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Eigenvalue estimates on homogeneous manifolds

by PETER LI

§0. Introduction

In the recent years, much work has been done on studying the first eigenvalue of the equation

$$\Delta f = -\lambda f$$

where f is a C^∞ function defined on a compact Riemannian manifold. In general, it is known that [1] the first eigenvalue λ_1 cannot be bounded by either the diameter or the volume alone. In [3] Cheng showed that λ_1 has an upper bound depending on the diameter, d, and the lower bound of the Ricci curvature, $(n-1)K$. Yau [12] later conjectured that one should be able to estimate λ_1 from below in terms of d and $(n-1)K$ also. This conjecture was shown to be true in [7] for a special case. The general case was later established by Yau and the author [9].

The purpose of the first part of this paper is to obtain a lower bound for λ_1 on a compact homogeneous manifold M. In fact, we will prove that $\lambda_1 \geq \pi^2/4d^2$. This is rather surprising that homogeneity is strong enough to guarantee a lower estimate of λ_1 in terms of d alone.

One can improve this estimate of λ_1 by assuming $K \geq 0$ (i.e. Ricci curvature ≥ 0). Actually, we will show that by a method in [9], if a general compact manifold is non-negatively Ricci-curved and also the first eigenvalue has multiplicity greater than one, then $\lambda_1 \geq \pi^2/d^2$. In particular, if M is homogeneous, the multiplicity condition on λ_1 is shown to be automatically satisfied. Hence in addition if $K \geq 0$, then $\lambda_1 \geq \pi^2/d^2$. Further more, this estimate is sharp. If, in addition, we assume that M is an irreducible homogeneous manifold then $\lambda_1 \geq n\pi^2/4d^2$.

In the third section, we will give an estimate on the differences of any two consecutive eigenvalues of a homogeneous manifold in terms of its lower eigenvalues. The method was also used in [10], [2] and [11]. In fact, if $\Lambda = \sum_{i=1}^{m-1} \lambda_i$ then

$$\lambda_m - \lambda_{m-1} \leq \frac{2}{m}\left(\sqrt{\Lambda^2 + m\Lambda\lambda_1} + \Lambda\right) + \lambda_1.$$

Finally, the last section is devoted to the studying of the spectrum of differential p-forms. When a homogeneous manifold is also assumed to have non-vanishing Euler number, we will show that the first eigenvalue for 1-forms λ_1^1 has a lower bound depending on d and K. A sufficient condition for a homogeneous manifold to have its p^{th} Betti number no greater than $\binom{n}{p}$ will also be derived.

Throughout this paper we will assume the M is a compact homogeneous manifold with isometry group G and isotropy subgroup H, unless specified.

§1. Basic estimates

PROPOSITION 1. *Let E be a finite dimensional G-invariant subspace of the space of L^2 p-forms on M. Suppose* dim $E = k$, *then for all $\omega \in E$ and $x \in M$*

$$|\omega|^2(x) \le \frac{k}{V} \|\omega\|_2^2$$

where $|\omega|$ denotes the pointwise norm of ω, and $V =$ volume of M.

Proof. Let $\{\omega_i\}_{i=1}^k$ be an orthonormal basis of E with respect to the L^2 inner product. We define the function

$$F(x) = \sum_{i=1}^{k} |\omega_i|^2(x) \qquad x \in M. \tag{1.1}$$

Clearly $F(x)$ is well defined under orthogonal change of basis. Let $x_0 \in M$ be fixed, then

$$\begin{aligned} F(x_0) &= \sum_{i=1}^{k} |\omega_i|^2(x_0) = \sum_{i=1}^{k} |g^*\omega_i(g^{-1}(x_0))| \\ &= \sum_{i=1}^{k} |\omega_i(g^{-1}(x_0))| \qquad g \in G. \end{aligned} \tag{1.2}$$

The last inequality follows from the fact that g is an isometry, hence $\{g^*\omega_i\}_{i=1}^k$ form an orthonormal basis of E. Since G acts transitively on M, there exists $g \in G$ such that $g(x) = x_0$. Hence (1.2) becomes

$$F(x_0) = \sum_{i=1}^{k} |\omega_i(x)| = F(x) \tag{1.3}$$

which shows F is a constant function. Integrating both sides of (1.1) yields

$$V \cdot F(x_0) = \int_M \sum_{i=1}^{k} |\omega_i|^2 = k. \tag{1.4}$$

Therefore

$$\sum_{i=1}^{k} |\omega_i|^2(x) = F(x) = F(x_0) = \frac{k}{V} \tag{1.5}$$

and the proposition follows directly.

