \text{if } i \neq l \\ Z/q & \text{if } i = l \end{cases}$$

SUBLEMMA 4.7. *Let $n \geq 5$ and $1 \leq l \leq n-4$. Then S^n has a splitting of the form $M_n(l, q) \cup M_n(n-l-2, q)$.*

Proof. Let $X = S^1 \bigcup_q e^2$ where q denotes a map of degree q. Then X embeds in S^5 and by suspending, we obtain embeddings $S^l \bigcup_q e^{l+1} = \Sigma^{l-1} X \to \Sigma^{l-1} S^5 = S^{l+4} \subset S^n$. Let F_1 be a regular neighborhood of $S^l \bigcup_q e^{l+1}$ and $F_2 = S^n - \operatorname{Int} F_1$. Then F_1 is an $M_n(l, q)$; while F_2 is an $M_n(n-l-2, q)$ by Alexander duality.

We shall denote S^n together with the splitting of 4.7 by $S^n(l, q)$. If $F = F_1 \cup F_2$ is a splitting of F^n, we may take the connected sum of this splitting with $S^n(l, q)$ along $F_0 = F_1 \cap F_2$ and $M_n(l, q) \cap M_n(n-l-2, q) = M_0$ to obtain the new splitting $F_1 \# M_n(l, q) \cup F_2 \# M_n(n-l-2, q)$ of F. We denote this splitting by $F \# S^n(l, q)$.

Proof of 4.4. Let $F_1 \cup F_2$ be an admissible strong balanced splitting of F. To prove (i), consider the splitting $F \# S^n(l, q) \# S^n(n-l-2, q) = F_1' \cup F_2'$. A simple computation shows that the homomorphism $H_i(F_j) \to H_i(F_j')$ $(j = 1, 2)$ is an isomorphism for $i \neq l$, $n-l-2$ and that the sequence

$$0 \to H_i(F_j) \to H_i(F_j') \to H_i(F_j', F_j) \to 0$$

is split exact for $i = l$, $n-l-2$. In this case, $H_i(F_j', F_j) = Z/q$ and it follows easily that $\chi(F_1'; G) = \chi(F_1; G) + (-1)^l 2[Z/q]$ when $n = 2m$. It is also easy to see that $F_1' \cup F_2'$ is an admissible strong balanced splitting of F; hence, (i) follows.

The proof of part (ii) is similar. One takes $F \# S^n(m-1, q)$.

5. The proof of Theorem C

In this section, we prove Theorem C by constructing the appropriate examples. We first set some notation.

Let X be a finite CW complex. Then $F^n(X)$ will denote the double, $N^n(X) \cup N^n(X)$, where $N^n(X)$ is a smooth regular neighborhood of X in R^n where $n \geq 2 \dim X + 2$.

We will let $j_*: H^n(Z/2; A(G) \to H^n(Z/2; \tilde{K}_0(G))$ be the map induced by the inclusion $j: A(G) \to \tilde{K}_0(G)$ and $H: H^n(Z/2; \tilde{K}_0(G)) \to L_n^h(G)$ be the hyperbolic map of Ranicki [12; Theorem 4.3]. We recall that there exists an exact sequence

$$\cdots \longrightarrow L_{n+1}^p(G) \longrightarrow H^n(Z/2; \tilde{K}_0(G)) \xrightarrow{H} L_n^h(G) \longrightarrow L_n^p(G) \longrightarrow \cdots$$

due to Shaneson [15], and Ranicki [12; Theorem 4.3]. (The reader will note that the usual dimension, $n+1$, for the cohomology group has been replaced by n since we are using the involution $[P] \to -[P^*]$ instead of the usual involution.)

Theorem C will be deduced from the following propositions whose proofs are temporarily deferred.

PROPOSITION 5.1. *Let G be a finite group admitting a free linear representation of dimension d. Let X be a finite CW complex such that $\tilde{H}_i(X) \in \mathcal{D}(G)$ for all $i > 0$. Let $n \geq 2 \dim X + 2$ and $k \geq n + 6$ be such that $k \equiv 0 \pmod d$. Suppose that either*

(i) *$n + k$ is even and $Hj_*[\chi(X; G)] = 0$; or*
(ii) *$n + k$ is odd, $|G|$ is even, $2\chi(X; G) = 0$ in $A(G)$, and $Hj_*[\chi(X; G)] = 0$.*

Then there exists a semi-free action of G on a homotopy sphere Σ^{n+k} with $\Sigma^G = F^n(X)$.

In this proposition $[\chi(X; G)]$ denotes the class of $\chi(X; G)$ in $H^n(Z/2; A(G))$. The condition that $2\chi(X; G) = 0$ in (ii) above is needed to insure that $[\chi(X; G)]$ is defined.

