On the Euler class of representations of finite groups over real fields.

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Introduction

For representations of finite groups over the rationals **Q** there is a uniform bound, depending on the degree m of the representation only, for the order of the Euler class. This has been proved in [E–M], and the best possible such bound was shown there to be E_m = denominator of B_m/m if m is even, where B_m is the m-th Bernoulli number (and, of course, $E_m = 2$ if m is odd). The Euler class of a representation $\rho: G \to \mathrm{GL}_m(\mathbf{R})$ is an element of $H^m(G; \mathbf{Z}(\rho))$, $\mathbf{Z}(\rho)$ being the group of integers turned into a G-module by multiplication with sgn det ρ and hence a trivial G-module if and only if ρ is "orientable."

In the present paper we discuss analogous bounds for representations realizable over an arbitrary real field $K \subset \mathbf{R}$ instead of the rationals **Q**. The universal bound is expressed in terms of a certain operator $\mathscr{E}_K(m)$ on finite Abelian groups, depending on K and m only. $\mathscr{E}_K(m)$ is defined (cf. Section 3.1), for each prime p, by its action on p-torsion. This action depends on the degree $\varphi_K(p)$ of the p-th cyclotomic extension of K, and on a further invariant $\gamma_K(p) \in \mathbf{N} \cup \infty$ attached to K and p, cf. Section 2.2. The main theorem states that if the representation ρ of a finite group G, of degree m, is realizable over K then

$$\mathscr{E}_K(m)e(\rho) = 0. \tag{*}$$

Moreover $\mathscr{E}_K(m)$ is best possible in that sense.

We mention here some properties of the operator $\mathscr{E}_K(m)$. If m is not divisible by $\varphi_K(p)$, then $\mathscr{E}_K(m)$ is the identity operator on p-torsion; thus (*) just expresses the fact (Proposition 2.1) that in that case the p-component of $e(\rho)$ is 0. If m is divisible by $\varphi_K(p)$, one has two different possibilities. Either $\gamma_K(p) = \infty$; then $\mathscr{E}_K(m)$ annihilates p-torsion, and (*) tells nothing about the p-component of $e(\rho)$: in fact, there is, in that case, *no* universal bound for the order of the p-component of $e(\rho)$ (Corollary 2.4). Or $\gamma_K(p) < \infty$; then $\mathscr{E}_K(m)$ is, on p-torsion, multiplication by $p^{\gamma_K(p)+\nu_p}$, where ν_p is the exponent of p in the prime decomposition of m.

If we assume $\gamma_K(p)<\infty$ for all primes p, and if $\varphi_K(p)$ divides m for a finite number of primes p only, then $\mathscr{E}_K(m)$ can be replaced by multiplication with the integer $E_K(m)=lcm\{n \mid m\equiv 0 \bmod \varphi_K(n)\}$. For $K=\mathbf{Q}$, $E_{\mathbf{Q}}(m)=E_m$ is the integer mentioned above. The assumption is fulfilled for all real *number fields* K. Statement (*) then tells that the order of $e(\rho)$ divides $E_K(m)$, for all finite groups and all K-representations of degree m; and this bound is best possible.

If a representation $\rho: G\to GL_m(\mathbf{R})$ is not known to be realizable over a subfield of $\mathbf{R}$ fixed in advance, we show that (*) still holds if one takes for K a field containing the values of the character of ρ (without assuming ρ to be defined over $K\subset\mathbf{R}$). In particular we show (Theorem 3.8) that

$$E_{\mathbf{Q}(\chi)}(m)e(\rho)=0$$

where $\mathbf{Q}(\chi)$ denotes the field obtained from $\mathbf{Q}$ by adjoining the values of the character χ of ρ.

We also obtain a bound for the order of $e(\rho)$ of an arbitrary real representation ρ in terms of the exponent $\exp(G)$ of G (Theorem 3.9):

$$\frac{m}{2}\exp(G)e(\rho)=0$$

for $\rho: G\to GL_m(\mathbf{R})$, m even.

