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Crossed n-fold extensions of groups and cohomology

JOHANNES HUEBSCHMANN

1. Introduction

Crossed modules (§2 below) were introduced by J. H. C. Whitehead [22], [25], and also by Peiffer [19] and Reidemeister [20]. Whitehead was lead to the definition of a crossed module when he investigated the structure of a second relative homotopy group (cf. [8 p. 39]).

The concept of a crossed module admits a natural generalisation to that of a *crossed complex* (§5). Complexes of this kind were considered in [1], [2], [3], [6], [9], [23], [25] and [26].

An exact crossed complex involving only finitely many non-zero groups and modules may be thought of as a *crossed n-fold extension*

$$0 \to A \to C_{n-1} \to \cdots \to C_1 \to G \to Q \to 1, \qquad n \geqq 1,$$

with Q a group and A a Q-module (see §3). The purpose of this paper is to show that under a suitable similarity relation the classes of crossed n-fold extensions of A by Q constitute an Abelian group $\mathrm{Opext}^n(Q, A)$ naturally isomorphic to the cohomology group $H^{n+1}(Q, A)$ (main Theorem in §7). Thereby the group composition is given by a "Baer sum". This generalises MacLane's interpretation of $H^2(Q, A)$ as group of operator extensions of A by Q [16].

Our major tools are the concepts of *free crossed modules* (§4), of *free (projective) crossed resolutions of groups* (§5), and that of *homotopy between morphisms of crossed complexes* (§6). The main Theorem is proved in §§7 and 8. In §9 we introduce the *crossed standard resolution* which will be used in [13], [14] and [15]. In §10 we give an illustrative application which will be needed in [13].

As crossed n-fold extensions do occur in mathematics, our interpretation seems to cast new light on group cohomology. We (hope to) demonstrate the significance of our theory in [11], [12], [13], [14] and [15].

Similar results as ours were obtained by other people; we refer to MacLane's Historical Note [17].

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2. Crossed modules

A *crossed module* (C, G, ∂) [25] consists of groups C and G, an operation of G on the left of C, written $(g, c) \mapsto {}^{g}c$, and a homomorphism $\partial: C \to G$ of G-groups, where G acts on the left of itself by conjugation. The map ∂ must satisfy the rule

$$bcb^{-1} = {}^{\partial(b)}c, \ b, \ c \in C.$$

A *morphism* $(\alpha, \beta): (C, G, \partial) \to (C', G', \partial')$ of crossed modules consists of homomorphisms $\alpha : C \to C', \beta : G \to G'$ of groups such that $\beta\partial = \partial'\alpha$ and $\alpha({}^{g}c) = {}^{\beta(g)}\alpha(c)$, $c \in C$, $g \in G$. If (C, G, ∂) is a crossed module, then C is called a *crossed G-module.*

A crossed module generalises the concepts of both an ordinary module and that of a normal subgroup. For if Q is a group and A a Q-(left-) module, then $(A, Q, 0)$ is a crossed module with 0 the trivial map $0(a) = 1 \in Q$, $a \in A$. If G is a group and N a normal subgroup, then (N, G, i) is a crossed module, with i the inclusion and G acting on N by conjugation.

We note at once certain consequences of the definition of a crossed module:

(a) The image ∂C is a normal subgroup of G.

(b) The kernel ker (∂) lies in the center Z of C.

(c) The operation of G on C induces a natural $(G/\partial C)$-module structure on Z, and ker (∂) is a submodule of Z.

(d) The action of G on C induces a natural $(G/\partial C)$-module structure on the commutator factor group $C^{Ab} = C/[C, C]$.

It is clear that the crossed modules constitute a category **XMod:** if G is a fixed group, the crossed G-modules constitute a (full) subcategory **G-XMod.**

3. Crossed *n*-fold extensions

Let Q be a group and A a Q-module. A *crossed n-fold extension of A by Q* $(n \geqq 1)$ is an exact sequence

$$e: 0 \longrightarrow A \xrightarrow{\gamma} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} G \longrightarrow Q \longrightarrow 1$$

of groups with the following properties:

(i) (C_1, G, ∂_1) is a crossed module,
(ii) for $1<k<n$, C_k is a Q-module, and ∂_k and γ are Q-linear.