PROPOSITION 5.2. *Let $G = Q2^l$ be the generalized quaternion group of order 2^l and let $H: H^n(Z/2; \tilde{K}_0(G)) \to L_n^h(G)$ be the hyperbolic map.*

(i) *If $n \not\equiv 1 \pmod 4$, then $H = 0$.*
(ii) *If $n \equiv 1 \pmod 4$, then H is injective.*

Proof of Theorem C. Let X be the Moore space $S^1 \cup_f e^2$ where $f: S^1 \to S^1$ has degree 3. Then $\chi(X; Q2^l) = [\langle 3, N \rangle] \neq 0$ in $A(Q2^l) = Z/2$ by [18; Proposition 3.5]. Theorem C now follows directly from 5.1 and 5.2.

Proof of 5.1. Since $k \equiv 0 \pmod d$, there exists a free orthogonal representation

W^k of G of dimension k. Furthermore, k is even since d is. Let G act trivially on R^n and diagonally on $R^n \times W^k$. Then $R^n \times 0$ is the fixed point set. Embed X in $R^n \times 0$ with smooth regular neighborhood $N^n(X)$. Let the pair $(R^n \times W^k, N(X))$ play the role of the pair (M, N_1) in Proposition 3.2.

Following the notation of 3.2, we let $\tilde{Q} = R^n \times (W - \operatorname{Int} D(W))$, $\tilde{Q}_1 = N(X) \times S(W)$, $Q = \tilde{Q}/G$, and $Q_1 = \tilde{Q}/G$ where $D(W)$ and $S(W)$ are the unit disk and sphere of W respectively. We also let $V_0 \subset Q$ be the submanifold, constructed in 3.2, that was denoted there by V_i for $i = [n + k/2] - 1$. The proof now breaks up into two cases:

Case I. n is even. In this case, we set $n = 2p$, $k = 2q$, and $n + k = 2r$. The restrictions on n and k imply that $p + 2 \leq q$ and that $n + 3 \leq r \leq k - 3$. By 3.2 and its proof, (Q, V_0) is r-connected and $H_r(\tilde{V}_0) = P_0 \in \mathcal{P}(G)$ and satisfies $(-1)^r[P_0] = \chi(X; G)$. Furthermore, since Q is $k-2$ connected and $r \leq k-3$, there are isomorphisms

$$H_r(\tilde{V}_0) \xleftarrow{\approx} \pi_r(\tilde{V}_0) \xrightarrow{\approx} \pi_r(V_0) \xleftarrow{\approx} \pi_{r+1}(Q, V_0)$$

Let $\sigma : H_r(\tilde{V}_0) \to \pi_{r+1}(Q, V_0)$ be the composite isomorphism. By the methods of [20; Chap. 1] each class $x \in H_r(\tilde{V}_0)$ determines a preferred class of immersions $f_x : S^r \times D^r \to V_0$ which may be used to define mutual and self intersections, λ and μ respectively, as in [20; Chap. 5].

LEMMA 5.3. *For all* $x, y \in H_r(\tilde{V}_0)$, $\lambda(x, y) = 0$.

Proof. Let S_x^r and S_y^r be immersed spheres in V_0 representing $\sigma(x)$ and $\sigma(y)$ respectively. Then by [20; Chap. 5], $\lambda(x, y) \in ZG$ is given by

$$\lambda(x, y) = \sum_{g \in G} \langle \tilde{S}_x^r, \tilde{S}_y^r g^{-1} \rangle g$$

where $\tilde{S}_x$ and $\tilde{S}_y$ are immersed spheres in $\tilde{V}_0$ covering S_x and S_y and $\langle\, , \rangle$ denotes the usual intersection pairing on $H_r(\tilde{V}_0)$.

Let $j : \tilde{V}_0 \to (\tilde{V}_0, \partial\tilde{V}_0)$ be the inclusion. Then for all $u, v \in H_r(\tilde{V}_0)$, $\langle u, v \rangle = \langle u, j_* v \rangle = 0$ since the following commutative shows that j_* is the zero map:

$$\begin{tikzcd}
H_r(\tilde{V}_0) \arrow[r, "j_*"] \arrow[d] & H_r(\tilde{V}_0, \partial V_0) \arrow[d, "e_*"', "\approx"] \\
0 = H_r(R^n \times W) \arrow[r] & H_r(R^n \times W, R^n \times W - \operatorname{Int}(\tilde{V}_0))
\end{tikzcd}$$

The lemma now follows.

It follows from 5.3 and formal properties of self-intersections that $\mu: H_r(\tilde{V}_0) \to ZG/\{v+(-1)^{r+1}\bar{v} \mid v \in ZG\}$ is a homomorphism of groups where "bar" is the anti-involution of ZG that sends $\Sigma_{g\in G} n_g\, g$ to $\Sigma_{g\in G} n_g\, g^{-1}$. But also 5.3 shows that $\mu(x)=(-1)^{r+1}\overline{\mu(x)}$ for all x. Hence, $\mu: H_r(\tilde{V}_0)\to H^{r+1}(Z/2; ZG)=\{u\in ZG \mid u=(-1)^{r+1}\bar{u}\}/\{v+(-1)^{r+1}\bar{v} \mid v\in ZG\}$; and, in fact, μ is a ZG homomorphism.

