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Autor(en): **Talelli, Olympia**

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## On cohomological periodicity for infinite groups

OLYMPIA TALELLI

### Introduction

The phenomenon of cohomological periodicity for finite groups has long been understood. Here we introduce the notion of “a group  $G$  having period  $q$  after  $k$ -steps” so as to allow infinite groups to have periodic cohomology. The definition is given in terms of a projective resolution of  $G$  and, as expected, it is equivalent to having the functors  $H^n(G, -)$  and  $H^{n+q}(G, -)$  naturally isomorphic for all  $n \geq k + 1$ . We then show that this definition coincides with the classical one for finite groups and moreover, we obtain that if an infinite group  $G$  has period  $q$  after  $k$ -steps then  $k \geq 1$ .

In §2 we investigate what it means for a countable locally finite group to have period  $q$  after  $k$ -steps. We obtain what one would expect, i.e. that a countable locally finite group  $G$  has period  $q$  after  $k$ -steps iff every finite subgroup of  $G$  has period  $q$ . Moreover, we have here that  $k = 1$ .

Then we show that there is an element  $g \in H^q(G, Z)$  such that cup product with  $g$

$$\cup g : H^i(G, -) \rightarrow H^{i+q}(G, -)$$

induces the natural isomorphism for all  $i \geq 2$ .

Finally, in §3 we characterize the infinite locally finite  $p$ -groups which have period  $q$  after  $k$ -steps. First, we point out two obvious candidates, i.e. the infinite locally cyclic  $p$ -group and the infinite locally quaternion group, and then we show that these are the only ones. This result depends heavily on the well known similar statement for periodic finite  $p$ -groups [Cartan + Eilenberg].

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### §1. Periodicity after some “steps”

Let  $G$  be a group and  $ZG$  its integral group ring. We work in the category of left  $ZG$ -modules. If  $A$  is a  $ZG$ -module, by a resolution of  $A$  we shall always mean a projective resolution of  $A$ .

DEFINITION. A group  $G$  is said to have period  $q$  after  $k$ -steps if there is an exact sequence

$$0 \longrightarrow R_{k+q} \xrightarrow{\beta} P_{k+q-1} \longrightarrow \cdots \longrightarrow P_k \xrightarrow{\partial_k} P_{k-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow Z \longrightarrow 0$$

$$\begin{array}{c} \nearrow \alpha \\ R_k \end{array}$$
(1)

where  $Z$  is regarded as a trivial  $ZG$ -module,  $R_{k+q} = R_k$  and  $P_i$   $0 \leq i \leq k + q - 1$  are projective  $ZG$ -modules. We take  $R_0 = Z$ . If  $k = 0$  then  $G$  is said to have period  $q$ . Having (1) we can form a resolution of  $G$

$$P'_* : \cdots \longrightarrow P'_i \xrightarrow{\partial'_i} P'_{i-1} \longrightarrow \cdots \longrightarrow P'_0 \longrightarrow Z \longrightarrow 0$$

$$\begin{array}{c} \searrow \\ R'_i \end{array}$$

by defining

$$P'_i = P_i \quad 0 \leq i \leq k - 1 \quad \partial'_i = \partial_i \quad 1 \leq i \leq k$$

and

$$P'_i = P_{k+\lambda_i} \quad i = k + nq + \lambda_i \quad \partial'_i = \partial_{k+\lambda_i} \quad i = k + nq + \lambda_i$$

$$n \geq 0, 0 \leq \lambda_i < q \quad n \geq 0, 0 < \lambda_i < q$$

$$\partial'_{k+\mu q} = \beta\alpha \quad \mu \geq 1$$

Such a resolution is naturally considered periodic after  $k$ -steps; we may refer to (1) itself as a resolution of  $G$  which is periodic after  $k$ -steps. If  $k = 0$  we call such a resolution a periodic resolution of  $G$ .

Evidently the functors  $H^n(G, -)$  and  $H^{n+q}(G, -)$  are naturally isomorphic for all  $n \geq k + 1$ .

Now let  $M, N$  be  $ZG$ -modules,  $g \in \text{Ext}_{ZG}^q(M, N)$  and consider the following diagram

$$\begin{array}{ccccccccccccccc} P_* : & \cdots & \rightarrow & P_{i+q} & \rightarrow & P_{i+q-1} & \rightarrow & \cdots & \rightarrow & P_{q+1} & \rightarrow & P_q & \longrightarrow & P_{q-1} & \rightarrow & \cdots & \rightarrow & P_0 & \rightarrow & M & \rightarrow & 0 \\ & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & \searrow & \nearrow & & & & & & & & & & \\ E_* : & \cdots & \rightarrow & E_i & \rightarrow & E_{i-1} & \rightarrow & \cdots & \rightarrow & E_1 & \rightarrow & E_0 & \rightarrow & N & \rightarrow & 0 & & & & & & & & \end{array}$$

$$\begin{array}{c} \searrow \\ R_q \\ \downarrow \theta \end{array}$$

where  $P_* \rightarrow M, E_* \rightarrow N$  are resolutions of  $M, N$  respectively and  $\theta : R_q \rightarrow M$  a

$q$ -cocycle representing  $g$ . Then  $\theta$  lifts to a chain map  $P_* \rightarrow E_*$  of degree  $-q$  and this chain map induces cup product with  $g$

$$\cup g : \text{Ext}_{ZG}^i(N, -) \rightarrow \text{Ext}_{ZG}^{i+q}(M, -) \quad i \geq 0.$$