Note that it makes sense to require ∂_2 to be Q-linear, since the kernel of ∂_1 is naturally a Q-module. Now a *morphism* $(\sigma, \alpha, \varphi) : e \to e'$ *of crossed n-fold extensions* consists of group homomorphisms $\varphi : Q \to Q'$, $\alpha_0 : G \to G'$, $\alpha_k : C_k \to C'_k$, $0<k<n$, and $\sigma : A \to A'$ such that $(\sigma, \alpha_{n-1}, \dots, \alpha_1, \alpha_0, \varphi)$ provides a commutative diagram of groups which preserves all the structure. So we have a category of crossed n-fold extensions of A by Q. For completeness, by a *crossed 0-fold extension* of A by Q we mean a derivation $d : Q \to A$.

Given a group K with center Z and automorphism group Aut (K), we have the crossed 2-fold extension

$$0 \to Z \to K \xrightarrow{\partial_K} \mathrm{Aut}\,(K) \to \mathrm{Out}\,(K) \to 1,$$

where ∂_K sends $k \in K$ to the corresponding inner automorphism; here Out (K) is the group of outer automorphisms. Now any abstract Q-kernel $\psi : Q \to \mathrm{Out}\,(K)$ (see [7]) provides a crossed 2-fold extension

$$e^{\psi} : 0 \to Z \to K \xrightarrow{\partial^{\psi}} G^{\psi} \to Q \to 1$$

with G^{ψ} the fibre product $\mathrm{Aut}\,(K) \times_{\mathrm{Out}\,(K)} Q$ and ∂^{ψ} the obvious map. Crossed 2-fold extensions of this kind with G^{ψ} a free group were studied in [16], see also [18]. An example of a crossed n-fold extension for $n>2$ will be given in [13].

4. Free crossed modules

Let **Grp(2)** denote the category whose objects are group homomorphisms and whose morphisms are commutative squares in the category of groups. The forgetful functor V:**XMod**$\to$**Grp(2)** which forgets the group action has a left adjoint $(\lambda : H \to G) \mapsto U(\lambda) = (C, G, \partial)$, the *free crossed module on* λ, see [4, p. 207]. If H is (as group) free on a set S, then C coincides with Whitehead's *free crossed G-module* [25, p. 455]; in this case S is called a *basis* for C. Thereby the use of the word "basis" is justified by the fact that the induced map $S \to C$ is injective. This follows from

LEMMA 1. *If C is the free crossed G-module with basis S, then C^{Ab} is as ordinary $(G/\partial C)$-module free on the elements $s[C, C]$, $s \in S$.*

Note, however, that the induced map $H \to C$ need not be injective (where still H is free on S).

Let now $(X; R)$ be a presentation of a group Q. Let N_0 be free on a set $\hat{R}$ in one-one correspondence with R (via $\hat{r} \mapsto r$), and let $\lambda : N_0 \to F$ be the map that is induced by the relators, where F is free on X; we denote by N the normal closure of R in F.

PROPOSITION 1. *Any presentation $(X; R)$ of a group Q determines a crossed module (C, F, ∂) which is unique up to isomorphism; thereby F is (as group) free on X and C is the free crossed F-module with basis R (resp. $\hat{R}$). Moreover, the following holds:*

(*a*) *If F has at least two free generators, then the center of C coincides with the kernel of ∂.*

(*b*) *The elements $\hat{r}[C, C]$, $r \in R$, constitute a Q-basis of C^{Ab}.*

(*c*) *The induced map ker $(\partial) \to C^{Ab}$ is injective, and*

$$0 \to \ker(\partial) \to C^{Ab} \to N^{Ab} \to 0$$

is a Q-free presentation of N^{Ab}.

Proof of (*c*). Since $\ker(\partial)$ is central in C, and since N is a free group, C is a direct product $\ker(\partial) \times \bar{N}$, where $C \to N$ induces an isomorphism $\bar{N} \to N$; hence $\ker(\partial) \to C^{Ab}$ is injective. Q.E.D.