In general, μ is not the zero homomorphism. The next step in the proof of 5.1 is to replace V_0 by a new manifold V_1 of the same homotopy type such that all mutual and self-intersections of the classes in $H_r(\tilde{V}_1)$ vanish. In order to do this, we let $V\subset Q$ be the manifold corresponding to $i=[n+k/2]-2=r-2$ in the proof of 3.2. Then (Q, V) is $(r-1)$ connected and there is a short exact sequence

$$0\to H_{r-1}(\tilde{Q}_1)\to H_{r-1}(\tilde{V})\to P\to 0$$

where $P\in\mathcal{P}(G)$ satisfies $(-1)^{r-1}[P]=\Sigma_{j=1}^{r-2}(-1)^j[H_j(X)]$. In fact, our numerical restrictions imply that $H_{r-1}(\tilde{Q}_1)=0$, that $H_{r-1}(\tilde{V})=P$, and that $(-1)^{r-1}[P]=\chi(X; G)$.

We can regard V_0 as having been obtained from V by choosing a resolution

$$0\longrightarrow P_0\longrightarrow F\xrightarrow{\rho} P\longrightarrow 0$$

where F is a finitely generated free ZG module with base $x_1,\ldots,x_s$; by letting ρ' be the unique homomorphism that makes the diagram

$$\begin{tikzcd} & & F \arrow[d, "\rho"] \arrow[lld, "\rho'"'] \\ \pi_r(\mathrm{Cl}\,(Q-V), \partial_+V) \arrow[r, "\approx"] & \pi_r(Q, V) \arrow[r, "\approx"] & \pi_{r-1}(V)=H_{r-1}(\tilde{V}) \end{tikzcd}$$

commute; by representing each class $\rho'(x_j)$ by an embedding $f'_j:(D^r_j\times D^r, S^{r-1}_j\times D^r)\to(\mathrm{Cl}\,(Q-V), \partial_+V)$; and by then attaching the handles $f_j(D^r_j\times D^r)$ to V via the embeddings $f_j=f'_j\mid: S^{r-1}_j\times D^r\to\partial_+V$ where $\partial_+V=\mathrm{Cl}\,(\partial V-Q_1)$.

Let $\mu': F\to H^{r+1}(Z/2; ZG)$ be a homomorphism such that $\mu'\mid P_0=\mu$. Since P_0 is a direct summand of F, such a homomorphism exists. Following [20; Chap. 5], we let $h_j: S^{r-1}_j\times D^r\times I\to\partial_+V\times I$ be a regular homotopy from f_j to a new embedding g_j such that the mutual intersection of the h_j vanish, while the self intersections of h_j are given by $-\mu'(x_j)$. Let $V_1=V\cup\bigcup_{j=1}^{s} D^r_j\times D^r$ where the attaching map for $D^r_i\times D^r$ is g_j.

In effect, V_1 is obtained from V by doing surgery on classes $\rho'(x_1),\ldots,\rho'(x_j)\in$

$\pi_r(Q, V)=\pi_r(i)$ to make the inclusion map $i: V\to Q$ r-connected. Thus, there exists a map $\psi_1: V_1\to Q$ and a stable framing F_1 of $\tau_{V_1}\oplus\psi_1^*\nu_Q$ where τ_M and ν_M respectively denote the stable tangent and normal bundles of the manifold M. In fact, there exists an immersion $\psi: V_0\to V_1$ such that $\psi_1\psi$ is homotopic to i; for we may map $V\subset V_0$ into $V\subset V_1$ by the identity on the complement of a collar $\partial V\times I$ of ∂V and by shrinking $\partial V\times I$ onto $\partial V\times[\frac{1}{2}, 1]$ leaving $\partial V\times 1$ fixed (where $\partial V\subset V$ corresponds to $\partial V\times 0$). This immersion may then be extended over the jth handle $D_j^r\times D^r=S_j^{r-1}\times D^r\times I\cup \bar{D}_j^r\times D^r$ of V_0 by mapping the collar $S_j^{r-1}\times D^r\times I$ into $\partial V\times[0,\frac{1}{2}]$ via h_j (with the parameter changed) and by mapping $\bar{D}_j\times D^r$ onto the corresponding handle of V_1 via the identity.

It is easy to see that ψ is a homotopy equivalence that makes the following diagram commute

$$\begin{tikzcd}
\pi_{r+1}(Q, V_0) \arrow[r, "\approx"] \arrow[d] & \pi_r(V_0) \arrow[d, "\psi_*", "\approx"'] & \pi_r(\tilde V_0) \arrow[l, "\approx"'] \arrow[r, "\approx"] \arrow[d, "\tilde\psi_*", "\approx"'] & H_r(\tilde V_0) \arrow[d, "\tilde\psi_*", "\approx"'] \\
\pi_{r+1}(Q, V_1) \arrow[r, "\approx"] & \pi_r(V_1) & \pi_r(\tilde V_1) \arrow[l, "\approx"'] \arrow[r, "\approx"] & H_r(\tilde V_1)
\end{tikzcd}$$

In particular, if $\phi_x: S^r\times D^r\to V_0$ is an immersion in the preferred class of immersions corresponding to $x\in H_r(\tilde V_0)$, then the composite immersion $\psi\phi_x$ is in the preferred class of immersions determined by $\tilde\psi_*(x)\in H_r(\tilde V_1)$. We may combine this observation with the following lemma to compute the mutual and self intersections of classes in $H_r(\tilde V_1)$.

