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Uniquely ergodic quadratic differentials

HOWARD MASUR

Introduction

It has been of interest to know to what extent the Teichmuller spaces of genus $g>1$ with the Teichmuller metric has the geometric properties of a hyperbolic space. An example of such a property is that for every line L and point P not on L there should be a unique line through P which approaches L in the positive direction asymptotically. This property is what we study here in the context of Teichmüller space. This means examining particular examples of Teichmüller extremal maps.

For any line L there is an isometric embedding of the unit disc with the Poincaré metric into Teichmüller space such that the image contains L. The uniquely determined image disc is called a Teichmüller disc. We refer to [9] for details. If P is on this disc, the existence and uniqueness are trivial as the question reduces to considering the Poincaré disc. In his Princeton thesis, Kerckhoff [6], proved uniqueness in the general situation. If L is determined by a quadratic differential with closed trajectories and these trajectories sweep out $3g$-3 cylinders, then an asymptotic line through P will always exist [6]. On any Riemann surface the quadratic differentials with closed trajectories are a countable union of sets of positive codimension so it is of interest to study this asymptotic property for a wider class of quadratic differentials. These are the quadratic differentials whose horizontal trajectory flow is uniquely ergodic. Our main result is that if L is determined by a uniquely ergodic q with no closed critical trajectories, then there is always an asymptotic line through any P.

Thurston [13] and Bers [3], found examples of hyperbolic axes in T_g. As Thurston showed, the horizontal and vertical trajectory structures of the quadratic differential are attracting and repelling fixed points of the action of a diffeomorphism on a sphere of foliations. Using this characterization one can prove the asymptotic property for these lines directly. On the other hand, as Thurston showed, the trajectory flows are uniquely ergodic so our theorem gives a different proof.

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For a detailed treatment of Teichmüller extremal maps we refer to [2]. For a discussion of Teichmüller geodesics see [8] and [10]. We mention here one bit of terminology. If q is a quadratic differential, then the positive Teichmüller ray determined by q is given by the Teichmüller maps f_k with dilatation $k\bar{q}/|q|$, $-1<k\leq 0$. This means for each k the stretch is along the vertical trajectories, the contraction is along the horizontal trajectories.

§2. A Preliminary counterexample

We begin with the following result.

THEOREM 1. *If the line L in T_g is determined by a quadratic differential q with closed trajectories determining one cylinder of homotopy type γ, there is a line L' through P positively asymptotic to L if and only if P is on the same Teichmüller disc as L.*

Proof. by the remarks in the introduction, we need only consider the situation of P not on the disc, and suppose L' through P exists. There are two cases depending on whether or not L' is determined by the unique normalized differential with closed trajectoris of homotopy class γ on the Riemann surface at P.

Suppose first that it is. The theorem of [9] associates endpoints Q and Q', $Q\neq Q'$ to L and L' on the boundary Teichmüller space obtained by pinching along the curve γ. Then Proposition 2 of [10] shows the asymptotic distance between L and L' is at least as great as the boundary Teichmüller distance between Q and Q' which is positive.

Now suppose L' is determined by a q' not as above. Let β be any simple closed curve disjoint from γ. With respect to the metric $|q|^{1/2}\,|dz|$, the geodesic in the homotopy class of β is represented by a union of critical horizontal trajectories on the boundary of the cylinder. Fix an annulus homotopic to β near the boundary. For any $k<0$ this annulus can be embedded in the image surface under the Teichmüller map. This shows the extremal length of β is bounded above along the ray. Now consider the geodesic for β with respect to $|q'|^{1/2}\,|dz|$. If it is not represented by horizontal trajectories alone, then as $k\to -1$ its length measured with respect to the terminal differential with unit norm becomes unbounded. Therefore the extremal length of β on the image surface which is at least as great is also unbounded. However M-quasiconformal mapping change extremal length by a factor at most M. Therefore β must be horizontal and since it was an arbitrary curve disjoint from γ, q' has closed trajectories in the homotopy class of γ and we are back to the first case, a contradiction.