For a group G, the notion of a free crossed G-module may be generalised: A *projective crossed G-module* is a projective object in **G-XMod.**

5. Crossed complexes and free (projective) crossed resolutions of groups

A *crossed complex* **C** (over a group) is a sequence

$$\mathbf{C}: \cdots \longrightarrow C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} G$$

of groups with the following properties:

(C1) The triple (C_1, G, ∂_1) is a crossed module;

(C2) for $k \geqq 2$ each C_k is a Q-module, where $Q = G/(\partial_1 C_1)$, and each ∂_k is a Q-map (for $k = 2$ this shall mean that ∂_2 commutes with the action of G; note, however, that the image $\partial_2(C_2) \subset C_1$ is a Q-module);

(C3) $\partial\partial = 0$.

A crossed complex **C** is called *free* (*projective*) if G is a free group, if C_1 is a free (projective) crossed G-module, and if each C_k, $k \geqq 2$, is a free (projective)

Q-module ($Q = G/(\partial_1 C_1)$). If a crossed complex $\mathbf{C}$ is exact, and if a group Q, given in advance, is isomorphic to the quotient $G/(\partial_1 C_1)$, then $\mathbf{C}$ is called a *crossed resolution* of Q (a *free* resp. a *projective* crossed resolution, if $\mathbf{C}$ is free resp. projective). Now a *morphism* $\alpha : \mathbf{C} \to \mathbf{C}'$ of *crossed complexes* consists of group homomorphisms $\alpha_0 : G \to G'$, $\alpha_k : C_k \to C'_k$, $k \geqq 1$, such that $(\ldots, \alpha_k, \alpha_{k-1}, \ldots, \alpha_1, \alpha_0)$ provides a commutative diagram of groups which preserves all the structure.

Clearly, crossed n-fold extensions yield special examples of (exact) crossed complexes with $C_k = 0$, $k > n$. The standard example of a crossed complex is given by the sequence of relative homotopy groups of a filtered space [3], [6], [25] ("homotopy system").

As for a given group Q any Q-module has a free (projective) resolution, from Proposition 1 we infer

PROPOSITION 2. *Any group has a free (projective) crossed resolution.*

The following is clear:

PROPOSITION 3. *Let* $\mathbf{C}$ *be a free (projective) crossed complex with* $Q =$ *coker* (∂_1), *and let* $\mathbf{C}'$ *be a crossed resolution of a group* Q'. *Then any homomorphism* $\varphi : Q \to Q'$ *may be lifted to a morphism* $\alpha : \mathbf{C} \to \mathbf{C}'$ *of crossed complexes.*

If $\mathbf{C}$ is a free (projective) crossed resolution of Q, denote by $\mathbf{C}^n$ the crossed complex (for $n \geqq 2$ it is a crossed n-fold extension)

$$\mathbf{C}^n : 0 \to J_n \to C_{n-1} \to \cdots \to C_1 \to F \to Q \to 1,$$

where $J_n = \ker (C_{n-1} \to C_{n-2})$ (with $C_0 = F$ and $C_{-1} = Q$). We shall refer to $\mathbf{C}^n$ as a *free (projective) crossed* n*-fold extension* (as to $\mathbf{C}^1$ see Note added in proof).

PROPOSITION 3′. *Let* e' *be a crossed* n*-fold extension with* $Q' =$ *coker* (∂_1). *Then any homomorphism* $\varphi : Q \to Q'$ *may be lifted to a morphism* $(\sigma, \alpha, \varphi) : \mathbf{C}^n \to e'$ *of crossed* n*-fold extensions.*

6. Homotopy

Let there be given two crossed complexes $\mathbf{C}, \mathbf{C}'$ with $Q = \operatorname{coker} (\partial_1)$ and $Q' = \operatorname{coker} (\partial'_1)$; let further α and β be morphisms $\mathbf{C} \to \mathbf{C}'$ of crossed complexes.