1. K-representations of finite p-groups

1.1. Let G be a finite group, and K a subfield of the field $\mathbf{C}$ of complex numbers. For a complex character χ of G we denote by $K(\chi)$ the Galois field extension obtained by adjoining to K all values of χ. In case χ is $\mathbf{C}$-irreducible, $K(\chi)$ is isomorphic to the center of $A_K(\chi)$, the unique simple component of the group algebra $K[G]$ on which χ is non-zero. If χ_1 and χ_2 are two $\mathbf{C}$-irreducible characters of G, then $A_K(\chi_1)=A_K(\chi_2)$ if and only if χ_1 and χ_2 are *Galois-conjugate over* K, which means that there is a $\sigma\in\mathrm{Gal}(K(\chi_1)/K)$ such that $\chi_2(g)=\sigma\chi_1(g)$ for all $g\in G$. The K-irreducible characters of K-representations of G are the characters of the form

$$\psi=s_K(\chi)\sum_{\sigma}\sigma\chi$$

where χ is $\mathbf{C}$-irreducible and the sum is extended over all $\sigma\in\mathrm{Gal}(K(\chi)/K)$, and where $s_K(\chi)$ denotes the Schur index of χ over K (we recall that $A_K(\chi)$ is a matrix algebra over a division algebra D, and that $s_K(\chi)^2$ is the dimension of D over its center $K(\chi)$).

1.2. The following result (cf. [$E-M$], Theorem 1.3) reduces the discussion of K-representations of finite p-groups to p-groups of very special types.

THEOREM 1.1. *Let G be a finite p-group, and $\rho: G \to \mathrm{GL}_m(K)$ an irreducible representation over $K \subset \mathbf{C}$. Then either ρ is induced, or ρ factors through a faithful representation $\bar{\rho}: \bar{G} \to \mathrm{GL}_m(K)$ of a factor group $\bar{G}$ of G which is of one of the following types*:

C_{p^α}, $\alpha \geq 0$ *(cyclic of order p^α)*;

Q_{2^α}, $\alpha \geq 3$ *(generalized quaternion group of order 2^α)*;

D_{2^α}, $\alpha \geq 4$ *(dihedral group of order 2^α)*; *or*

SD_{2^α}, $\alpha \geq 4$ *(semidihedral group of order 2^α)*.

In order to determine the degrees of the faithful irreducible K-representations of these groups of special type, we use two invariants of K:

DEFINITION 1.2. Let $K(n)$ denote the "n-th cyclotomic extension of K"; i.e., the field obtained by adjoining to K the n-th roots of unity. Then we write $\varphi_K(n)$ for the dimension of $K(n)$ over K and we put

$$\gamma_K(p) = \sup\{\alpha \mid K(p) = K(p^\alpha)\} \text{ for an odd prime } p,$$

and

$$\gamma_K(2) = \sup\{\alpha \mid K(4) = K(2^{\alpha+1})\}.$$

We write sometimes γ for $\gamma_K(p)$, if no confusion can arise; there are, of course, cases with $\gamma = \infty$.

If p is an odd prime and $\alpha \geq 1$ is such that $K(p^\alpha) \neq K(p^{\alpha+1})$ (i.e., $(K(p^{\alpha+1}): K(p^\alpha)) = p$) then $K(p^{\alpha+1}) \neq K(p^{\alpha+2})$. This follows from the commutative diagram of Galois groups (the maps being induced by restriction)

$$\begin{array}{ccc} \mathrm{Gal}\,(K(p^{\alpha+2})/K(p^\alpha)) & \to & \mathrm{Gal}\,(\mathbf{Q}(p^{\alpha+2})/\mathbf{Q}(p^\alpha)) \cong \mathbf{Z}/p^2\mathbf{Z} \\ \downarrow & & \downarrow \\ \mathrm{Gal}\,(K(p^{\alpha+1})/K(p^\alpha)) & \to & \mathrm{Gal}\,(\mathbf{Q}(p^{\alpha+1})/\mathbf{Q}(p^\alpha)) \cong \mathbf{Z}/p\mathbf{Z} \end{array}$$

Similarly, if $\alpha \geq 2$, then $K(2^\alpha) \neq K(2^{\alpha+1})$ implies $K(2^{\alpha+1}) \neq K(2^{\alpha+2})$. Note also that for $K \subset \mathbf{R}$, $\varphi_K(p)$ is even for p odd, and $(K(4):K) = 2$. The following lemma is now immediate.