§3. Uniquely ergodic quadratic differentials

To begin the discussion we employ a device of Strebel's [12]. Given a quadratic differential q on X, fix a small vertical segment β containing no zeroes and label the two sides β_+ and β_-. For each $x \in \beta$ consider the horizontal trajectory leaving x on the + side. If the trajectory is dense it returns to β a first time, either to β_+ or β_-. We will assume every *noncritical trajectory is dense.* Then X decomposes into a union of rectangles R_i as in the following figures. These are

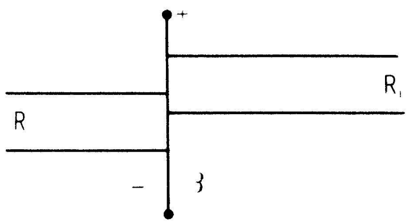


Figure 1

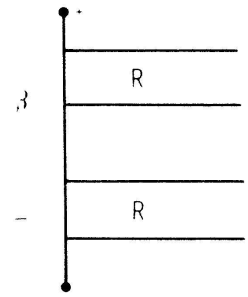


Figure 2

rectangles in the natural coordinates of q. In figure 1, a trajectory leaving a point $x \in R_i \cap \beta$ on the + side returns on the − side, in figure 2 it returns on the + side. There are possibly rectangles of both types. The total height of the rectangles of the second kind leaving and returning to β_+ is the same as the height of those leaving and returning to β_-. These rectangles R_i are identified to each other along various pieces of the top and bottom horizontal edges. The endpoints of the identifications are the zeroes x_i of q.

If all rectangles are of the first kind we may define a map $T : \beta \to \beta$; for $x \in \beta$, $T(x)$ is the first return for a trajectory through x leaving on the + side. If there are rectangles of the second kind we must define $T : \beta_+ \cup \beta_- \to \beta_+ \cup \beta_+$. For $x \in \beta_+$ if the first return is to $\beta_-(\beta_+)$, $T(x)$ is the corresponding point on $\beta_+(\beta_-)$. There is a similar definition for $x \in \beta_-$. It is possible to define T at the vertices of the rectangles to be either right or left continuous depending on the type of the rectangle. If all rectangles are of the first kind, T is defined to right continuous and is called an interval exchange map.

Now β and $\beta_+ \cup \beta_-$ may be given the measure μ defined by $|q^{1/2}|\,|dz|$. It is clear μ is invariant under T and we say T is uniquely ergodic if it is the only invariant measure up to scalar multiples. Although a different vertical interval determines a different map T, an invariant measure for one induces an invariant measure for the other. Therefore it makes sense to say the quadratic differential is uniquely ergodic.

We now formulate a topological definition. Two quadratic differentials q_1 and q_2 on X have *topologically equivalent* horizontal trajectory structures if by a finite sequence of homeomorphisms homotopic to the identity and a finite number of operations of collapsing and expanding of compact critical segments, the horizontal trajectories of q_1 can be transformed to the trajectories of q_2. In [5], p. 232, the definition is given of the strong equivalence of two measured foliations. The definition here is the same except that we do not require vertical distances to be preserved.

PROPOSITION 1. *The quadratic differential q on X is uniquely ergodic if and only if the only topologically equivalent quadratic differentials are real multiples.*

Proof. If ν is another (nonmultiple) invariant measure for T then ν defines a vertical measure for the topological foliation defined by the horizontal trajectories of q. The main theorem in [5] says this measured foliation is realized as the horizontal trajectories of a quadratic differential q' on X. Conversely, topologically equivalent quadratic differentials define the same map T but different invariant measures.

Remark. It is possible for topologically inequivalent quadratic differentials to define the same first return map, for instance if they correspond under a homeomorphism of the surface.

An important and seemingly difficult question is whether almost all interval exchange maps are uniquely ergodic. See [14].

EXAMPLES. 1. As mentioned in the introduction, Thurston found homeomorphisms which fix transverse foliations. These foliations define a uniquely ergodic quadratic differential. 2. Any interval exchange map with two or three intervals is uniquely ergodic. Starting with an interval exchange map one can always construct quadratic differentials inducing that interval exchange map. We will give an example of such a construction in §4. Keynes and Newton [7] found nonuniquely ergodic interval exchange maps with dense orbits.

We define the critical graph Γ of a quadratic differential to consist of the union of the compact critical segments.