**THEOREM 1.1.** *Let  $G$  be a group. Then the following statements are equivalent:*

- (i)  $G$  has period  $q$  after  $k$ -steps
- (ii) There is a resolution of  $G$

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_i & \longrightarrow & P_{i-1} & \longrightarrow & \cdots \longrightarrow P_0 \longrightarrow Z \longrightarrow 0 \\ & & \searrow & & \nearrow & & \\ & & & & R_i & & \end{array}$$

and an element  $h \in \text{Ext}_{ZG}^q(R_k, R_k)$  such that for every  $ZG$ -module  $A$  cup product with  $h$

$$\cup h : \text{Ext}_{ZG}^i(R_k, A) \longrightarrow \text{Ext}_{ZG}^{i+q}(R_k, A)$$

is an isomorphism for  $i \geq 1$  and an epimorphism for  $i = 0$ . Note that  $\text{Ext}_{ZG}^i(R_k, A) = H^{i+k}(G, A)$  for  $i \geq 1, k \geq 0$ .

*Proof.* (i)  $\Rightarrow$  (ii). Consider a resolution  $P'_*$  of  $G$  which is periodic, or period  $q$ , after  $k$ -steps

$$\begin{array}{ccccccc} P'_* : \cdots & \longrightarrow & P'_i & \longrightarrow & P'_{i-1} & \longrightarrow & \cdots \longrightarrow P'_0 \longrightarrow Z \longrightarrow 0 \\ & & \searrow & & \nearrow & & \\ & & & & R'_i & & \end{array}$$

and the resolution  $P'_* | R'_k$  of  $R'_k$

$$P'_* |_{R'_k} : \cdots \longrightarrow P'_i \longrightarrow P'_{i-1} \longrightarrow \cdots \longrightarrow P'_k \xrightarrow{\alpha} R'_k \longrightarrow 0.$$

Now there is an element  $h \in \text{Ext}_{ZG}^q(R'_k, R'_k)$  defined by  $P'_{k+q} \xrightarrow{\alpha} R'_k$  or in Yoneda's interpretation of  $\text{Ext}_{ZG}^i(-, -)$  by the multiple extension

$$0 \longrightarrow R'_k \longrightarrow P'_{k+q-1} \longrightarrow \cdots \longrightarrow P'_k \xrightarrow{\alpha} R'_k \longrightarrow 0.$$

Clearly, for any  $ZG$ -module  $A$  cup product with  $h$

$$\cup h : \text{Ext}_{ZG}^i(R_k, A) \longrightarrow \text{Ext}_{ZG}^{i+q}(R_k, A)$$



is the identity for  $i \geq 1$  and an epimorphism for  $i = 0$ . (ii)  $\Rightarrow$  (i) can be easily deduced from the following Theorem. It is due to C. T. C. Wall [[7], Theorem. 1.2]; however, we give here a different proof.

**THEOREM 1.2.** *Let  $A, B$  be  $ZG$ -modules. If  $g \in \text{Ext}_{ZG}^n(A, B)$  is such that for any  $ZG$ -module  $E$  cup product with  $g$*

$$\cup g : \text{Ext}_{ZG}^i(B, E) \longrightarrow \text{Ext}_{ZG}^{i+n}(A, E)$$

*is injective for  $i = 1$  and surjective for  $i = 0$ , then  $g$  is represented by a multiple extension*

$$0 \longrightarrow B \longrightarrow K \longrightarrow P_{n-2} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

*with  $K, P_i$   $0 \leq i \leq n - 2$  projective  $ZG$ -modules.*

*Proof.* Let  $P_* \rightarrow A, E_* \rightarrow B$  be resolutions of  $A$  and  $B$  respectively, and let  $\theta : R_n \rightarrow B$  be a cocycle representing  $g$

$$\begin{array}{ccccccccccccccc} P_* : & \cdots & \longrightarrow & P_{n+1} & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow & \nearrow & \downarrow & \nearrow & & & & & & & & & & \\ & & & & & & R_n & & & & & & & & & & & & \\ & & & & & & \downarrow & & & & & & & & & & & & \\ E_* : & \cdots & \longrightarrow & E_1 & \longrightarrow & E_0 & \longrightarrow & B & \longrightarrow & 0. & & & & & & & & & \end{array}$$

Now  $\theta$  lifts to a chain map  $P_* \rightarrow E_*$  of degree  $-n$  which induces cup product with  $g$ .

Let

$$\begin{array}{ccc} R_n & \xrightarrow{\gamma} & P_{n-1} \\ \theta \downarrow & & \downarrow \alpha \\ B & \xrightarrow{\beta} & K \end{array}$$

be a push-out diagram. Then  $\beta$  is injective and  $\text{coker } \beta \cong \text{coker } \gamma \cong R_{n-1}$ .

Now  $g$  is represented by  $0 \rightarrow B \rightarrow K \rightarrow P_{n-2} \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0$ . We shall show that  $K$  is a projective  $ZG$ -module.

Consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & R_n & \longrightarrow & P_{n-1} & \longrightarrow & R_{n-1} & \longrightarrow & 0 \\ & & \downarrow \theta & & \downarrow \alpha & & \downarrow id & & \\ 0 & \longrightarrow & B & \longrightarrow & K & \longrightarrow & R_{n-1} & \longrightarrow & 0. \end{array}$$

This gives rise to the following commutative diagram

$$\begin{array}{ccccccccccc}
 \cdots & \rightarrow & \text{Hom}_{ZG}(B, E) & \rightarrow & \text{Ext}_{ZG}(R_{n-1}, E) & \rightarrow & \text{Ext}_{ZG}(K, E) & \rightarrow & \text{Ext}_{ZG}(B, E) & \rightarrow & \cdots \\
 & & \theta^* \downarrow & & \text{id} \downarrow & & \alpha^* \downarrow & & \theta^* \downarrow & & \\
 \cdots & \rightarrow & \text{Hom}_{ZG}(R_n, E) & \xrightarrow{\delta} & \text{Ext}_{ZG}(R_{n-1}, E) & \rightarrow & \text{Ext}_{ZG}(P_{n-1}, E) & \rightarrow & \text{Ext}_{ZG}(R_n, E) & \rightarrow & \cdots
 \end{array}$$

By hypotheses  $\theta^* : \text{Ext}_{ZG}(B, E) \rightarrow \text{Ext}_{ZG}(R_n, E) \simeq \text{Ext}_{ZG}^{1+n}(A, E)$  is injective and

$$\delta\theta^* : \text{Hom}_{ZG}(B, E) \rightarrow \text{Ext}_{ZG}(R_{n-1}, E) \simeq \text{Ext}_{ZG}^n(A, E)$$

is surjective. Moreover,  $\text{Ext}_{ZG}(P_{n-1}, E) = 0$  since  $P_{n-1}$  is projective. Thus by the commutativity of the diagram we obtain that  $\text{Ext}_{ZG}(K, E) = 0$ , and this holds for any  $ZG$ -module  $E$ . Hence  $K$  is a projective  $ZG$ -module.

*Remark.* If  $P_{n-1}, B$  are finitely generated  $ZG$ -modules then  $K$  is a finitely generated module since from the push-out diagram we have an epimorphism  $P_{n-1} \oplus B \xrightarrow{\alpha+\beta} K \rightarrow 0$ .

**PROPOSITION 1.3.** *If  $G$  has period  $q$  after  $k$ -steps then so does every subgroup  $H$  of  $G$ .*

*Proof.* This follows from the fact that a projective  $ZG$ -module is a projective  $ZH$ -module.

**PROPOSITION 1.4.** *If  $G$  is an infinite group and has period  $q$  after  $k$ -steps then  $k \geq 1$ .*

*Proof.* Let  $H$  be a group which has period  $q$  and consider a periodic resolution of  $H$

$$0 \longrightarrow Z \xrightarrow{\beta} P_{q-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow Z \longrightarrow 0.$$

Now  $\text{im } \beta \simeq H^0(H, P_{q-1})$ . But  $P_{q-1}$  is a direct summand of an induced  $ZH$ -module, and it is well known that if  $A$  is a non-trivial induced  $ZH$ -module then  $H^0(H, A) \neq 0$  iff  $H$  is finite. The result follows.

**LEMMA 1.5.** *If  $G$  is a finite group and has period  $q$  after  $k$ -steps, then  $G$  has period  $q$ .*

*Proof.* By Theorem 1.1 (i)  $\Rightarrow$  (ii) there is a resolution of  $G$

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & P_i & \longrightarrow & P_{i-1} & \longrightarrow & \cdots \longrightarrow P_0 \longrightarrow Z \longrightarrow 0 \\
 & & \searrow & & \nearrow & & \\
 & & & & R_i & & 
 \end{array} \tag{2}$$

and an element  $h \in \text{Ext}_{ZG}^q(R_k, R_k)$  such that for every  $ZG$ -module  $A$  cup product with  $h$

$$\cup h : \text{Ext}_{ZG}^i(R_k, A) \rightarrow \text{Ext}_{ZG}^{i+q}(R_k, A)$$

is an isomorphism for  $i \geq 1$ .

Now since  $G$  is a finite group, a projective  $ZG$ -module  $P$  is a direct summand of a coinduced  $ZG$ -module; hence  $H^i(G, P) = 0$  for all  $i \geq 1$ . Thus from (2) we obtain an isomorphism

$$\delta : \text{Ext}_{ZG}^q(R_k, R_k) \xrightarrow{\cong} H^q(G, Z).$$

Let  $h' \in H^q(G, Z)$  be the image of  $h \in \text{Ext}_{ZG}^q(R_k, R_k)$  under  $\delta$ . Then clearly for any  $ZG$ -module  $A$  cup product with  $h'$

$$\cup h' : H^i(G, A) \rightarrow H^{i+q}(G, A)$$

is an isomorphism for  $i \geq k + 1$ .

Let  $0 \rightarrow C \rightarrow P \rightarrow A \rightarrow 0$  be a short exact sequence with  $P$  a projective  $ZG$ -module. Then we obtain the following commutative diagram

$$\begin{array}{ccc} H^{n-1}(G, A) & \xrightarrow{\cup h'} & H^{n+q-1}(G, A) \\ \delta_n \downarrow & & \downarrow \delta_{n+q} \\ H^n(G, C) & \xrightarrow{\cup h'} & H^{n+q}(G, C) \end{array}$$

where  $\delta_n, \delta_{n+q}$  are connecting homomorphisms. But since  $H^i(G, P) = 0$  for all  $i \geq 1$ ,  $\delta_i$  is an isomorphism for  $i > 1$  and an epimorphism for  $i = 1$ . Thus it follows that cup product with  $h'$

$$\cup h' : H^i(G, A) \longrightarrow H^{i+q}(G, A)$$

is an isomorphism for  $i \geq 1$  and an epimorphism for  $i = 0$ . The result now follows from Theorem 1.1 (ii)  $\Rightarrow$  (i).