LEMMA 1.3. (a) *For an odd prime p one has, for any* $K \subset \mathbf{C}$,

$$\varphi_K(p^\alpha) = \begin{cases} \varphi_K(p) & \text{if } 1 \leqslant \alpha \leqslant \gamma = \gamma_K(p), \\ \varphi_K(p) \cdot p^{\alpha-\gamma} & \text{if } \alpha \geqslant \gamma. \end{cases}$$

(b) *If* $K \subset \mathbf{R}$ *and* $p = 2$, *then*

$$\varphi_K(2^\alpha) = \begin{cases} 1 & \text{if } \alpha = 1 \\ 2 & \text{if } 1 < \alpha \leqslant \gamma + 1 (\gamma = \gamma_K(2)), \\ 2^{\alpha-\gamma} & \text{if } \alpha \geqslant \gamma + 1. \end{cases}$$

1.3. We now describe the degrees of the faithful irreducible representations of the p-groups listed in Theorem 1.1, and their orientability.

PROPOSITION 1.4. *Let* K *be a subfield of* $\mathbf{R}$, *and let* ρ *be a faithful irreducible* K*-representation of one of the* p*-groups* G *of special type. Then the degree* m *of* ρ *is*:

$m = \varphi_K(p^\alpha)$ *in case* $G = C_{p^\alpha} (\alpha \geqslant 0)$;

$m = 2\varphi_K(2^{\alpha-1})$ *in case* $G = Q_{2^\alpha} (\alpha \geqslant 3)$;

$m = \varphi_K(2^{\alpha-1})$ *in case* $G = D_{2^\alpha} (\alpha \geqslant 4)$;

$m = \varphi_K(2^{\alpha-1})$ *or* $2\varphi_K(2^{\alpha-1})$ *in case* $G = SD_{2^\alpha} (\alpha \geqslant 4)$.

Moreover, ρ *is orientable (i.e., lies in* $\mathrm{SL}_m(K)$) *except for* $G = C_2$.

Proof. The character ψ of ρ is of the form $\psi = s_K(\chi)\Sigma\sigma\chi$, $\sigma \in \mathrm{Gal}\,(K(\chi)/K)$, where χ is faithful and $\mathbf{C}$-irreducible. The faithful and $\mathbf{C}$-irreducible representations of the groups of special types were discussed in [E–M]; we will make use of their properties without further reference. The following four cases have to be considered.

C_{p^α} : $s_K(\chi) = 1$, χ is of degree one and $K(\chi) = K(p^\alpha)$. The degree of ψ is therefore $m = |\mathrm{Gal}\,(K(p^\alpha)/K)| = \varphi_K(p^\alpha)$.

Q_{2^α}: for any $K \subset \mathbf{R}$, one has $s_K(\chi) = 2$, and χ has degree 2. Since $K(\chi) = K(2^{\alpha-1}) \cap \mathbf{R}$ and $\alpha \geqslant 3$, we have $(K(2^{\alpha-1}) : K(\chi)) = 2$. The degree of ψ is thus given by $m = 2 \cdot 2 \cdot |\mathrm{Gal}\,(K(\chi)/K)| = 2|\mathrm{Gal}\,(K(2^{\alpha-1})/K)| = 2\varphi_K(2^{\alpha-1})$.

D_{2^α} (or SD_{2^α} respectively): $s_K(\chi) = 1$ and χ has degree 2. Again we have $(K(2^{\alpha-1}) : K(\chi)) = 2$ (or possibly $K(2^{\alpha-1}) = K(\chi)$ in the case SD_{2^α}) and thus $m = 2\,|\mathrm{Gal}\,(K(\chi)/K)| = \varphi_K(2^{\alpha-1})$ (or possibly $2\varphi_K(2^{\alpha-1})$ in the case SD_{2^α}).

If p is odd, ρ is certainly orientable. For $p=2$ we note that, except for the faithful representation of C_2 of degree 1, ψ is a sum of an even number of Galois conjugate representations $\sigma\chi$ which are all orientable in cases C_{2^α}, $\alpha \geqslant 2$ and Q_{2^α}, $\alpha \geqslant 3$; and which are all non-orientable in the other cases (cf. [E–M]). Hence ψ is orientable except for $G = C_2$.