THEOREM 2. *Suppose the line L is determined by a uniquely ergodic q and Γ contains no simple closed curves. Then for any P not on L there is a (unique) line through P positively asymptotic to L.*

Remark. The set of q on X with nonempty Γ is of measure zero in $H^0(X, \Omega^{\otimes 2})$ so if the conjecture on almost all interval exchange maps being uniquely ergodic is true, almost all quadratic differentials will satisy the hypothesis of the theorem. Our example in §4 will show the hypothesis to be necessary.

Proof of Theorem 2. We will first prove the theorem in the case that Γ is empty. By the main theorem of [5], there is a unique quadratic differential q' on P whose horizontal structure is measure equivalent to that of q. In this case this means there is a homeomorphism of the horizontal trajectories homotopic to the identity which also preserves the vertical distances between trajectories. Now q' may not have unit norm but taking terminal quadratic differentials along the line L' determined by q' we can find one with unit norm. Since the terminal quadratic differential determines L' as well we may assume q' has unit norm to begin with.

Pick small vertical segments β and β' for q and q' joining the same horizontal trajectories and having one endpoint in common. Since q and q' are measure equivalent, the first return maps T and measures μ on β and β' are the same. The rectangles R_i and R_i' have the same height and are identified in the same way; only their lengths are different. Now let $\epsilon>0$. We must show for K large enough the points at distance $\frac{1}{2}\log K$ on L from the base point are within ϵ of points at distance $\frac{1}{2}\log K$ along L' from P.

In all estimates to follow $O(\epsilon)$ refers to any function such that $O(\epsilon)/\epsilon \leq B$ as $\epsilon \to O$ where B depends only on the base points and not on K.

Since T is uniquely ergodic, for any continuous f on β or $\beta_+ \cup \beta_-$ (β' or $\beta'_+ \cup \beta'_-$), $1/n \sum_{j=0}^{n-1} f(T^j(x))$ converges uniformly to $\int f du$ as $n \to \infty$, [15, p. 136]. A routine approximation shows the same to be true if f is replaced by the characteristic function of an open interval. Pick N large enough so that all $n \geq N$,

$$\left|\frac{1}{n}\sum_{j=0}^{n-1} \chi_{R_i} T^j(x) - \mu(R_i)\right| < \epsilon \tag{1}$$

for each i and any $x \in \beta$. Of course the same holds for R_i' and β'. For any $\delta>0$, we can find intervals $\sigma \subset \beta$ and $\sigma' \subset \beta'$ joining the same trajectories of equal length less than δ such that for any $x \in \sigma$, $T^j(x) \notin \sigma$ and $T^j(x)$ not a vertex of R_i for $0<j\leq N-1$ and $-N+1\leq j<0$. We require the same condition on σ'.

Consider the induced return map and decomposition for these intervals giving rectangles S_i and S_i' of equal height. For $x \in S_j$ let $v_i(x)$ be the number of visits of x to R_i before returning to σ. This is the same as the number of visits of x to R_i' before returning to σ' for $x \in S_j'$. We wish to compute the lengths denoted $|\ |$ of S_j and S_j'. Then

$$|S_j| = \sum_{i=1}^{m} |R_i| v_i(x), \quad \text{and} \quad |S_i'| = \sum_{i=1}^{m} |R_i'| v_i(x)$$

where the sum is over all rectangles R_i and R_i'. Let $v=\sum_{i=1}^{m} v_i$. Then by (1)

$|v_i/v-\mu(R_i)|<\epsilon$. Therefore

$$\frac{1-\epsilon\sum_{i=1}^{m}|R_i|}{1+\epsilon\sum_{i=1}^{m}|R_i'|}=\frac{\sum_{i=1}^{m}|R_i|(\mu(R_i)-\epsilon)}{\sum_{i=1}^{m}|R_i|(\mu(R_i)+\epsilon)}\le\frac{\sum_{i=1}^{m}|R_i|\frac{v_i}{v}}{\sum_{i=1}^{m}|R_i'|\frac{v_i}{v}}$$

$$\le\sum_{i=1}^{m}\frac{|R_i|(\mu(R_i)+\epsilon)}{|R_i'|(\mu(R_i)-\epsilon)}=\frac{1+\epsilon\sum_{i=1}^{m}|R_i|}{1-\epsilon\sum_{i=1}^{m}|R_i'|}.$$

Therefore

$$\frac{|S_i|}{|S_i'|}=1+O(\epsilon)\quad\text{as}\quad\epsilon\to 0. \tag{2}$$

Let y be a vertex of a rectangle S_j and $|y|$ the distance to a zero along a trajectory. Let y' be the vertex for S_j'. Then an argument exactly as above shows $|y|/|y'|=1+O(\epsilon)$.