Let $V_0=V'\cup H$ where $V'=V$ minus an open collar of ∂V and $H=V_0-\operatorname{Int} V'$.

LEMMA 5.4. *Let $x\in H_r(\tilde V_0)$. Then there exists an embedding $\phi_x^+:(D_+^r\times D^r, S^{r-1}\times D^r)\to(H,\partial V')$ and an immersion $\phi_x^-:(D_-^r\times D^r, S^{r-1}\times D^r)\to(V',\partial V')$ such that $\phi_x^+|S^{r-1}\times D^r=\phi_x^-|S^{r-1}\times D^r$ and such that $\phi_x=\phi_x^+\cup\phi_x^-$ is an immersion representing x.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_{r+1}(Q, V_0) \arrow[r, "\partial"] & \pi_r(V_0, V') \arrow[r] & \pi_r(Q, V') \arrow[r] & 0 \\
 & H_r(\tilde V_0) \arrow[u, "\approx"'] & \pi_r(H, \partial V') \arrow[u, "\approx"'] \arrow[r] \arrow[dr, "\partial'"'] & \pi_r(Q, \partial V') \arrow[u, "\approx"'] \arrow[d, "\approx"] & \\
 & & & \pi_{r-1}(\partial V') &
\end{tikzcd}$$

It follows from the previous discussion and homotopy excision that the row is exact and that the indicated maps are isomorphisms. Hence $\partial(x)$ can be represented by a map $f_+:(D^r_+, S^{r-1})\to(H, \partial V')$ such that $f_+ \mid S^{r-1}$ extends to a map $f_-:(D^r_-, S^{r-1})\to(V', \partial V')$. But then x itself can be represented by a map $g:(D^{r+1}; S^r, D^r_+ D^r_-)\to(Q; V_0, H, V')$ such that $g \mid D^r_\pm = f_\pm$. In particular, the stable bundle representation $\tau_{S^r\times D^r}\to\tau_{V_0}$ that defines the regular homotopy class of immersions representing x restricts to stable bundle representations $\tau(D^r_+\times D^r)\to\tau(H)$ and $\tau(D^r_-\times D^r)\to\tau(V')$ that yield the same representation of $\tau(S^{r-1}\times D^r)\to\tau(\partial V')$.

Let $\psi_+:(D^r_+\times D^r, S^{r-1}\times D^r)\to(H, \partial V')$ and $\psi_-:(D^r_-\times D^r, S^{r-1}\times D^r)\to(V', \partial V')$ be immersions corresponding to these representations. Since $\pi_1(\partial V')\to\pi_1(H)$ is an isomorphism, we may regularly homotope ψ_+ to an embedding $\phi^+_x:(D^r_+\times D^r, S^{r-1}\times D^r)\to(H, \partial V')$. Since $\phi^+_x \mid S^{r-1}\times D^r$ and $\psi_- \mid S^{r-1}\times D^r$ correspond to the same representation of stable tangent bundles, they are regularly homotopic. By splicing the regular homotopy onto ψ_-, we may replace ψ_- with a new immersion $\phi^-_x:(D^r_-\times D^r, S^{r-1}\times D^r)\to(V', \partial V')$ such that $\phi^+_x \mid S^{r-1}\times D^r = \phi^-_x \mid S^{r-1}\times D^r$. It follows immediately from the construction that $\phi^+_x\cup\phi^x_-$ is an immersion representing x.

LEMMA 5.5. *All mutual and self-intersections of classes in $H_r(\tilde V_1)$ vanish.*

Proof. Let $x'\in H_r(\tilde V_1)$ and suppose $x'=\psi_*(x)$. Let $\phi_x:S^r\times D^r\to V_0$ be the immersion representing x constructed in 5.4. Then $\psi\phi_x$ represents x' and the self-intersections of $\psi\phi_x(S^r\times 0)$ determine $\mu(x')$. The descriptions of ψ and ϕ_x, however, show that the self-intersections of $\psi\phi_x(S^r\times 0)$ are just the sum of those of $\psi\phi_x(D^r_-\times 0)$ and $\psi\phi_x(D^r_+\times 0)$. The self-intersections of $\psi\phi_x(D^r_-\times 0)$ are just those of $\phi_x(D^r_-\times 0)$ which are the same as those of $\phi_x(S^r\times 0)$ since ϕ^+_x is an embedding. Hence, the self-intersections of $\psi\phi_x(D^r_-\times 0)$ equal $\mu(x)$. On the other hand, since $\phi_x(D^r_+\times 0)$ is embedded, the self-intersections of $\psi\phi_x(D^r_+\times 0)$ all arise from the behavior of ψ. The description of ψ given above, however, shows that the self-intersections of $\psi\phi_x(D^r_+\times 0)$ are $-\mu'(x)=-\mu(x)$. Hence, $\mu(x')=0$.