COROLLARY 1.5. *Let K be a subfield of $\mathbf{R}$. The degree of a K-irreducible representation ρ of a finite p-group G is either* 1 *or of the form $\varphi_K(p)p^\beta$, $\beta \geqslant 0$.*

Proof. We consider the alternative in Theorem 1.1.

If ρ is induced from a representation τ of degree 1, then $p=2$ and therefore the degree of ρ is of the form $2^\beta = \varphi_K(2)2^\beta$ (p odd would imply that τ is a permutation representation, thus reducible). If ρ is induced from a representation τ of degree >1, the degree of τ is of the form $\varphi_K(p)p^\beta$, by induction, and thus the degree of ρ has the desired form.

If ρ factors through a faithful representation $\bar{\rho}$ of C_{p^α}, Q_{2^α}, D_{2^α} or SD_{2^α}, the degree of $\bar{\rho}$ is $\varphi_K(p^\alpha)$, $2\varphi_K(2^{\alpha-1})$ or $\varphi_K(2^{\alpha-1})$, which is 1 or of the form $\varphi_K(p)p^\beta$, $\beta \geqslant 0$. The assertion of the Corollary thus follows.

2. The Euler class of K-representations of p-groups

2.1. For a K-representation $\rho: G \to \mathrm{GL}_m(K)$, where K is a subfield of $\mathbf{R}$, the Euler class $e(\rho) \in H^m(G; \mathbf{Z}(\rho))$ is defined as the Euler class of the flat real vector bundle over $K(G, 1)$, associated with $\rho \otimes \mathbf{R}$; $\mathbf{Z}(\rho)$ stands for the G-module $\mathbf{Z}$ with G-action defined by $g \cdot 1 = \operatorname{sgn}\det \rho(g)$. The general properties of this (twisted) Euler class were discussed in [E–M].

Our main objective is to find universal bounds, depending on the field $K \subset \mathbf{R}$ and the degree m only, for the order of the Euler class of K-representations of finite groups. We proceed by dealing first with p-groups and then (Section 3) with arbitrary finite groups.

2.2. We start with the following simple observation.

PROPOSITION 2.1 *Let G be a finite p-group and let $\rho: G \to \mathrm{GL}_m(K)$ be a representation of degree $m \not\equiv 0 \bmod \varphi_K(p)$. Then the Euler class of ρ is* $=0$.

Proof. The assumption implies that $\varphi_K(p) > 1$ and thus p odd ($\varphi_K(2) = 1$). Let $\rho = \bigoplus_{i=1}^n \rho_i$, with ρ_i irreducible; then $e(\rho) = e(\rho_1)e(\rho_2)\cdots e(\rho_n)$. At least one of the ρ_i must have degree 1, for otherwise m would be divisible by $\varphi_K(p)$ (Corollary 1.5). Thus the corresponding $e(\rho_i)$ is 0 and whence $e(\rho) = 0$.

We may thus, for a p-group G, assume that the degree m of ρ is $\equiv 0 \bmod \varphi_K(p)$. It turns out that the situation is quite different according to whether $\gamma_K(p)$ is finite or infinite.

Let m be even and $\equiv 0 \bmod \varphi_K(p)$, and assume $\gamma_K(p) = \infty$. Then no uniform bound can exist for the order of the Euler class of K-representations of p-groups. This will be illustrated by Corollary 2.4 below. We first prove a lemma concerning the cyclic group C_n.

LEMMA 2.3. *Let $K \subset \mathbf{R}$ be an arbitrary real field. There exists, for any integer $l > 0$, a K-representation ρ of C_n of degree $l\varphi_K(n)$ and with Euler class $e(\rho)$ of (maximal possible) order n.*

Proof. C_n has a faithful irreducible representation τ over K of degree $m = \varphi_K(n)$ (its character is $= \sum_\sigma \sigma\chi$, where χ is faithful $\mathbf{C}$-irreducible and σ varies through $\mathrm{Gal}\,(K(n)/K)$). For the Euler class $e(\tau)$ one has $e(\tau)^2 = \pm c_m(\tau \otimes \mathbf{C})$, the top Chern class of $\tau \otimes \mathbf{C}$; since $\tau \otimes \mathbf{C}$ is a sum of m faithful one-dimensional $\mathbf{C}$-representations, $c_m(\tau \otimes \mathbf{C})$ has order n, and so has $e(\tau)$. If we take for ρ the l-fold direct sum of such K-representations τ, the order of $e(\rho)$ will be n and the degree $l \cdot \varphi_K(n)$.