Now a family $\Sigma=\{\Sigma_k, k\geqq 0\}$ of maps $\Sigma_0: G\to C_1'$, $\Sigma_k: C_k'\to C_{k+1}$, $k\geqq 1$, is called a *homotopy* between α and β, denoted $\Sigma: \alpha\simeq\beta$, if

(i) $\Sigma_0: G\to C_1$ is a (left-) derivation (crossed homomorphism) associated with β_0, i.e. $\Sigma_0(xy)=\Sigma_0(x)({}^{\beta_0(x)}\Sigma_0(y))$, $x, y\in G$, such that

$$\partial_1\Sigma_0(x)=\alpha_0(x)\beta_0(x)^{-1}, \qquad x\in G,$$

(ii) $\Sigma_1: C_1\to C_2'$ is a G-homomorphism, with G acting on C_2' via α_0 (or β_0, which yields the same action in view of (i)), such that

$$\partial_2\Sigma_1(x)=\beta_1(x)^{-1}(\Sigma_0\partial_1(x))^{-1}\alpha_1(x), \qquad x\in C_1,$$

(iii) for $k\geqq 2$, Σ_k is a Q-homomorphism, with Q acting on the C_k' via the induced map $Q\to Q'$ (note that α and β induce the same map $Q\to Q'$ in view of (i)), such that

$$\partial_{k+1}\Sigma_k+\Sigma_{k-1}\partial_k=\alpha_k-\beta_k.$$

LEMMA 2. *Homotopy is an equivalence relation.*

PROPOSITION 4. *Let* $\mathbf{C}$ *be a free (projective) crossed complex with* $Q=$ *coker* (∂_1), *and let* $\mathbf{C}'$ *be a crossed resolution of* Q'; *let further* $\alpha, \beta: \mathbf{C}\to\mathbf{C}'$ *be morphisms of crossed complexes. If* α *and* β *induce the same homomorphism* $\varphi: Q\to Q'$, *there is a homotopy* $\Sigma: \alpha\simeq\beta$.

It is clear that we also have the notion of a homotopy $\Sigma: (\sigma, \alpha, \varphi)\simeq(\tau, \beta, \varphi)$ of morphisms $e\to e'$ of crossed n-fold extensions with the same *right end* $\varphi: Q\to Q'$: it is a family $(\Sigma_{n-1}, \ldots, \Sigma_0)$ of maps satisfying (i), (ii) and (iii) above; thereby $\partial_n=\gamma$, $\Sigma_n=0=\partial_{n+1}$, $\alpha_n=\sigma$, $\beta_n=\tau$, $C_n=A$.

PROPOSITION 4′. *Let* $\mathbf{C}^n$ *be a free (projective) crossed n-fold extension with* $Q=$ *coker* (∂_1), *and let* e' *be a crossed n-fold extension with* $Q'=$ *coker* (∂_1). *If* $(\sigma, \alpha, \varphi)$ *and* (τ, β, φ) *are morphisms* $\mathbf{C}^n\to e'$ *of crossed n-fold extension with the same right end* φ, *then there is a homotopy* $\Sigma: (\sigma, \alpha, \varphi)\simeq(\tau, \beta, \varphi)$.

Proofs are routine and left to the reader. If we combine the above with Proposition 3 resp. Proposition 3′, we obtain

PROPOSITION 5. *The set* Hom (Q, Q') *classifies the homotopy classes of morphisms* $\mathbf{C}\to\mathbf{C}'$ *resp. of morphisms* $\mathbf{C}^n\to e'$ *with the same right end.*

It is now clear how to introduce the notion of *homotopy equivalence* of crossed complexes, and we have the

COROLLARY. *Any two free (projective) crossed resolutions of a group are homotopy equivalent.*

7. Opext^n-groups and cohomology; the main Theorem.

Let Q be a given group, and let

$$\mathbf{C}: \cdots \to C_k \xrightarrow{\partial} C_{k-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_1 \to F \dashrightarrow Q$$

be a free (projective) crossed resolution of Q. For any Q-module A, consider the complex (the arrows are the obvious maps)

$$\mathrm{Hom}(\mathbf{C}, A): \mathrm{Der}(F, A) \to \mathrm{Hom}_F(C_1, A) \to \mathrm{Hom}_Q(C_2, A) \to \cdots.$$

(For a group G and a G-module A, "$\mathrm{Der}(G, A)$" denotes the Abelian group of derivations from G to A.) Its cohomology groups are as follows:

PROPOSITION 6. $H^0(\mathrm{Hom}(\mathbf{C}, A)) = \mathrm{Der}(Q, A)$, $H^q(\mathrm{HOM}(\mathbf{C}, A)) = H^{q+1}(Q, A)$, $q \geqq 1$.