A similar argument shows that $\lambda(x, y)=0$ for all $x, y\in H_r(\tilde V_1)$.

The proof of 5.1 in the case when n is even is now completed as follows: Let $M_1=\tilde V_1\cup(N(X)\times D(W))$ with corners equivariantly rounded and let M^{2r} be the double of M_1. Then M^{2r} is $(r-1)$ connected and supports a semi-free G-action with $M^G=F^n(X)$. If we delete a G-tubular neighborhood of M^G and pass to the orbit space, we obtain $V_1\cup V_1$ where the union is along $\partial_+V_1=\mathrm{Cl}\,(\partial V_1-Q_1)$. We note that $H=\pi_{r+1}(f)=H_r(D(\tilde V_1))$ and λ and μ are the mutual and self-intersection forms on H. But now, the inclusion $V_1\to D(V_1)$ induces a monomorphism $H_r(V_1)\to H$ whose image is totally isotropic by 5.5. It follows that (H, λ, μ) is the hyperbolic form on $H_r(V_1)$. Since $(-1)^r[H_r(V_1)] = \chi(X; G)$, it now follows that if $Hj_*[\chi(X; G)] = 0$, f may be made $(r+1)$ connected via surgery relative to $\partial D(V_1)$. Let $g: V_2 \to Q$ be the resulting map.

Since $\partial V_2 = \partial D(V_1)$ is the double of Q_1, $\partial \tilde{V}_2$ is the double of $\tilde{Q}_1 = N(X) \times S(W)$ where $\tilde{V}_2$ is the universal cover of V_2. We now let $\Sigma^{n+k} = \tilde{V}_2 \cup (F^n(X) \times D(W))$. Then Σ^{n+k} supports a smooth semi-free G-action with $\Sigma^G = F^n(X)$. On the other hand standard arguments show that Σ^{n+k} is a homotopy sphere. This completes the proof of 5.1 in the case when n is even.

Case II. n is odd. In this case, we let $n = 2p+1$, $k = 2q$, and $n+k = 2r+1$. Here, the manifold V_0 constructed at the beginning of this proof has the following properties which may be derived easily from 3.2 and properties of $\tilde{Q}_1$:

(a) V_0 is a smooth regular neighborhood of a CW complex $X = Q_1 \cup$ cells of dimensions $\leq r$.

(b) (Q, V_0) is r-connected.

(c) $H_i(\tilde{V}_0) = 0$ for $0 < i < r$ and $H_r(\tilde{V}_0) = P_0$ is in $\mathcal{P}(G)$ and satisfies $(-1)^r[P_0] = \chi(X; G)$.

(d) For $i > r$, $k_*: H_i(\tilde{Q}_1) \to H_i(\tilde{V}_0)$ is an isomorphism.

It follows immediately from (c) that $[P_0] = k_*[Z/p] = [\langle p, N\rangle]$ for some integer p prime to $|G|$ (cf. section 2 for notation). By stabilizing, if necessary, we may assume that $P_0 = \langle p, N\rangle \oplus F_0$ where F_0 is a free ZG module. Combining this with the exact sequence (*) of 2.3, we obtain an exact sequence

$$0 \to F_1 \to P_0 \to Z/p \to 0$$

where $F_1 = F_0 \oplus ZG$ is a free ZG module and Z/p is trivial ZG module.

We now proceed as in the proof of 3.2. We identify P_0 with $\pi_{r+1}(\mathrm{Cl}\,(Q - V_0), \partial_+ V_0)$ where $\partial_+ V_0 = \mathrm{Cl}\,(\partial V_0 - Q_1)$. We pick free generators $x_1, \ldots, x_s$ of F_1 and represent them by piecewise linear embeddings $f_j: (D^{r+1}, S^r) \to (\mathrm{Cl}\,(Q - V_0), \partial_+ V_0)$ with mutually disjoint images. We then attach the cells $f_j(D^{r+1})$ $(j = 1, \ldots, s)$ to V_0 and let V_1 be a smooth regular neighborhood of $V_0 \cup \bigcup_{j=1}^{s} f_j(D^{r+1})$ in Q. It is easy to see that V_1 has the following properties:

(a′) (Q, V_1) is r-connected.

(b′) $H_i(\tilde{V}_1) = 0$ for $0 < i < r$ and $H_r(\tilde{V}_1) = Z/p$ where $(-1)^r k_*[Z/p] = \chi(X; G)$.

(c′) For $i > r$, $k_*: H_i(\tilde{Q}_1) \to H_i(\tilde{V}_1)$ is an isomorphism.