**PROPOSITION 1.6.** *If  $G$  has period  $q$  after  $k$ -steps then every finite subgroup  $H$  of  $G$  has period  $q$ .*

*Proof.* This follows from Proposition 1.3 and Lemma 1.5.

**COROLLARY 1.7.** *If  $G$  has period 2 after  $k$ -steps then every finite subgroup  $H$  of  $G$  is cyclic.*

*Proof.* It is known [[6], Lemma 5.2] that a finite group  $H$  has period 2 iff it is cyclic; hence the result follows from Proposition 1.6.

Let  $G$  be a group such that the functors  $H^j(G, -)$  and  $H^{j+q}(G, -)$  are naturally isomorphic for some  $j \geq 1$ . Consider a resolution of  $G$

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_i & \longrightarrow & P_{i-1} & \longrightarrow & \cdots \longrightarrow P_0 \longrightarrow Z \longrightarrow 0. \\ & & \searrow & & \nearrow & & \\ & & & & R_i & & \end{array} \tag{3}$$

Then clearly  $\text{Ext}_{ZG}(R_{j-1}, -) \cong^{\text{nat.}} \text{Ext}_{ZG}(R_{j+q-1}, -)$ . Thus by [[4], Thm. 2.6] there exist projective  $ZG$ -modules  $Q_1, Q_2$  such that

$$R_{j-1} \oplus Q_1 \xrightarrow{\alpha} R_{j+q-1} \oplus Q_2. \tag{*}$$

Now (3) gives rise to the following exact sequence

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & R_{j+q-1} \oplus Q_2 & \xrightarrow{e} & P_{j+q-2} \oplus Q_2 & \longrightarrow & P_{j+q-3} & \longrightarrow & \cdots & \longrightarrow & P_{j-2} & \longrightarrow \\ & & & & \searrow & & \nearrow & & & & \searrow & & \nearrow \\ & & & & & & & & & & P_{j-1} \oplus Q_1 & \longrightarrow & P_{j-2} \oplus Q_1 & \longrightarrow & P_{j-3} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & Z & \longrightarrow & 0 \end{array} \tag{4}$$

Hence it follows from (4) and (\*) that:

**PROPOSITION 1.8.** *If the functors  $H^\lambda(G, -)$  and  $H^{\lambda+q}(G, -)$  are naturally isomorphic for some  $\lambda \geq 2$ , then  $G$  has period  $q$  after  $\lambda-1$ -steps.*

Now let  $j = 1$  and  $q > 0$ . Clearly it follows that  $H^n(G, -) \cong^{\text{nat.}} H^{n+q}(G, -)$  for all  $n \geq 1$ . Thus by Proposition 1.8  $G$  has period  $q$  after 1-step. But  $G$  is finite since there is a monomorphism  $\rho\alpha : Z \oplus Q_1 \rightarrow P_{q-1} \oplus Q_2$  (same argument as for the proof of Proposition 1.4). Thus by Lemma 1.5  $G$  has period  $q$ . Hence we have proved:

**PROPOSITION 1.9.** *If the functors  $H^1(G, -)$  and  $H^{1+q}(G, -)$  are naturally isomorphic for some  $q > 0$ , then  $G$  is finite and has period  $q$ .*

### 2. Periodic countable locally finite groups

In this section we state and prove our main Theorem. The proof is based on direct limit arguments.

**PROPOSITION 2.1** [[1]]. *Let  $(P_i, \lambda_{ij})_I$  be a countable direct system of projective ZG-modules. Then there exists an exact sequence  $0 \rightarrow Q \rightarrow Q \rightarrow P \rightarrow 0$  where  $P = \varinjlim (P_i, \lambda_{ij})_I$  and  $Q$  is a projective ZG-module.*

**LEMMA 2.2.** *Let  $\cdots \rightarrow A_n \xrightarrow{\alpha_n} A_{n-1} \rightarrow \cdots \rightarrow A_1 \xrightarrow{\alpha_1} A_0 \xrightarrow{\pi} \text{coker } \alpha_1 \rightarrow 0$  be an exact sequence of ZG-modules such that for every  $n \geq 0$  there is a resolution  $P_{n*} \rightarrow A_n$  of the form  $0 \rightarrow P_{n,1} \xrightarrow{\partial_1^n} P_{n,0} \xrightarrow{\partial_0^n} A_n \rightarrow 0$ . If  $\{\alpha_*^j\}: P_{j*} \rightarrow P_{j-1*}$  are chain maps lifting  $\alpha_j: A_j \rightarrow A_{j-1}$   $j \geq 1$  then there is a resolution  $P_* \rightarrow \text{coker } \alpha_1$*

$$P_*: \cdots \longrightarrow P_k \xrightarrow{\partial_k} P_{k-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \text{coker } \alpha_1 \longrightarrow 0$$

where

$$P_k = P_{k-1,1} \oplus P_{k,0} \quad k \geq 0$$

$$\partial_k = \begin{cases} P_{k,0} & \xrightarrow{\alpha_0^k - (\partial_1^{k-2})^{-1} \alpha_0^{k-1} \alpha_0^k} & P_{k-1,0} \oplus P_{k-2,1} \\ P_{k-1,1} & \xrightarrow{\partial_1^{k-1} - \alpha_1^{k-1}} & P_{k-1,0} \oplus P_{k-2,1} \end{cases} \quad k \geq 1$$

and  $\varepsilon = \pi \partial_0^0$ .