We would like to map S_j to S_j' by an $e^{O(\epsilon)}$ quasiconformal map preserving the zeroes which is linear along the edges so the maps would glue together to a map between the surfaces. The lengths and heights have ratios which are $e^{O(\epsilon)}$. However the positioning of the zeroes presents difficulties and here is where we must let $K\to\infty$ for then the heights of S_j and S_j' go to infinity. The first case of the theorem follows from the lemma.

LEMMA. *Let R_ϵ and R_ϵ' be two rectangles with vertices A_i, A_i' $i=1,\dots,4$ such that $l(R_\epsilon)/l(R_\epsilon')=|A_1A_2|/|A_1'A_2'|=e^{O(\epsilon)}$ as $\epsilon\to 0$. Suppose there are points P_1, P_2 on the top (A_1A_2) and bottom (A_3A_4), P_1', P_2' similarly on R_ϵ' such that $|A_1P_1|/|A_1'P_1'|=e^{O(\epsilon)}$ and $|P_1A_2|/|P_1'A_2'|=e^{O(\epsilon)}$ with similar equalities for P_2 and P_2'. Finally suppose the heights $h(R_\epsilon)=h(R_\epsilon')$ satisfy $|A_1A_2|/h(R_\epsilon)=O(\epsilon)$ as $\epsilon\to 0$. Then there is an $e^{O(\epsilon)}$ quasiconformal map R_ϵ to R_ϵ' which is linear on all sides and sends P_1 to P_1', P_2 to P_2'.*

Proof. By dividing each rectangle in half we may assume there are no points P_2 and P_2' on the bottom. With a simple affine stretch we may assume $|A_1A_2|=|A_1'A_2'|$. Therefore let the A_i and A_i' have coordinates $(0,b)$, (a,b), $(a,0)$, and $(0,0)$ in the z and w planes, resp., and suppose P_1 and P_1' have coordinates (c,b)

and (c', b) resp., where $c/c' = e^{O(\epsilon)}$ and $(a-c)/(a-c') = e^{O(\epsilon)}$. The quasiconformal map is

$$u = x\left[\left(\frac{c'}{c} - 1\right)\frac{y}{b} + 1\right], \qquad v = y \qquad 0 \le x \le c$$

$$u = a + (x-a)\left[\left(\frac{c'-a}{c-a} - 1\right)\frac{y}{b} + 1\right], \qquad v = y \qquad c \le x \le a.$$

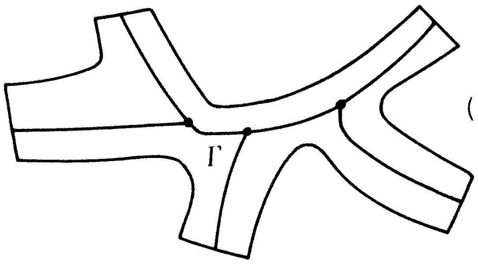
Here $w = u + iv$, $z = x + iy$. One checks easily that this has the desired mapping properties. Now for $x \le c$

$$u_x = \left(\frac{c'}{c} - 1\right)\frac{y}{b} + 1, \qquad v_y = 1, \qquad u_y = \left(\frac{c'}{c} - 1\right)\frac{x}{b}, \qquad v_x = 0.$$

Since $y/b \le 1$ and $c'/c - 1 = O(\epsilon)$ we have $u_x = e^{O(\epsilon)}$. Recalling a/b is $O(\epsilon)$ we have $u_y = O(\epsilon)$ so the map is $e^{O(\epsilon)}$ quasiconformal. We get similar estimates for $x \ge c$, proving the lemma.

The above proof fails if Γ is nonempty. For then there are segments of Γ on the top or bottom of some S_j giving two or more dividing points. As $K \to \infty$ the lengths have fixed ratio with corresponding lengths on S_j'. Instead we have to map neighborhoods of Γ onto each other by $e^{O(\epsilon)}$ quasiconformal maps for each K and map their complements as before. The compact trajectories will in general not correspond.