Proof. Assume for convenience that, in case $\mathbf{C}$ is a proper projective crossed resolution, the crossed F-module C_1 is free. The case of a proper projective crossed F-module C_1 is left as an exercise. Now the crossed complex $\mathbf{C}$ may be transformed into the complex

$$\hat{\mathbf{C}}: \cdots \to C_k \to \cdots \to C_2 \to C_1^{Ab} \to \mathbb{Z}Q \otimes_F IF,$$

where $C_2 \to C_1^{Ab}$ is the obvious map, and where $C_1^{Ab} \to \mathbb{Z}Q \otimes_F IF$ is given by the rule $x[C_1, C_1] \mapsto 1 \otimes (\partial x - 1)$, $x \in C_1$. (Here "IG" denotes the augmentation ideal of a group G.) By Proposition 2, C_1^{Ab} is a free Q-module, and the cokernel of $C_2 \to C_1^{Ab}$ is the relation module N^{Ab}, where $N = \ker(F \to Q)$. Hence $\hat{\mathbf{C}}$ is a free (projective) resolution of IQ. Applying the functor $\mathrm{Hom}_Q(-, A)$ to $\hat{\mathbf{C}}$ yields a complex canonically isomorphic to $\mathrm{Hom}(\mathbf{C}, A)$ whence the cohomology of $\mathrm{Hom}(\mathbf{C}, A)$ is as stated. Q.E.D.

The fact that $H^2(Q, A)$ is $H^1(\mathrm{Hom}(\mathbf{C}, A))$ was already proved by MacLane [16, Theorem A'].

We now divide the crossed n-fold extensions of A by Q ($n \geqq 1$) into classes as follows: Two crossed n-fold extensions e, e' of A by Q are related if there is a morphism $(1, \alpha, 1): e \to e'$ of crossed n-fold extensions; this relation generates an equivalence relation which shall be denoted by "$\equiv$". The equivalence class of e, also called *similarity class*, is to be denoted by $[e]$.

We next consider a crossed n-fold extension e of A by Q. If $\mathbf{C}$ is a projective crossed resolution of Q, it follows from Proposition 3 that the identity map of Q lifts to

$$\begin{array}{lccccccccccccc}
\mathbf{C}: \cdots \to & C_n & \to & C_{n-1} & \to & \cdots & \to & C_1 & \to & F & \to & Q & \to & 1 \\
 & \downarrow \zeta & & \downarrow \alpha_{n-1} & & & & \downarrow \alpha_1 & & \downarrow \alpha_0 & & \| & & \\
e: \quad 0 \to & A & \to & A_{n-1} & \to & \cdots & \to & A_1 & \to & G & \to & Q & \to & 1
\end{array}$$

In view of the above, ζ represents a class $[\zeta] \in H^{n+1}(Q, A)$. If $\mathbf{C}$ is replaced by $\mathbf{C}^n$ (introduced in §5), the above induces a morphism $(\nu, \alpha, 1): \mathbf{C}^n \to e$ of crossed n-fold extensions. Now, for $n \geqq 2$, the coequaliser $C_{n-1,\nu}$, say, of $J_n \underset{i}{\overset{\nu}{\rightrightarrows}} A \times C_{n-1}$, where i denotes the inclusion $J_n \to C_{n-1}$, yields the crossed n-fold extension

$$\nu\mathbf{C}^n: 0 \to A \to C_{n-1,\nu} \to C_{n-2} \to \cdots \to C_1 \to F \to Q \to 1,$$

with $C_{n-1,\nu} \to C_{n-2}$ the obvious map. If $n=1$, the coequaliser $C_{0,\nu}$ of $J_1 \rightrightarrows A]F$ ($J_1 = N = \ker(F \to Q)$) yields the ordinary group extension

$$\nu\mathbf{C}^1: 0 \to A \to C_{0,\nu} \to Q \to 1;$$

here "]" denotes the semi-direct product. Clearly, there is a morphism $(1, \beta, 1): \mathbf{C}^n \to e$ of crossed n-fold extensions; hence

PROPOSITION 7. *Each equivalence class of crossed n-fold extensions of A by Q has a representative of the form $\nu\mathbf{C}^n$.*

It is now clear that the Abelian group $\mathrm{Hom}_F(J_n, A)$ ($=\mathrm{Hom}_Q(J_n, A)$, if $n \geqq 2$) maps onto the classes of crossed n-fold extensions of A by Q by rule $\nu \mapsto \nu\mathbf{C}^n$. Consequently, these classes constitute a set, denoted henceforth by $\mathrm{Opext}^n(Q, A)$.