Note that  $P_{ij}$  is understood to be zero if  $i < 0$  or  $j < 0$ .

*Proof.* We have the following commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & A_n & \xrightarrow{\alpha_n} & A_{n-1} & \longrightarrow & \cdots & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & A_0 & \xrightarrow{\pi} & \text{coker } \alpha_1 & \longrightarrow & 0 \\ & & \uparrow \partial_0^n & & \uparrow \partial_0^{n-1} & & & & \uparrow \partial_0^1 & & \uparrow \partial_0^0 & & & & \\ \cdots & \longrightarrow & P_{n,0} & \xrightarrow{\alpha_0^n} & P_{n-1,0} & \longrightarrow & \cdots & \longrightarrow & P_{1,0} & \xrightarrow{\alpha_0^1} & P_{0,0} & & & & \\ & & \uparrow \partial_1^n & & \uparrow \partial_1^{n-1} & & & & \uparrow \partial_1^1 & & \uparrow \partial_1^0 & & & & \\ \cdots & \longrightarrow & P_{n,1} & \xrightarrow{\alpha_1^n} & P_{n-1,1} & \longrightarrow & \cdots & \longrightarrow & P_{1,1} & \xrightarrow{\alpha_1^1} & P_{0,1} & & & & \end{array}$$

Now it is easily seen that this gives rise to the resolution  $P_* \rightarrow \text{coker } \alpha_1$ .

We shall need the following notion.

Let  $(G_i, \gamma_{ij})_I$  be a direct system of groups. For  $i \in I$  let  $(C_*^i, \partial_*^i)$  be a  $ZG_i$ -chain complex. If  $i \leq j$  then let  $c_{ij}: (C_*^i, \partial_*^i) \rightarrow (C_*^j, \partial_*^j)$  be a  $ZG_i$ -chain map, where  $(C_*^j, \partial_*^j)$  is considered as a  $ZG_i$ -chain complex via  $\gamma_{ij}: G_i \rightarrow G_j$ , such that

- (1)  $c_{ii}$  is the identity chain map of  $(C_*^i, \partial_*^i)$ , for all  $i \in I$  and
- (2) if  $i \leq j \leq k$  then  $c_{jk}c_{ij} = c_{ik}$  as  $ZG_i$ -chain maps.

We call  $((C_*^i, \partial_*^i), c_{ij})_I$  a direct system of chain complexes over the direct system of groups  $(G_i, \gamma_{ij})_I$ . In particular let  $(A_i, \alpha_{ij})_I$  be a direct system of modules over the direct system of groups  $(G_i, \gamma_{ij})_I$ . Clearly  $(A_i, \alpha_{ij})_I$  is a direct system of abelian groups and if  $A = \varinjlim (A_i, \alpha_{ij})_I$  then  $G = \varinjlim (G_i, \gamma_{ij})_I$  acts on  $A$  in a natural way [[2], ch. 2, §6, nos 6, 7].

Moreover, if  $B_i = \underset{ZG_i}{ZG} \otimes A_i$  and  $\beta_{ij} : B_i \rightarrow B_j$

$$x \underset{ZG_i}{\otimes} a_i \rightarrow x \underset{ZG_i}{\otimes} \alpha_{ij} a_i$$

then it follows that  $(B_i, \beta_{ij})_I$  is a direct system of  $ZG$ -modules and  $\varinjlim (B_i, \beta_{ij})_I \simeq A$  as  $ZG$ -modules.

Recall that if  $X$  is a class of groups, then a group  $G$  is said to be locally an  $X$ -group if every finite set of elements of  $G$  is contained in some  $X$ -subgroup of  $G$ .

**THEOREM 2.3.** *Let  $G$  be a countable locally finite group all of whose finite subgroups have period  $q$ . Then  $G$  has period  $q$  after 1-step.*

*Proof.* It is easily seen that there is a direct system of finite subgroups of  $G$  over  $I = \{1, 2, 3, \dots\}$ ,  $(G_i, e_{ij})_I$ , with  $e_{ij} : G_i \rightarrow G_j$  inclusions and  $G = \varinjlim (G_i, e_{ij})_I$ . By hypothesis for each  $i \in I$  we have a periodic resolution  $P_*^{i'}$  of  $G_i$

$$P_*^{i'} : 0 \longrightarrow Z \longrightarrow P_{q-1}^{i'} \longrightarrow \dots \longrightarrow P_1^{i'} \longrightarrow ZG_i \xrightarrow{\varepsilon_i} Z \longrightarrow 0.$$

Now consider the following diagram

$$\begin{array}{ccccccccccccccc} P_*^{i'} & 0 & \longrightarrow & Z & \longrightarrow & P_{q-1}^{i'} & \longrightarrow & \dots & \longrightarrow & P_1^{i'} & \longrightarrow & ZG_i & \xrightarrow{\varepsilon_i} & Z & \longrightarrow & 0 \\ \varrho_{ii+1}' \downarrow & & & \downarrow h & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow id_Z & & \\ P_*^{i+1'} & 0 & \longrightarrow & Z & \longrightarrow & P_{q-1}^{i+1'} & \longrightarrow & \dots & \longrightarrow & P_1^{i+1'} & \longrightarrow & ZG_{i+1} & \xrightarrow{\varepsilon_{i+1}} & Z & \longrightarrow & 0. \end{array}$$

Since  $P_*^{i+1'}$  is evidently a resolution of  $G_i$  via  $e_{ii+1} : G_i \rightarrow G_{i+1}$  we can lift  $id_Z$  to a  $ZG_i$ -chain map  $\varrho_{ii+1}' : P_*^{i'} \rightarrow P_*^{i+1'}$ .