Let l be the sum of the orders of the zeroes contained in Γ. Since Γ contains no closed curves there are $l+2$ trajectories leaving Γ which are arbitrarily long. Consider a neighborhood U of Γ as in the following drawing.



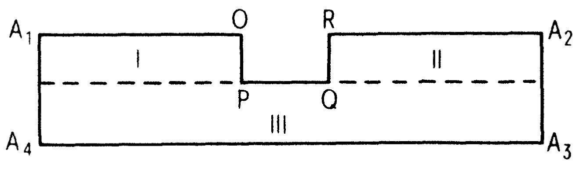
The boundary of U consists alternately of horizontal and vertical trajectories. We choose the horizontal trajectories leaving the graph to have common length h. Then the horizontal trajectories on δU have length $2h$ plus possibly one or more lengths of the pieces of Γ.

For h fixed and large the vertical segments on δU must be short and in fact, can be made arbitrarily small. For large enough K depending on h, we give them each length h/K, half on each side of the horizontal trajectory. Along the ray L at distance $\frac{1}{2}\log K$, the corresponding trajectories leaving the graph Γ_K have length $K^{-1/2}h$ and the vertical pieces on δU have length $K^{1/2}h/K = K^{-1/2}h$. By renormalizing the terminal quadratic differential q_K we can take all lengths to be h. Consider then this neighborhood U_K of Γ_K on the terminal surface. Each U_K embedds conformally in the Riemann sphere in such a way that $q_K\,dz^2$ is the restriction of $(z^l + p(z))\,dz^2$ for some polynomial $p(z)$ of degree at most $l-2$ (See Lemma 3.15 of [5]).

If we let $h \to \infty$ with respect to $|q^{1/2}\,dz|$ and set $\bar{q}_K = Kq_K/h^2$ then the segments leaving Γ_K have $\bar{q}_K$ length 1 and the critical segments have lengths which approach zero. In particular then by taking h, K large enough we can make $\bar{q}_K\,dz^2$ arbitrarily close to $z^l\,dz^2$ on U_K.

On the surface defined by P take a similar neighborhood of Γ' taking $h' = h$ and for the vertical segments $v' = v$. Then for large K, $\bar{q}'_K\,dz^2 = (z^l + p_2(z))\,dz^2$ on U'_K with $p_2(z)$ near zero. Now since the vertical lengths on $\delta U'_K$ are equal to the vertical lengths on δU_K and the horizontal lengths differ only by lengths on Γ_K and Γ'_K which approach zero, $\delta U'_K$ can be made arbitrarily close to δU_K in the z-plane. We may therefore map U_K to U'_K by an $e^{O(\epsilon)}$ quasiconformal map which is linear along the pieces of δU_K. A complete justification can be given by mapping the two regions to the upper half-plane. The desired boundary map satisfies an M condition where $M \to 1$ as $h, K \to \infty$. One can then apply an Ahlfors-Beurling extension. [1].

We continue with the proof of the theorem. Given $\epsilon > 0$ we fix the neighborhoods in the above discussion so that there is an $e^{O(\epsilon)}$ quasiconformal map between them for all large K. This means in particular the length h above is fixed. Now we proceed as in the first case. We can assume the rectangles S_j and S'_j are as in the following drawing (drawn for S_j).



The "missing" rectangle $QPQR$ is part of U_K. Since h is fixed, σ and σ' can be picked small enough so that

$$|A_1A_2|/|A'_1A'_2|, \qquad |A_1O|/|A'_1O'|, \qquad |RA_2|/|R'A'_2|, \qquad |PQ|/|P'Q'|$$

are all $1 + O(\epsilon)$ while $|OP| = |O'P'|$, $|A_1A_4| = |A'_1A'_4|$ and again $|A_1A_2|/|A_1A_4| = O(\epsilon)$.