COROLLARY 2.4. *Let $K \subset \mathbf{R}$, and let p be a prime such that $\gamma_K(p) = \infty$. If m is even and $m \equiv 0 \bmod \varphi_K(p)$, then there exists an m-dimensional K-representation of C_{p^α} with Euler class of order p^α.*

Proof. If p is odd, $\gamma_K(p) = \infty$ implies that $\varphi_K(p) = \varphi_K(p^\alpha)$ for $\alpha \geq 1$ and the result follows from Lemma 2.3. If $p = 2$, $\varphi_K(2^\alpha) = 2$ or 1 for $\alpha \geq 1$. Hence for any even m one can find a K-representation of C_{2^α} of degree m and Euler class of order 2^α (cf. Lemma 2.3).

2.3. We now turn to the case $\gamma_K(p) < \infty$, where the situation is different.

THEOREM 2.5. *Let K be a subfield of $\mathbf{R}$ and p a prime with $\gamma = \gamma_K(p) < \infty$. For any finite p-group G and any K-representation $\rho: G \to \mathrm{GL}_m(K)$ the Euler class $e(\rho) \in H^m(G: \mathbf{Z}(\rho))$ satisfies*

$$p^\gamma m e(\rho) = 0.$$

Proof. We first assume that ρ is irreducible. According to Theorem 1.2 we distinguish two possibilities.

(a) ρ factors as $G \to \bar{G} \xrightarrow{\bar{\rho}} GL_m(K)$ where $\bar{G}$ is one of the p-groups of special type and $\bar{\rho}$ faithful. If $\bar{G}$ is of order p^α, $\alpha \leq \gamma$ then plainly $p^\gamma m e(\rho) = 0$;

thus we may assume $\alpha > \gamma$. If p is odd, ρ is of degree $m = \varphi_K(p^\alpha) = \varphi_K(p) \cdot p^{\alpha-\gamma}$, and hence $p^\gamma m e(\rho) = 0$. In case $p = 2$ and $\alpha = \gamma + 1$, 2^α divides $2^\gamma m$ for m even; thus $2me(\rho) = 0$ (the case m odd is trivial, since then always $2e(\rho) = 0$). It remains to consider the case $p = 2$, $\alpha \geq \gamma + 2$. According to Proposition 1.4 the degree of ρ is then $2^{\alpha-\gamma}$ for the groups C_{2^α}, Q_{2^α}; and $2^{\alpha-\gamma-1}$ for D_{2^α}, $2^{\alpha-\gamma}$ or $2^{\alpha-\gamma-1}$ for SD_{2^α}. For the first two groups, $2^\gamma \cdot 2^{\alpha-\gamma} = 2^\alpha = |\bar{G}|$ annihilates $e(\rho)$, and for the latter ones $2^\gamma \cdot 2^{\alpha-\gamma-1} = 2^{\alpha-1} = |\bar{G}|/2$ annihilates $e(\rho)$ (since the cohomology of D_{2^α} and SD_{2^α} with **Z**-coefficients contains no elements of order 2^α).

(b) ρ is induced from $\tau : H \to \mathrm{GL}_{m/p}(K)$, where $H \subset G$ is of index p. Let *tr* denote the cohomology transfer. The Euler class of the restriction ρ_H satisfies $\mathrm{tr}\, e(\rho_H) = pe(\rho)$. Since we may assume by induction that $p^\gamma (m/p) e(\tau) = 0$, and since ρ_H is of the form $\tau \oplus \nu$, we infer $p^\gamma (m/p) e(\rho_H) = p^\gamma (m/p) e(\tau) e(\nu) = 0$. It follows that

$$p^\gamma m e(\rho) = \mathrm{tr}\left(p^\gamma \frac{m}{p} e(\rho_H)\right) = 0.$$