Let $b: H_r(\tilde{V}_1) \times H_r(\tilde{V}_1) \to QG/ZG$ denote the linking form as defined in [19; Section 5]. Similarly let $q: H_r(\tilde{V}_1) \to QG/\{v + (-1)^{r+1}\bar{v} \mid v \in ZG\}$ be the quadratic map defined by self-linking.

LEMMA 5.6. (i) *For all* $x, y \in H_r(\tilde{V}_1)$, $b(x, y) = 0$.
(ii) *If* $|G|$ *is even, then* $q(x) = 0$ *for all* $x \in H_r(\tilde{V}_1)$.

Proof. Since $H_r(\tilde{V}_1) = Z/p$ is cyclic, it suffices to show that $b(x, x) = 0$ for x a generator of Z/p. It follows immediately from the construction of V_1 from V_0 that there exist embeddings $f: S^r \to \text{Int } V_0$ and $g: (D^{r+1}, S^r) \to (\text{Cl}(V_1 - V_0), \partial_+ V_0)$ whose lifts to $\tilde{V}_1$ represent the class x and a chain whose boundary is px. Since $b(x, x)$ is just the intersection of $f(S^r)$ with $g(D^{r+1})$ and these sets are disjoint, $b(x, x) = 0$ and (i) is established.

To prove (ii), we note first that (i) and [19; p. 252] imply that q is a homomorphism. On the other hand, $q(x) = b(x, x) = 0$ modulo ZG by [19; p. 252]. Hence $q: H_r(\tilde{V}_1) \to ZG/\{v + (-1)^{r+1}\bar{v} \mid v \in ZG\}$. If $|G|$ is even, then $H_r(\tilde{V}) = Z/p$ has odd order. Furthermore, it is well known that the only torsion in $ZG/\{v + (-1)^{r+1}\bar{v} \mid v \in ZG\}$ has order two. Hence $q(x) = 0$.

The proof now proceeds as in the even dimensional case. We let $M_1 = \tilde{V}_1 \cup (N(X) \times D(W))$ and M^{2r+1} be the double of M_1. Then M^{2r+1} is $(r-1)$ connected, has $H_r(M) = Z/p \oplus Z/p$, and supports a semi-free G-action with $M^G = F^n(X)$. As before, the complement of a G-tubular neighborhood of M^G covers $V_1 \cup V_1$ where the union is along $\partial_+ V_1$ and there is an r-connected map $f: V_1 \cup V_1 \to Q$, representing a surgery problem, with $H = \pi_{r+1}(f) = H_r(M)$. Furthermore, since $i_*: H_r(\tilde{V}_1) \to H_r(M)$ is monomorphic with totally isotropic image by 5.6, the form (H, b, q) is the hyperbolic form on Z/p where b (respectively, q) is the linking (respectively, self-linking) form on $\pi_{r+1}(f)$. (cf. [19; Section 5]). It now follows that the class of (H, b, q) in $L^{fr}_{2r+2}(ZG, Z - \{0\})$ is $H'(\langle p, N \rangle)$ where $L^{fr}_{2r+2}(ZG, Z - \{0\})$ is the group $L_{2r+2}(A, S, \varepsilon)$ of Ranicki [13; Proposition 2.4 and Section 7] for $A = Z\pi$, $S = Z - \{0\}$, $\varepsilon = 1$, and $X = \{0\}$ arising from (split) ε-quadratic linking forms and $H': H^{2r+1}(Z/2; \tilde{K}_0(G)) \to L^{fr}_{2r+2}(ZG, Z - \{0\})$ is the hyperbolic map.

On the other hand, there is a commutative diagram [13; Proposition 7.1]

$$\begin{array}{ccc} H^{2r+1}(Z/2; \tilde{K}_0(G)) & \xrightarrow{H'} & L^{fr}_{2r+2}(ZG, Z - \{0\}) \\ \| & & \downarrow \sigma \\ H^{2r+1}(Z/2; \tilde{K}_0(G)) & \xrightarrow{H} & L^{h}_{2r+1}(G) \end{array}$$

and it follows from the geometric interpretation of σ given by Pardon [10] that $\sigma(H, b, q) = H(\langle p, N \rangle)$ is the obstruction to doing surgery relative to $\partial(V_1, \cup V_1)$ to make f $(r+1)$-connected. Hence, if $H(\langle p, N \rangle = Hj_*[\chi(X; G)] = 0$, there exists an $(r+1)$-connected map $g: V_2 \to Q$ where $\partial V_2 = \partial(V_1 \cup V_1)$ and $g \mid \partial V_2 = f \mid \partial(V_1 \cup V_2)$. The remainder of the proof for the odd dimensional case now follows exactly the proof of the even dimensional case.

Proof of 5.2. By the results of [8] and [3] the 2-primary component of $\tilde{K}_0(Q2^l)$ is $Z/2$ (cf. also [1; Section 4]) and the restriction map $r^*: H^n(Z/2; \tilde{K}_0(Q2^l)) \to H^n(Z/2; \tilde{K}_0(Q8))$ is an isomorphism. Hence, $H^n(Z/2; \tilde{K}_0(Q2^l)) = Z_2$ for all n.