Given two crossed n-fold extensions e, e' of A by Q, it is routine to construct their "Baer- sum" $e+e'$. We refrain from writing down details. Moreover, the Baer- sum induces a sum on similarity classes, and the surjection $\mathrm{Hom}_F(J_n, A) \to \mathrm{Opext}^n(Q, A)$ is a homomorphism with respect to the Baer- sum, i.e.$(\mu+\nu)\mathbf{C}^n \equiv \mu\mathbf{C}^n + \nu\mathbf{C}^n$, $\mu, \nu: J_n \to A$ operator maps. Consequently, under the Baer- sum, $\mathrm{Opext}^n(Q, A)$ is an Abelian group, with zero element $0\mathbf{C}^n$, i.e. the image of the zero map $J_n \to A$, and $\mathrm{Hom}_F(J_n, A) \to \mathrm{Opext}^n(Q, A)$ is an epimorphism of Abelian groups.

LEMMA 3. *Let* $\nu : J_n \to A$, $n \geqq 1$, *be an operator map which may be extended over* C_{n-1} *to*

(i) *a derivation* $F \to A$, *if* $n = 1$,

(ii) *an* F*-map* $C_1 \to A$, *if* $n = 2$, *and*

(iii) *a* Q*-map* $C_{n-1} \to A$, *if* $n \geqq 3$.

Then the extension

$$E: 0 \to A \to C_{n-1,\nu} \to J_{n-1} \to 1$$

($J_1 = N$, $J_0 = Q$) *splits, i.e. there is a section* $J_{n-1} \to C_{n-1,\nu}$ *which is a group homomorphism, if* $n = 1$, *an* F*-homomorphism, if* $n = 2$, *and a* Q*-homomorphism, if* $n \geqq 3$.

The proof is straightforward.

If, given an operator map $\nu : J_n \to A$, $n \geqq 2$, the extension E splits (as in Lemma 3), there is a morphism $(1, \alpha, 1): \nu\mathbf{C}^n \to \mathbf{0}$ of crossed n-fold extensions, where $\mathbf{0}$ denotes

$$\mathbf{0}: 0 \to A \overset{=}{\to} A \to 0 \to \cdots \to 0 \to Q \overset{=}{\to} Q \to 1,$$

whence $\nu\mathbf{C}^n$ and $\mathbf{0}$ are equivalent; since $\mathbf{0}$ represents $0 \in \operatorname{Opext}^n(Q, A)$, so does $\nu\mathbf{C}^n$. By Proposition 6, the cokernel of $\operatorname{Hom}_F(C_{n-1}, A) \to \operatorname{Hom}_Q(J_n, A)$ ($\operatorname{Hom}_F(C_{n-1}, A) = \operatorname{Hom}_Q(C_{n-1}, A)$ if $n \geqq 3$) is the cohomology group $H^{n+1}(Q, A)$. It follows from Lemma 3 that for $n \geqq 2$ the rule $\nu \mapsto \nu\mathbf{C}^n$, $\nu : J_n \to A$ an operator map, induces an epimorphism $\Phi : H^{n+1}(Q, A) \to \operatorname{Opext}^n(Q, A)$ of Abelian groups; this also follows for $n = 1$, as $H^2(Q, A)$ is the cokernel of $\operatorname{Der}(F, A) \to \operatorname{Hom}_F(N, A)$.