We now assume that ρ is reducible, $\rho = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_k$, the ρ_i being K-irreducible. Then $e(\rho) = e(\rho_1)e(\rho_2) \cdots e(\rho_k)$, and

$$p^\gamma m e(\rho) = p^\gamma m_1 e(\rho_1)e(\rho_2) \cdots e(\rho_k) + \cdots + p^\gamma m_k e(\rho_1)e(\rho_2) \cdots e(\rho_k)$$

where m_i is the degree of ρ_i. Since ρ_i is irreducible, we have $p^\gamma m_i e(\rho_i) = 0$, and thus $p^\gamma m e(\rho) = 0$.

Remark 2.6. If m is even, $m = \varphi_K(p)p^\beta \cdot f$ with $(f, p) = 1$ and $\gamma_K(p) = \gamma < \infty$, then there exists a K-representation of $C_{p^{\gamma+\beta}}$ of degree m with Euler class satisfying $p^{\gamma-1} m e(\rho) \neq 0$. This follows immediately from Lemma 2.3.

3. Arbitrary finite groups

3.1. We define for a subfield K of **R** and an integer $m > 0$, an additive operator $\mathscr{E}_K(m)$ on finite Abelian groups. If m is odd, $\mathscr{E}_K(m) : A \to A$ is multiplication by 2. For m even, $\mathscr{E}_K(m)$ is given by its action on p-torsion groups as follows.

(1) $\mathscr{E}_K(m)$ is the identity on p-torsion, if $m \not\equiv 0 \bmod \varphi_K(p)$.

(2) $\mathscr{E}_K(m)$ is zero on p-torsion if $m \equiv 0 \bmod \varphi_K(p)$ and $\gamma_K(p) = \infty$.

(3) $\mathscr{E}_K(m)$ is multiplication by $p^{\gamma+\alpha}$ on p-torsion, if $m \equiv 0 \bmod \varphi_K(p)$, $\gamma = \gamma_K(p) < \infty$ and $m = p^\alpha \cdot f$, f prime to p.

For instance, if $K = \mathbf{R}$, then $\mathscr{E}_{\mathbf{R}}(m)$ is the zero operator for all even m.

If K is a field such that $\gamma_K(p) < \infty$ for all p, and if only finitely many $\varphi_K(p)$ divide m, we define a numerical function by

$$E_K(m) = lcm\{n \mid m \equiv 0 \bmod \varphi_K(n)\}.$$

Note that, for any prime p, p^β divides $E_K(m)$ if and only if $m \equiv 0 \bmod \varphi_K(p^\beta)$. In one direction this is part of the definition; conversely, if p^β divides $E_K(m)$ there is an n divisible by p^β with $m \equiv 0 \bmod \varphi_K(n)$ and thus, since $\varphi_K(p^\beta)$ divides $\varphi_K(n)$, one has $m \equiv 0 \bmod \varphi_K(p^\beta)$. The prime decomposition of $E_K(m)$ is now obtained as follows, for $K \subset \mathbf{R}$ and m even ($E_K(m) = 2$ if m is odd):

Let $m = \Pi p^{\nu_p}$ be the decomposition of m into powers of different primes. By Lemma 1.3, $\varphi_K(p^{\beta+\gamma}) = \varphi_K(p)p^\beta$ ($\gamma = \gamma_K(p)$, $\beta \geq 0$ in case p odd, and $\beta \geq 1$ if $p = 2$); thus, for a prime p with $m \equiv 0 \bmod \varphi_K(p)$, m even, the greatest power dividing $E_K(m)$ is $p^{\nu_p+\gamma}$. We thus have

PROPOSITION 3.1. *If for* $K \subset \mathbf{R}$ *and* $m = \Pi p^{\nu_p}$ *the integer* $E_K(m)$ *is defined, then*

$E_K(m) = 2$ *if* m *is odd,*

$E_K(m) = \Pi' p^{\nu_p + \gamma_K(p)}$ *if* m *is even, the product* Π' *being taken over all those primes* p *for which* $m \equiv 0 \bmod \varphi_K(p)$.