If $n \equiv 3 \pmod 4$, H is the zero map by [4]. Assuming that $H=0$ for $n\equiv 0 \pmod 4$ the result for $n\equiv 1 \pmod 4$ and $l=3$ follows easily from the exact sequence

$$H^1(Z/2; \tilde{K}_0(Q8)) \xrightarrow{H} L_1^h(Q8) \longrightarrow L_1^p(Q8) \xrightarrow{j} H^0(Z/2; \tilde{K}_0(Q8))$$

using the facts that j is onto, $L_1^p(Q8)$ has order 4 by [11; Theorem 9.24b], and $L_1^h(Q8)$ has order 4 by [21; Theorem 5.2.4]. The general case follows from this case by applying the restriction map.

The proofs for the cases when $n=2m$ use the commutative diagram [13; Proposition 7.1]

$$\begin{tikzcd} L^p_{2m+2}(ZQ2^l, Z-\{0\}) \arrow[r, "j"] \arrow[d] & H^0(Z/2; \tilde{K}_0(Q2^l)) \arrow[d, "="] & \\ L^p_{2m+1}(Q2^l) \arrow[r] & H^0(Z/2; \tilde{K}_0(Q2^l)) \arrow[r, "H"] & L^h_{2m}(Q2^l) \end{tikzcd}$$

where the upper left hand term is the torsion L-theory of [13; Section 7] for $Y=\tilde{K}_0(Q2^l)$. We shall show that j is onto by constructing an appropriate ε-quadratic linking form. This construction is facilitated by the following result.

LEMMA 5.7. *Let $\varepsilon = \pm 1$ and (M, λ) be an ε-symmetric linking form over $(ZG, Z-\{0\})$. If M has odd order, then there exists a unique function $\mu: M \to QG/\{x+\varepsilon\bar{x} \mid x \in ZG\}$ such that (M, λ, μ) is an ε-quadratic linking form.*

Proof. Since this result is well known, we only sketch its proof. Let $A_\varepsilon = \{y \in QG/\{x + \varepsilon \bar{x} \mid x \in ZG\} \mid y = \varepsilon \bar{y}\}$ and $B_\varepsilon = \{z \in Q\pi/Z\pi \mid z = \varepsilon \bar{z}\}$. Then $\lambda(x,x) \in B_\varepsilon$ for all $x \in M$. Furthermore, since M has odd order, $\lambda(x,x) \in (B_\varepsilon)_{\text{odd}}$, the odd torsion subgroup of B_ε. The natural map $QG/\{x+\varepsilon\bar{x} \mid x \in ZG\} \to QG/ZG$ induces an isomorphism $\rho: (A_\varepsilon)_{\text{odd}} \to (B_\varepsilon)_{\text{odd}}$. Set $\mu(x) = \rho^{-1}\lambda(x,x)$.

The proof of 5.2 in the case when $n \equiv 2 \pmod 4$ is now completed as follows: Let Z/r be endowed with the trivial $ZQ2^l$ module structure and define $\lambda: Z/r \times Z/r \to QQ2^l/ZQ2^l$ by $\lambda(s,t) = (st/r)N$ where $N \in ZQ2^l$ is the norm element. Then λ is a symmetric linking form on Z/r. Hence, $(Z/r, \lambda)$ determines an element $[Z/r, \lambda, \mu]$ of $L^p_{2m+2}(ZQ2^l; Z-\{0\})$ such that $j[Z/r, \lambda, \mu] = [\langle r, N\rangle] \in H^0(Z/2; \tilde{K}_0 Q2^l)$. Since $[\langle 3, N\rangle]$ generates the latter group the result follows.

Suppose now that $n \equiv 0 \pmod 4$. We consider first the case when $l = 3$. In this case we make use of a construction shown to us by R. Oliver. Note first that there is a pull-back diagram.

$$\begin{tikzcd} ZQ8 \arrow[r, "\sigma_1"] \arrow[d, "\sigma_2"'] & ZD4 \arrow[d, "\rho_2"] \\ Z[i, j, k] \arrow[r, "\rho_1"] & F_2Dr \end{tikzcd}$$

where $D4 = Z/2 \oplus Z/2$ is the dihedral group of order 4 with elements 1, S, T, ST and F_p is the field with p elements. By [7], the map $\partial : K_1(F_2D4) \to \tilde{K}_0(Q8)$ in the corresponding Mayer–Victoris sequence is onto and carries any unit $a + bS + cT + dST$ of F_2D4 with exactly three of a, b, c, d odd onto the generator $\langle 3, N\rangle$ of $\tilde{K}_0(Q8)$.