Again we may assume $|A_1A_2| = |A_1'A_2'|$. We map I to I' and II to II' by affine maps which are $e^{O(\epsilon)}$ quasiconformal. Now we map III to III' as in the lemma. Here the base corresponds to $y = 0$, the top to $y = b$. The map is

$$u = x\left[\left(\frac{c'}{c} - 1\right)\frac{y}{b} + 1\right] \qquad 0 \le x \le c$$

$$u = d' + (x - d)\left[\left(\frac{d' - c'}{d - c} - 1\right)\frac{y}{b} + 1\right] \qquad c \le x \le d$$

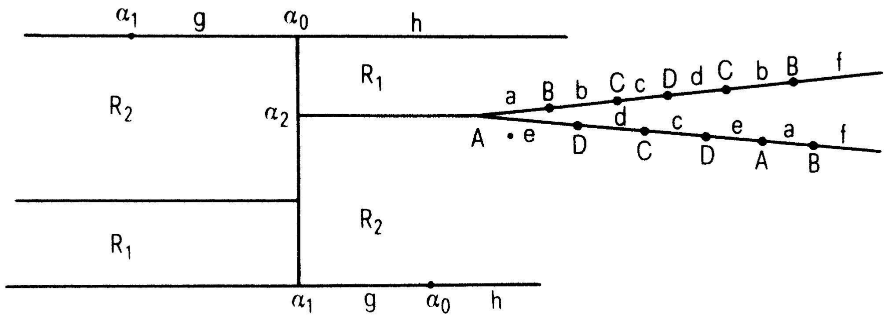
$$u = a + (x - a)\left[\left(\frac{d' - a'}{d - a} - 1\right)\frac{y}{b} + 1\right] \qquad d \le x \le a$$

$$v = y.$$

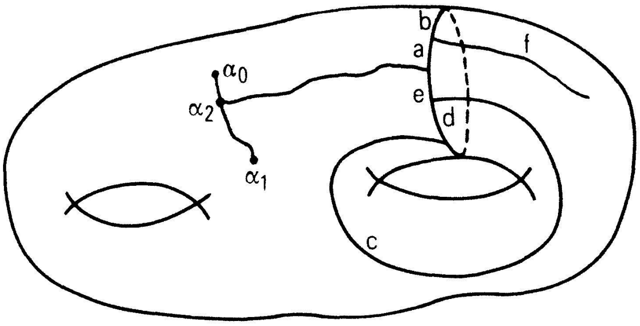
Here P and P' have coordinates (c, b) and (c', b') and Q and Q' have coordinates (d, b) and (d', b'). Estimates as in the lemma and (3) show that the map is $e^{O(\epsilon)}$ quasiconformal finishing the proof.

§4. A counterexample

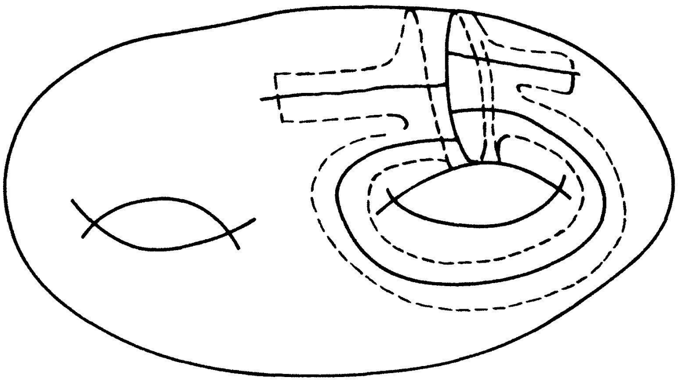
Theorem 2 fails when there are closed curves in Γ as a neighborhood like U does not exist. Consider an interval exchange map with two intervals. Attach rectangles R_1 and R_2 as indicated in the drawings



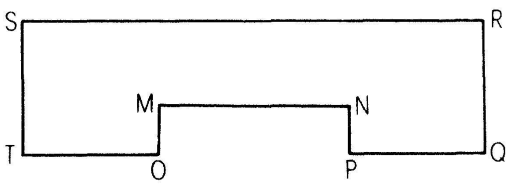
The points A, B, C, D are simple zeroes for the quadratic differential and the rectangles are attached along a, b, c, d, e, f, g, and h. Assign some lengths to these segments and to the rectangles to form a surface of genus 2 and a differential q as in the following figure. If $|\alpha_0\alpha_1| = 1$ and α_2 is irrational, the trajectories f and h are dense. Since this is an interval exchange map on two intervals, the flow is uniquely ergodic [4]. The segments a, b, d, e form a curve γ_1, while c, d form γ_2. We show there are points P with no lines asymptotic to the line L determined by q.