The main Theorem. *The map* Φ *is an isomorphism of Abelian groups. In other words, the classes of crossed* n*-fold extensions of* A *by* Q *constitute an Abelian group* $\operatorname{Opext}^n(Q, A)$ *naturally isomorphic to the cohomology group* $H^{n+1}(Q, A)$. *The group composition is given by the Baer-sum. The zero element of this group is the class of the crossed* n*-fold extension* $\mathbf{0}$, *whereas the inverse of the class of*

$$e: 0 \to A \overset{\gamma}{\to} C_{n-1} \to \cdots \to C_1 \to G \to Q \to 1$$

is the class of

$$-e: 0 \to A \xrightarrow{(-\gamma)} C_{n-1} \to \cdots \to C_1 \to G \to Q \to 1.$$

8. The proof of the main Theorem

We have to prove that $\Phi : H^{n+1}(Q, A) \to \text{Opext}^n(Q, A)$, $n \geqq 2$, is injective (the case $n = 1$ is classical). This amounts to show that if $\mu\mathbf{C}^n \equiv \nu\mathbf{C}^n$, with $\mathbf{C}^n$ a free (projective) crossed n-fold extension and μ, ν operator maps $J_n \to A$, then $\mu - \nu$ extends over C_{n-1} as in (ii) resp. (iii) of Lemma 3. We argue as follows:

Since $\mu\mathbf{C}^n \equiv \nu\mathbf{C}^n$, there are crossed n-fold extensions $e_1, e_2, \ldots, e_m$ of A by Q, with $e_m = \nu\mathbf{C}^n$, and morphisms $(1, \alpha^1, 1) : \mu\mathbf{C}^n \to e_1$, $(1, \alpha^2, 1) : e_2 \to e_1$, $(1, \alpha^3, 1) : e_2 \to e_3$, $(1, \alpha^4, 1) : e_4 \to e_3$, and so forth. By construction, there are morphisms $(\mu, \beta^0, 1) : \mathbf{C}^n \to \mu\mathbf{C}^n$ and $(\nu, \beta^m, 1) : \mathbf{C}^n \to \nu\mathbf{C}^n$. Moreover, it follows from Proposition 3′ that for $1 \leq k < m$ the identity map of Q lifts to a morphism $(\nu^k, \beta^k, 1) : \mathbf{C}^n \to e_k$ of crossed n-fold extensions. We may assume that m is even (otherwise we add the identity morphism $e_{m+1} \to e_m$). It follows from Proposition 4′ that the morphisms $(\mu, \alpha^1\beta^0, 1)$ and $(\nu^2, \alpha^2\beta^2, 1) : \mathbf{C}^n \to e_1$ are homotopic; likewise, $(\nu^2, \alpha^3\beta^2, 1)$ and $(\nu^4, \alpha^4\beta^4, 1) : \mathbf{C}^n \to e_3$ are homotopic also, and so forth. We ultimately arrive at $(\nu^{m-2}, \alpha^{m-1}\beta^{m-2}, 1)$ and $(\nu, \alpha^m\beta^m, 1) : \mathbf{C}^n \to e_{m-1}$ which again are homotopic. Now $\mu - \nu = \mu - \nu^2 + \nu^2 - \nu^4 + \cdots + \nu^{m-2} - \nu$ extends over C_{n-1} as desired.

The proofs of naturality of Φ and of the assertion as to the inverse of a class $[e] \in \text{Opext}^n(Q, A)$ are left to the reader. Q.E.D.

9. The (inhomogenous) crossed standard resolution

The following section will be needed in [13], [14] and [15]; it will provide the bridge between our interpretation of group cohomology and the classical description in terms of cocycles.

Let Q be a group, and let $(Q^*; Q^* \times Q^*)$ be its standard presentation $(Q^* = Q \backslash \{1\})$; hence the relator $[q_1, q_2]$, $q_1, q_2 \in Q^*$, corresponds to the word ${}^{[q_1]}[q_2][q_1q_2]^{-1}$. Next, let F be the free group on Q^*, and let C be the free crossed F-module with basis $Q^* \times Q^*$; it is then clear that the elements

$$(*) \qquad {}^{[q_1]}[q_2, q_3][q_1, q_2q_3][q_1q_2, q_3]^{-1}[q_1, q_2]^{-1}, \qquad q_1, q_2, q_3 \in Q^*,$$

lie in the kernel of $\partial : C \to F$. By Proposition 1, we have the Q-free presentation