Let p be a prime, $p \equiv 3 \pmod 4$ and choose a, b, c, d such that $p = a^2 + b^2 + c^2 + d^2$. Let $u = a + bi + cj + dk \in Z[i, j, k]$. Then right multiplication by u, $\cdot\, u : Z[i, j, k] \to Z[i, j, k]$ is injective. A tedious computation shows that this map has cokernel $Z/p \oplus Z/p$ (additively); hence,

$$0 \longrightarrow Z[i, j, k] \xrightarrow{\cdot u} Z[i, j, k] \longrightarrow Z/p \oplus Z/p \longrightarrow 0$$

is exact. Furthermore, $Z/p \oplus Z/p$ inherits a $ZQ8$ module structure via the epimorphisms $ZQ8 \to Z[i, j, k] \to Z/p \oplus Z/p$. Let P be kernel of this composite and M denote $Z/p \oplus Z/p$ with this $ZQ8$ module structure.

LEMMA 5.8. (due to R. Oliver). *The $ZQ8$ module P is a projective module representing the non-trivial element of $\tilde{K}_0(Q8)$.*

Proof. Let $\bar{u} \in F_2D4$ be given by $\bar{u} = \bar{a} + \bar{b}S + \bar{c}T + \bar{d}ST$ where $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ are the mod 2 reductions of a, b, c, d respectively. Since $p \equiv 3 \pmod 4$, exactly three of $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ are odd and $\bar{u}$ is a unit of F_2D4. Consider now the following commutative diagram

$$\begin{tikzcd} P \arrow[dr] \arrow[drrr, "\tau"] \arrow[dd] & & & \\ & ZQ8 \arrow[rr, "\sigma_1"] \arrow[dd, "\sigma_2"] \arrow[dddr, "\rho"] & & ZD4 \arrow[dd, "\rho_2"] \\ Z[i, j, k] \arrow[rr, "\rho_1"] \arrow[dr, "\cdot u"'] & & F_2D4 \arrow[dr, "\cdot \bar{u}"] & \\ & Z[i, j, k] \arrow[rr, "\rho_1"] \arrow[dr] & & F_2D4 \\ & & Z/p \oplus Z/p & \end{tikzcd}$$

where the unlabelled maps are inclusions and τ makes the triangle on the top commute. A diagram chase shows that

$$\begin{tikzcd} P \arrow[r, "\tau"] \arrow[d] & ZD4 \arrow[d] \\ Z[i, j, k] \arrow[r, "\cdot\bar{u}\rho_1"] & F_2D4 \end{tikzcd}$$

is a pull-back diagram. Hence P is a projective module representing $\partial(\bar{u})$. The lemma now follows from the remarks above.

LEMMA 5.9. *The module M supports a $ZQ8$ invariant skew-symmetric linking form $\phi : M \times M \to Q/Z$.*

Proof. Define ϕ by setting $\phi((r, s), (r', s')) = (rs' - r's)/p$ where r, s, r', s' are integers representing classes in $M \times M = (Z/p \oplus Z/p) \times (Z/p \oplus Z/p)$. Clearly ϕ is skew-symmetric; in fact, ϕ is the form with matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

On the other hand, M is an indecomposable $ZQ8$ module. Since $F_pQ8 = (F_p)^4 \oplus M_2(F_p)$, it follows that the $Q8$ representation on M is given via the inclusion $Q8 \to SL_2(p) = Sp_2(p)$ [2]. Since $Sp_2(p)$ is the group of isometrics of the form with matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

the lemma now follows.

The proof of 5.2 in the case when $n = 0 \pmod 4$ and $l = 3$, is now concluded as follows: We define $\lambda : M \times M \to QQ8/ZQ8$ by

$$\lambda(x, y) = \sum_{g \in Q8} \phi(x, yg^{-1})g.$$

Then λ is a -1 ($=$skew)-symmetric linking form in the terminology of [13]. Since M has odd order, 5.8 shows that (M, λ) determines an element $x = [M, \lambda, \mu] \in L_1^p(ZQ2^l, Z - \{0\})$ such that $j(x) = [\langle 3, N \rangle] \neq 0$.

The proof of 5.2 in the case when $n \equiv 0 \pmod 4$ is completed by observing that the 2-Sylow subgroup of $SL_2(p)$ is $Q2^l$ [2] when $p \equiv -1 \pmod{2^l}$. Hence, the action of $Q8$ on M given above extends to an action of $Q2^l$ and the form ϕ of 5.8 remains invariant under this extended action. Hence, we can construct a (-1)-quadratic linking form representing an element $y \in L_1^p(ZQ2^l, Z-\{0\})$ whose restriction to $ZQ8$ is x. The commutative diagram

$$\begin{array}{ccc} L_1^p(ZQ2^l, Z-\{0\}) & \xrightarrow{j} & H^0(Z/2; \tilde{K}_0Q2^l) \\ \downarrow & & \downarrow\approx \\ L_1^p(ZQ8), Z-\{0\}) & \longrightarrow & H^0(Z/2; \tilde{K}_0Q8) \end{array}$$

coming from restriction now shows that j is onto in general. This concludes the proof of 5.2.

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