First form for each K a neighborhood V_K of Γ_K as in the following figure.



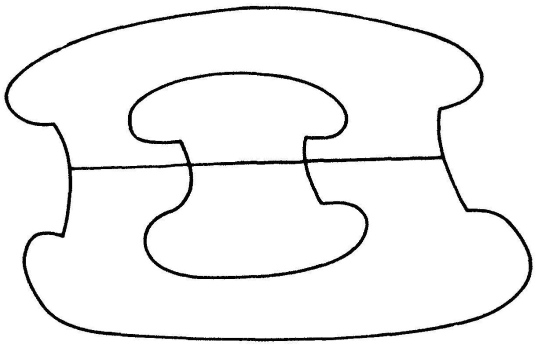
As before, we can make the vertical and horizontal segments on δV_K long and equal by taking K large. It is easy to see then that for $K_1 < K_2$, V_{K_1} embedds naturally in V_{K_2}. Also, δV_K bounds a torus T_K with one hole and for fixed K_0, δV_K and δV_{K_0} bound an annulus A_K whose modulus $\to\infty$ as $K\to\infty$. Cut this annulus along the horizontal trajectories leaving the graph, dividing A_K into two simply connected regions. Map each to the upper half-plane. We get two regions each as in the following figure.



As $K\to\infty$, the lengths $|PQ|$, $|QR|$, $|RS|$, $|ST|$ and $|OT|$ become unbounded while $|OM|$, $|MN|$ and $|NP|$ are fixed. Consider the maps

$$w=\frac{1}{z} \quad \text{and} \quad w=-\frac{1}{z}.$$

The image of the two together in the w-plane is as follows.



The quadratic differential is now $1/\omega^4\, dw^2$. The inner boundary is δV_K and collapses to 0 as $K \to \infty$. There is therefore for each K a canonical way of filling in T_K to a punctured torus T_0 and q_K to a quadratic differential q_0 which is $1/\omega^4\, dw^2$ in local coordinates at the puncture.

One proves by the method used in Theorem 3 in [10] that for every convergent sequence in the Bers embedding of T_g of points on L, there is a B-group with a noninvariant component representing T_0.

Now the differential q_0 on T_0 also has critical closed curves in the homotopy classes γ_1 and γ_2. We may vary q_0 and T_0 by varying the lengths of the segments a, b, c, d and e, giving a new punctured torus T_1 and a quadratic differential q_1 on T_1 with a pole of order 4 and closed critical trajectories in the classes γ_1 and γ_2. We construct the interval exchange map with these new lengths giving a compact surface and a quadratic differential q' with the same measured foliation as q. (The lengths of the rectangles are essentially arbitrary.) The line L' determined by q' cannot be asymptotic to the line L determined by q. Suppose there are M quasiconformal maps, $M \to 1$ between the surfaces on L and L'. This would give a conformal map between T_0 and T_1. The proof of this fact is precisely the same as the proof of Proposition 2 in [10]. Finally, suppose q' on this surface is any other quadratic differential. Then the horizontal foliations F' of q' and F of q are not measure equivalent. However, as in the proof of Theorem 1, the curves γ_1, γ_2 must still be critical for q' if L' is to be asymptotic to L. This forces the first return map to be a two interval exchange map and hence uniquely ergodic. Now let $\beta_n \to F$ be a sequence of simple closed curves converging to F in the sense of measured foliations (see [5] or [13]). The curves β_n are represented by geodesics with respect to q'. Suppose the vertical lengths v_n of $\beta_n \to 0$ as $n \to \infty$. Then there are subsequences which converge to trajectories of q'. Since these trajectories are equally distributed, as q' is uniquely ergodic, the β_n are equally distributed in the limit as well, and one concludes $\beta_n \to F'$, a contradiction.

Therefore v_n is bounded below and the extremal length of β_n on the surfaces on L' goes to infinity uniformly as $K \to \infty$. However, for each K we may take β_n close to F so that the extremal length of β_n is close to the extremal length of

$q_K\, dz^2$ on the terminal surface. However this latter is $1/K$ by Proposition 3 of [6], giving a contradiction.

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