Remarks. (1) If $K = \mathbf{Q}$, $E_{\mathbf{Q}}(m) = E_m$, the numerical function considered in [E–M] (which is equal to the denominator of B_m/m, m even).

(2) $E_K(m)$ is defined for all m if K is an algebraic number field.

COROLLARY 3.2. *If for* $K \subset \mathbf{R}$ *the integer* $E_K(m)$ *is defined, then the operator* $\mathscr{E}_K(m): A \to A$ *differs from "multiplication with* $E_K(m)$*" only by a canonical automorphism of* A. *In particular,* $\mathscr{E}_K(m)$ *and multiplication by* $E_K(m)$ *have the same kernel.*

We will make use later on of the following special case.

COROLLARY 3.3. *Let* $K = \mathbf{Q}(4n) \cap \mathbf{R}$ *and* p *a prime dividing* $4n$. *Then for even* m *the operator* $\mathscr{E}_K(m)$ *has the same kernel on any* p*-torsion group as multiplication by* $2nm$.

Proof. Let $m = \Pi p^{\nu_p(m)}$ and $n = \Pi p^{\nu_p(n)}$ be the prime decompositions. Since $p \mid 4n$ we have, for p odd, $\varphi_K(p) = 2$. Further we have $\gamma_K(p) = \nu_p(n)$ for p odd and $\gamma_K(2) = \nu_2(n) + 1$. Hence, for m even, $\mathscr{E}_K(m)$ acts on p-torsion by multiplication with $p^{\nu_p(n)+\nu_p(m)}$ if p is odd, and with $2^{\nu_2(n)+1+\nu_2(m)}$ if $p = 2$. Thus the kernel of $\mathscr{E}_K(m)$ on p-torsion is the same as the kernel of multiplication by $2nm$.

3.2. We now state and prove our main theorem.

THEOREM 3.4. *Let $K \subset \mathbf{R}$ be a real field and $\rho: G \to \mathrm{GL}_m(K)$ a K-representation of degree m of a finite group G. Then the Euler class $e(\rho) \in H^m(G; \mathbf{Z}(\rho))$ satisfies*

$$\mathscr{E}_K(m)e(\rho) = 0.$$

In particular, if $E_m(K)$ is defined (e.g., if K is a number field) the order of $e(\rho)$ divides $E_K(m)$.

Proof. Let $G(p)$ denote a p-Sylow subgroup of G. Since the cohomology restriction from G to $G(p)$ is injective on the p-primary component, it suffices to prove the theorem in the case where G is a p-group. If m is odd, $2e(\rho) = \mathscr{E}_K(m)e(\rho) = 0$. If m is even and $m \not\equiv 0 \bmod \varphi_K(p)$, $e(\rho) = 0$ by Proposition 2.1. It remains to consider the case m even, $m \equiv 0 \bmod \varphi_K(p)$: If $\gamma_K(p) = \infty$, then $\mathscr{E}_K(m)e(\rho) = 0$ by definition of $\mathscr{E}_K(m)$. If $\gamma = \gamma_K(p) < \infty$, we have $p^\gamma m e(\rho) = 0$ by Theorem 2.5; since, for a p-group G, $p^\gamma m e(\rho)$ and $p^{\gamma + \nu_p(m)} e(\rho)$ have the same order, we infer $\mathscr{E}_K(m)e(\rho) = 0$. In case $E_K(m)$ is defined, $E_K(m)e(\rho) = 0$ by Corollary 3.2.

Remark 3.5. The operator $\mathscr{E}_K(m)$ in Theorem 3.4 is best possible in the following obvious sense. Suppose $\mathscr{E}'_K(m)$ is another such operator (i.e., a natural transformation of the identity functor on the category of finite Abelian groups, such that $\mathscr{E}'_K(m)e(\rho) = 0$ for all K-representations ρ of degree m of finite groups) then

$$\ker(\mathscr{E}_K(m): A \to A) \subset \ker(\mathscr{E}'_K(m): A \to A) \qquad (*)$$

for all finite Abelian groups A. In order to prove this we observe that it suffices to check (*) in case A is a cyclic p-group; for that case (*) is an easy consequence of Lemma 2.3 and Remark 2.6 together with the definition of $\mathscr{E}_K(m)$.

In particular, if K is a number field, we obtain the following.