Then  $\varrho_{ii+1}'$  induces a map  $h : Z \rightarrow Z$  (multiplication by  $h$ ) which need not be the identity on  $Z$ .

Our aim is to find a periodic resolution  $P_*^{i+1'}$  of  $G_{i+1}$  and a  $ZG_i$ -chain map  $\varrho_{ii+1}' : P_*^{i'} \rightarrow P_*^{i+1'}$  which lifts  $id_Z$  on the right and induces  $id_Z$  on the left. Now the map  $\varrho_{ii+1}'$  induces an isomorphism between the cohomology groups of  $G_i$  defined using  $P_*^{i'}$  and those defined using  $P_*^{i+1'}$ . On  $H^q(G_i, Z)$  this map is obviously

multiplication by  $h$ . Since  $H^q(G_i, Z) \simeq Z/|G_i|Z$  by the periodicity, it follows that  $(h, |G_i|) = 1$ ; hence there are integers  $\lambda_1, \lambda_2$  such that  $1 = \lambda_1 h + \lambda_2 |G_i|$ .

Let  $\lambda = \lambda_1 + x |G_i|$  where  $x = p_1 \cdots p_m$  with  $p_j$   $1 \leq j \leq m$  primes such that  $p_j \mid |G_{i+1}|$  and  $p_j \nmid |G_i|$ ,  $p_j \nmid \lambda_1$  for all  $1 \leq j \leq m$ . If no such primes exist, take  $x = 1$ . Then

$$(\lambda, |G_{i+1}|) = 1 \text{ and } 1 = \lambda h + (\lambda_2 - xh) |G_i| (*).$$

Consider  $P_*^{i+1'}$  as a projective resolution of  $G_{i+1}$ . Then  $id_Z$  is a  $q$ -cocycle which defines an element  $g_{i+1} \in H^q(G_{i+1}, Z)$ . Clearly  $\lambda g_{i+1} \in H^q(G_{i+1}, Z)$  is represented by the  $q$ -cocycle  $\lambda : Z \rightarrow Z$  (multiplication by  $\lambda$ ). We shall show that  $\lambda g_{i+1} \in H^q(G_{i+1}, Z) = \text{Ext}_{ZG_{i+1}}^q(Z, Z)$  is represented by a multiple extension

$$P_*^{i+1} : 0 \longrightarrow Z \longrightarrow P_{q-1}^{i+1} \longrightarrow P_{q-2}^{i+1} \longrightarrow \cdots \longrightarrow P_1^{i+1} \longrightarrow ZG_{i+1} \longrightarrow Z \longrightarrow 0$$

with  $P_k^{i+1}$   $1 \leq k \leq q-1$  projective  $ZG_{i+1}$ -modules. By Theorem 1.2 it is enough to show that for any  $ZG_{i+1}$ -module  $A$  cup product with  $\lambda g_{i+1}$

$$\cup \lambda g_{i+1} : H^k(G_{i+1}, A) \longrightarrow H^{k+q}(G_{i+1}, A)$$

is injective for  $k=1$  and surjective for  $k=0$ . This follows easily since cup product with  $\lambda g_{i+1}$  is multiplication by  $\lambda$  for  $k=0, 1$  and we have that  $(\lambda, |G_{i+1}|) = 1$ . Note that  $|G_{i+1}| H^k(G_{i+1}, A) = 0$  for all  $k \geq 1$  and by the periodicity  $H^q(G_{i+1}, A) \simeq A^{G_{i+1}} / (\sum_{g \in G_{i+1}} g)A$  where  $A^{G_{i+1}} \simeq \text{Hom}_{ZG_{i+1}}(Z, A)$ .

Moreover, from the proof of Theorem 1.2, we obtain the following commutative diagram

$$\begin{array}{ccccccccccc}
 P_*^{i'} & 0 & \longrightarrow & Z & \xrightarrow{\mu_i} & P_{q-1}^{i'} & \longrightarrow & \cdots & \longrightarrow & P_1^{i'} & \longrightarrow & ZG_i & \longrightarrow & Z & \longrightarrow & 0 \\
 \downarrow \varrho_{i+1}^{i'} & & & \downarrow h & & \downarrow \alpha & & & & \downarrow & & \downarrow & & \downarrow id_Z & & \\
 P_*^{i+1'} & 0 & \longrightarrow & Z & \xrightarrow{\gamma} & P_{q-1}^{i+1'} & \longrightarrow & P_{q-2}^{i+1'} & \longrightarrow & \cdots & \longrightarrow & P_1^{i+1'} & \longrightarrow & ZG_{i+1} & \longrightarrow & Z & \longrightarrow & 0 \\
 \downarrow L' & & & \downarrow \lambda & & \downarrow \beta & & \parallel & & \parallel & & \parallel & & \parallel & & \downarrow id_Z & & \\
 P_*^{i+1} & 0 & \longrightarrow & Z & \xrightarrow{\mu_{i+1}} & P_{q-1}^{i+1} & \longrightarrow & P_{q-2}^{i+1} & \longrightarrow & \cdots & \longrightarrow & P_1^{i+1} & \longrightarrow & ZG_{i+1} & \longrightarrow & Z & \longrightarrow & 0
 \end{array}$$

Clearly  $L' \varrho_{i+1}^{i'} : P_*^{i'} \rightarrow P_*^{i+1}$  is a  $ZG_i$ -chain map.