$$0 \to \ker(\partial) \to C^{Ab} \to N^{Ab} \to 0$$

of N^{Ab}. Since C^{Ab} is the corresponding term of the (ordinary) inhomogenous standard resolution of the integers where it is known that the elements (*) generate the kernel of $C^{Ab} \to N^{Ab}$ (in the operator sense), it follows that the elements (*) generate $\ker(\partial)$. If we now "splice" our free crossed module $C \to F$ with the remaining part of the inhomogenous standard resolution of the integers (this is a resolution of $\ker(\partial)$), we obtain a free crossed resolution of Q, henceforth called the (*inhomogenous*) *crossed standard resolution* of Q. Now, if C_2 is the free Q-module on $Q^* \times Q^* \times Q^*$, and if $\partial_2 : C_2 \to C_1$ $(C_1 = C)$ is given by sending $[q_1, q_2, q_3]$ to (*), the kernel of ∂_2 is generated by the elements (written multiplicatively)

$$(**) \qquad {}^{q_1}[q_2, q_3, q_4][q_1q_2, q_3, q_4]^{-1}[q_1, q_2q_3, q_4][q_1, q_2, q_3q_4]^{-1}[q_1, q_2, q_3],$$
$$q_1, q_2, q_3, q_4 \in Q^*.$$

10. An illustration

If e^{ψ} is the crossed 2-fold extension obtained from an abstract Q-kernel $\psi : Q \to \mathrm{Out}(K)$ (§3), we may lift the identity map of Q to

$$\begin{array}{llllllll}
\mathbf{C}: \cdots \to & C_2 & \to & C_1 & \to & F & \to Q \to 1 \\
& \downarrow{\scriptstyle \zeta} & & \downarrow & & \downarrow & \| \\
e^{\psi}: \quad 0 \to & Z & \to & K & \to & G^{\psi} & \to Q \to 1,
\end{array}$$

where $\mathbf{C}$ is the crossed standard resolution. This yields an operator map $\zeta : C_2 \to Z$, which, in view of (**), is a 3-cocycle; it is the Eilenberg–MacLane cocycle [7]. This, together with our main Theorem shows that $[e^{\psi}] \in \mathrm{Opext}^2(Q, Z)$ is the Eilenberg–MacLane class of (K, ψ). Eilenberg–MacLane's extendibility criterion is now recovered by the following

THEOREM. *Let* $e : 0 \to A \to K \xrightarrow{\partial} G \to Q \to 1$ *be a crossed* 2-*fold extension. There is a group extension* $1 \to K \xrightarrow{j} E \to Q \to 1$ *together with a morphism* $(1, \alpha) : (K, E, j) \to (K, G, \partial)$ *of crossed modules inducing the identity map of* Q *if and only if* $[e] = 0 \in \mathrm{Opext}^2(Q, A)$.

This generalises Eilenberg–MacLane's extendibility criterion, since A need not coincide with the center of K.

Proof. We show that the condition suffices. To this end, let $\mathbf{C}^2$: $0 \to J \to C \to F \to Q \to 1$ be a free crossed 2-fold extension and let $(\nu, \beta_1, \beta_0, 1) : \mathbf{C}^2 \to e$ be a lifting of the identity map of Q. Since $[e] = 0 \in \mathrm{Opext}^2(Q, A)$, ν extends over C as in (ii) of Lemma 3; it follows that there is a morphism $(\beta, \beta_0) : (N, F, i) \to (K, G, \partial)$ of crossed modules, where $N = \ker(F \to Q)$. Now the coequaliser E of $N \underset{\beta}{\overset{i}{\rightrightarrows}} K]F$ (where F acts on K via β_0) yields the required extension. Q.E.D.

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Mathematisches Institut der Universität
Im Neuenheimer Feld 288
D- 69 Heidelberg, W- Germany

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Note added in Proof: Strictly speaking, the crossed complex $\mathbf{C}^1$ in §5 is not a crossed 1-fold extension since $J_1 = N = (\ker(F \to Q))$ is not a Q-module; however, $\mathbf{C}^1$ may always be replaced by

$$O \to N^{Ab} \to F/[N, N] \to Q \to 1.$$