COROLLARY 3.6. *Let $K \subset \mathbf{R}$ be a number field. Then the least common multiple of the orders of the Euler classes $e(\rho)$, where ρ ranges over all K-representations of degree m of finite groups, is equal to $E_K(m) = lcm\{n \mid m \equiv 0 \bmod \varphi_K(n)\}$.*

3.3. If a representation $\rho: G \to GL_m(\mathbf{R})$ is not known to be realizable over some subfield $K \subset \mathbf{R}$ fixed in advance, one can still obtain a bound on the order of $e(\rho)$, depending on the character field $\mathbf{Q}(\chi)$ (i.e. the field obtained from $\mathbf{Q}$ be adjoining the values of the character χ of ρ). We need first the following lemma.

LEMMA 3.7. *Let* $\rho: G \to \mathrm{GL}_m(\mathbf{R})$ *be a real representation of a finite p-group G. Then* ρ *is equivalent to a representation defined over* $\mathbf{Q}(\chi)$, *where* χ *is the character of* ρ.

Proof. If p is odd, all $\mathbf{C}$-irreducible characters ψ of G have Schur index 1 over $\mathbf{Q}$, and therefore ρ is defined over $\mathbf{Q}(\chi)$ (cf. [R]). In case $p=2$, the Schur index $s_{\mathbf{Q}}(\psi)$ is one or two; by [F; Prop. 4.2], $s_{\mathbf{Q}}(\psi)=s_{\mathbf{R}}(\psi)$ and therefore $s_K(\psi)=s_{\mathbf{R}}(\psi)$ for any subfield $K \subset \mathbf{R}$. It follows that an $\mathbf{R}$-representation of a 2-group whose character takes values in $K \subset \mathbf{R}$, is realizable over K.

THEOREM 3.8. *Let* $\rho: G \to \mathrm{GL}_m(\mathbf{R})$ *be a real representation of an arbitrary finite group G. Then the Euler class* $e(\rho)$ *satisfies*

$$E_{\mathbf{Q}(\chi)}(m)e(\rho)=0$$

where $\mathbf{Q}(\chi)$ *denotes the field obtained from* $\mathbf{Q}$ *by adjoining the values of the character* χ *of* ρ.

Proof. Let ρ' denote the restriction of ρ to a p-Sylow subgroup of G, and denote by χ' the character of ρ'. From Theorem 3.4 and Lemma 3.7 we infer that $E_{\mathbf{Q}(\chi')}e(\rho')=0$ and, as $\mathbf{Q}(\chi') \subset \mathbf{Q}(\chi)$, $E_{\mathbf{Q}(\chi)}e(\rho')=0$. The assertion of the theorem now follows, since the cohomology restriction from G to a p-Sylow subgroup is injective on the p-primary component.

3.4. Using Corollary 3.3 we can get bounds for the order of Euler classes of *arbitrary* real representations, in terms of the *exponent* of G.

THEOREM 3.9. *Let G be a finite group of exponent* $\exp(G)$ *and* $\rho: G \to \mathrm{GL}_m(\mathbf{R})$ *a real representation of even degree m. Then the Euler class satisfies*

$$\frac{m}{2}\exp(G)e(\rho)=0.$$

Proof. Since the cohomology restriction from G to a p-Sylow subgroup is injective on the p-primary component, we may assume that G is a p-group. Then ρ is realizable over $\mathbf{Q}(\exp(G)) \cap \mathbf{R}$ since the character of ρ takes its values in that field (Lemma 3.7). If p is odd, we apply Corollary 3.3 with $K=\mathbf{Q}(4\exp(G)) \cap \mathbf{R}$ and obtain $2m\exp(G)e(\rho)=0$; thus $(m/2)\exp(G)e(\rho)=0$, $e(\rho)$ being a p-torsion element. If $p=2$ and $\exp(G) \leqslant 2$, then G is an elementary Abelian 2-group and thus even $\exp(G)e(\rho)=0$. If $p=2$ and $\exp(G)=4n \geqslant 4$, we infer from Corollary 3.3 (with $K=\mathbf{Q}(4n) \cap \mathbf{R}$) that $2nme(\rho)=0$, whence the assertion.

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