Now consider  $P_*^{i'}$  as a resolution of  $G_i$ . Then  $id_Z$  is a  $q$ -cocycle which defines an element  $g_i \in H^q(G_i, Z)$ .

Clearly the  $q$ -cocycle  $\lambda h : Z \rightarrow Z$  represents  $\lambda h g_i$ . But from (\*)  $\lambda h g_i = g_i$  since  $|G_i| g_i = 0$ . Hence the cocycles  $h$  and  $id_Z$  represent the same element. So there

exists  $\gamma : P'_{q-1} \rightarrow Z$  such that  $id_Z - \lambda h = \gamma \mu_i$ . If we now take  $\beta\alpha + \mu_{i+1}\gamma$  instead of  $\beta\alpha$  we still have a chain map  $P_*^{i'} \rightarrow P_*^{i+1}$  which now induces  $id_Z$  on the left. We call this chain map  $\varrho_{i+1}$ , i.e. we have the following commutative diagram

$$\begin{array}{ccccccccccccccc}
 P_*^{i'} & 0 & \longrightarrow & Z & \longrightarrow & P_{q-1}^{i'} & \longrightarrow & \cdots & \longrightarrow & P_1^{i'} & \longrightarrow & ZG_i & \xrightarrow{\varepsilon_i} & Z & \longrightarrow & 0 \\
 \varrho_{i+1} \downarrow & & & \downarrow id_Z & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow id_Z & & \\
 P_*^{i+1} & 0 & \longrightarrow & Z & \longrightarrow & P_{q-1}^{i+1} & \longrightarrow & \cdots & \longrightarrow & P_1^{i+1} & \longrightarrow & ZG_{i+1} & \xrightarrow{\varepsilon_{i+1}} & Z & \longrightarrow & 0.
 \end{array}$$

Take  $P_*^{i'} = P_*^1$ . Then given  $P_*^1$  and  $P_*^{2'}$  we construct as above a periodic resolution  $P_*^2$  of  $G_2$  and a  $ZG_1$ -chain map  $\varrho_{12} : P_*^1 \rightarrow P_*^2$  which lifts  $id_Z$  on the right and induces  $id_Z$  on the left. In this way we construct inductively a direct system  $(P_*^i, \varrho_{ij})_I$  of periodic resolutions over the direct system of groups  $(G_i, e_{ij})_I$ . Clearly  $\varrho_{ij}(i \leq j) : P_*^i \rightarrow P_*^j$  are given as  $\varrho_{ij} = \varrho_{j-1j} \cdots \varrho_{i+1}$  and they lift  $id_Z$  on the right and induce  $id_Z$  on the left.

Now since direct limit preserves exactness, taking the direct limit of  $(P_*^i, \varrho_{ij})_I$  we obtain the following exact sequence of  $ZG$ -modules

$$0 \longrightarrow Z \xrightarrow{\theta} \varinjlim P_{q-1}^i \xrightarrow{\alpha_{q-1}} \varinjlim P_{q-2}^i \longrightarrow \cdots \longrightarrow \varinjlim P_1^i \xrightarrow{\alpha_1} ZG \xrightarrow{\varepsilon} Z \longrightarrow 0 \tag{*}'$$

By splicing together copies of  $(*)'$  we obtain an exact sequence of  $ZG$ -modules

$$\begin{aligned}
 \cdots \xrightarrow{\alpha_1} ZG \xrightarrow{\alpha_q} \varinjlim P_{q-1}^i \xrightarrow{\alpha_{q-1}} \cdots \longrightarrow \varinjlim P_1^i \xrightarrow{\alpha_1} ZG \longrightarrow \\
 \xrightarrow{\alpha_q} \varinjlim P_{q-1}^i \xrightarrow{\alpha_{q-1}} \cdots \longrightarrow \varinjlim P_1^i \xrightarrow{\alpha_1} ZG \xrightarrow{\varepsilon} Z \longrightarrow 0 \tag{1}
 \end{aligned}$$

where  $\alpha_q = \theta\varepsilon$ .

By Proposition 2.1 we have resolutions  $Q_{j*} \rightarrow \varinjlim P_j^i$  of the form

$$0 \longrightarrow Q_{j1} \longrightarrow Q_{j0} \longrightarrow \varinjlim P_j^i \longrightarrow 0 \quad \text{for all } 1 \leq j \leq q-1.$$

Now the hypotheses of Lemma 2.2 hold for (1). Moreover, it is clear that we can choose here the chain maps  $\{a_*^n\}$  of Lemma 2.2 so as

$$\{\alpha_*^{j+kq}\} = \{\alpha_*^j\}, \quad \{\alpha_*^{\lambda q}\} = \{\alpha_*^q\} \quad 1 \leq j \leq q-1, \quad k \geq 0, \quad \lambda \geq 1.$$





and an element  $g' \in \text{Ext}_{ZG}^q(R_1, R_1)$  such that for every  $ZG$ -module  $A$  cup product with  $g'$

$$\cup g' : \text{Ext}_{ZG}^i(R_1, A) \longrightarrow \text{Ext}_{ZG}^{i+q}(R_1, A)$$

is an isomorphism for all  $i \geq 1$ .

By Proposition 2.5 if  $P$  is a projective  $ZG$ -module then  $H^i(G, P) = 0$  for all  $j \geq 2$ . Having this result one obtains from (2) that  $\text{Ext}_{ZG}^q(R_1, R_1) \xrightarrow{\delta} H^q(G, Z)$ ; note that  $q \geq 2$  since by Proposition 1.6  $q$  is a period of every finite subgroup of  $G$ , and that is known to be even [[3], ch. XII, p. 261].

Now if  $g \in H^q(G, Z)$  corresponds to  $g' \in \text{Ext}_{ZG}^q(R_1, R_1)$  under  $\delta$  then one clearly has that for any  $ZG$ -module  $A$  cup product with  $g$

$$\cup g : H^i(G, A) \longrightarrow H^{i+q}(G, A)$$

is an isomorphism for all  $i \geq 2$ . Thus we have proved that:

**PROPOSITION 2.6.** *Let  $G$  be a countable locally finite group which has period  $q$  after  $k$ -steps. Then there is an element  $g \in H^q(G, Z)$  such that for any  $ZG$ -module  $A$  cup product with  $g$*

$$\cup g : H^i(G, A) \longrightarrow H^{i+q}(G, A)$$

*is an isomorphism for all  $i \geq 2$ .*

### §3. Periodic locally finite $p$ -groups

The following well known theorem characterizes the finite  $p$ -groups which have period  $q > 0$ .

**THEOREM 3.1** [[3], ch. XII, Thm. 11.6]. *For a finite  $p$ -group  $G$  the following statements are equivalent:*

- (i)  $G$  has period  $q > 0$
- (ii)  $G$  is either cyclic or is a (generalized) quaternion group.

Moreover, a cyclic group has period 2 and a (generalized) quaternion group has period 4.

We shall characterize the infinite locally finite  $p$ -groups which have period  $q > 0$  after  $k$ -steps.

I. Consider an infinite locally cyclic  $p$ -group. It is easily seen that such a group is uniquely determined up to isomorphism by the prime  $p$  and is given by

$$C_p^\infty = \langle c_1, c_2, \dots, c_i, \dots; c_1^p = 1, c_2^p = c_1, \dots, c_{i+1}^p = c_i, \dots \rangle.$$

II. Recall that a (generalized) quaternion group  $Q_{2i}$  is given by

$$Q_{2i} = \langle x, y; x^{2^{i-2}} = y^2, xyx = y \rangle \quad i \geq 3.$$

One easily shows that a (generalized) quaternion group  $Q_{2i}$  contains a normal cyclic subgroup  $C$  of index 2, which is characteristic if  $i > 3$ , and every element of  $Q_{2i} \setminus C$  is of order four and inverts every element of  $C$ . Having this result and following the proof of Proposition 1.I.2 [8], one shows:

**PROPOSITION 3.2.** *Let  $G$  be a locally quaternion group. Then  $G$  has a locally cyclic normal subgroup  $C$  of index 2, and every element of  $G \setminus C$  is of order four and inverts every element of  $C$ .*

**COROLLARY 3.3.** *Up to isomorphism there exists only one infinite locally quaternion group, namely*

$$Q_{2^\infty} = \langle c_1, \dots, c_n, \dots, y; c_1^2 = 1, \dots, c_{n+1}^2 = c_n, \dots, y^2 = c_1, \\ c_i^y c_i = 1 \quad \text{all } i \geq 1 \rangle.$$

*Proof.* This follows easily from Proposition 3.2 and the fact that up to isomorphism there exists only one infinite locally cyclic 2-group, namely,  $C_{2^\infty}$ .

**PROPOSITION 3.4.** (i) *An infinite locally cyclic  $p$ -group has period 2 after 1-step.*

(ii) *An infinite locally quaternion group has period 4 after 1-step.*

*Proof.* (i) Clearly it is enough to consider  $C_{p^\infty}$ . Now  $C_{p^\infty}$  is a countable group. Moreover, every finite subgroup of  $C_{p^\infty}$  is cyclic, hence by Theorem 3.1 it has period 2. The result now follows from Theorem 2.3.

(ii) By Corollary 3.3 it is enough to consider  $Q_{2^\infty}$ . Clearly  $Q_{2^\infty}$  is countable. Moreover, if  $K$  is a finite subgroup of  $Q_{2^\infty}$  then  $K$  is contained in some (generalized) quaternion group; hence by Theorem 3.1 and Proposition 1.3  $K$  has period 4. Now the result follows from Theorem 2.3.

**THEOREM 3.5.** *For an infinite locally finite  $p$ -group  $G$  the following statements are equivalent:*

(i)  $G$  has period  $q > 0$  after  $k$ -steps.

(ii)  $G$  is either  $C_{p^\infty}$  or  $Q_{2^\infty}$ .

Moreover,  $k = 1$ . If  $G = C_{p^\infty}$  then  $q = 2$  and if  $G = Q_{2^\infty}$  then  $q = 4$ .

*Proof.* (i)  $\Rightarrow$  (ii). Let  $G$  have period  $q$  after  $k$ -steps. By Proposition 1.6 every finite subgroup of  $G$  has period  $q$  and therefore by Theorem 3.1 every finite subgroup of  $G$  is either cyclic or is a (generalized) quaternion group. Thus  $G$  is either an infinite locally cyclic  $p$ -group or an infinite locally quaternion group, i.e.  $G$  is either  $C_{p^\infty}$  or by Corollary 3.3,  $G$  is  $Q_{2^\infty}$ . (ii)  $\Rightarrow$  (i) follows from Proposition 3.4.

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*Forschungsinstitut für Mathematik  
ETH Zentrum  
CH-8092 Zürich*

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