COROLLARY 2. *Let E^p_λ be an eigenspace of p-forms with eigenvalue λ on M. If $\omega \in E^p_\lambda$, then*

$$\|\omega\|^2_\infty \le \frac{\dim E^p_\lambda}{V} \|\omega\|^2_2.$$

Proof. Since the Laplacian commutes with isometries, E^p_λ is a finite dimensional G-invariant subspace, hence proposition 1 can be applied.

Remark. One can also apply the proposition to any G-invariant subspace of E^p_λ.

PROPOSITION 2. *If E is a finite dimensional G-invariant subspace of L^2 functions on M, then*

$$\|f\|^2_\infty \le \frac{k}{V} \|f\|^2_2 \quad \textit{for all} \quad f \in E$$

where $k = \dim E$. Moreover if $E \ne \{0\}$, there exists $f_0 \in E$ such that

$$\|f_0\|^2_\infty = \frac{k}{V} \|f_0\|^2_2.$$

Proof. The first part of the proposition is just a special case of proposition 1. The equality follows from the existence of "zonal functions" discovered by E. Cartan in the case of symmetric spaces. However for completeness sake, we will sketch its proof.

Define $E_0 \subset E$ to be the subspace

$$E_0 = \{f \in E \mid f(x_0) = 0\} \tag{1.6}$$

where $x_0 \in M$ is fixed. By homogeneity of M and the fact that $E \neq \{0\}$, we have $E_0 \neq E$. We claim that the perpendicular subspace $E_0^\perp$ of E_0 in E is of dimension 1. If not, let f_0 and f_1 be two linearly independent functions in $E_0^\perp$. On the other hand, there exists $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha f_0(x_0) + \beta f_1(x_0) = 0.$$

But this implies $\alpha f_0 + \beta f_1 \in E_0$, which is a contradiction. Hence there exists $f_0 \in E$ such that $E_0 \oplus \langle f_0 \rangle = E$ and $\|f_0\|_2 = 1$. Let $\{f_1\}_{i=1}^k$ be an orthonormal basis of E with $f_0 = f_1$. By equation (1.5), we have

$$\sum_{i=1}^k f_i^2(x_0) = \frac{k}{V}. \tag{1.7}$$

However $f_\alpha(x_0) = 0$, for $\alpha \geq 2$, therefore $f_0^2(x_0) = k/V$ which proves the proposition.

Remark. Let us denote H_0 to be the isotropic subgroup of G which leaves x_0 fixed. Then f_0 is invariant under the action of H_0 and hence takes constant value on each orbit of H_0. This was the original definition of zonal functions. We will call f_0 the zonal function of E at x_0.

COROLLARY 4. *Let E_λ^0 be an eigenspace of functions with eigenvalue λ. Then for a fixed point $x_0 \in M$, there exists a unique $f_0 \in E_\lambda^0$ which satisfies*

(i) $\|f_0\|_\infty = f_0(x_0) = \left(\dfrac{\dim E_\lambda^0}{V}\right)^{1/2}$

(ii) $\|f_0\|_2 = 1$

(iii) *f_0 is invariant under H_0*

(iv) $\|f_0\|_\infty^2 \geq f^2(x)$ *for all* $f \in E_\lambda^0$

(v) $\langle f_0 \rangle \oplus E_0 = E.$

§2. The first eigenvalue for functions

In this section we will utilize corollaries 2 and 4 of the above section to obtain a lower bound for λ_1. A sharp estimate can be obtained if in addition we assume the homogeneous manifold M is non-negatively Ricci-curved.

THEOREM 5. *Let E^0_λ be an eigenspace of functions on M. Suppose $f_0 \in E^0_\lambda$ is a zonal function we obtained in corollary 4. Then*

$$|\nabla f_0|^2(x) + \lambda f_0^2(x) \le \lambda \|f_0\|^2_\infty \quad \textit{for all} \quad x \in M.$$

Proof. Let $x_0 \in M$ be a point such that f_0 is a zonal function at x_0. Consider $g \in G$ such that $g(x_0) = x$. Then the action of g on f_0 is given by

$$g \cdot f_0(y) = f_0(g(y)) \quad \text{for all} \quad y \in M. \tag{2.1}$$

One can complete $\{f_0\}$ to $\{f_i\}_{i=1}^k$ an orthonormal basis for E^0_λ with $f_0 = f_1$ and $f_\alpha \in E_0$, for $\alpha \ge 2$. We may also assume that

$$g \cdot f_0 = af_0 + bf_2 \qquad a, b \in \mathbb{R}. \tag{2.2}$$

Since $\|g \cdot f_0\|_2 = 1$, we have $a^2 + b^2 = 1$. By the fact that g is an isometry

$$|\nabla f_0|^2(x) + \lambda f_0^2(x) = |\nabla(g \cdot f_0)|^2(x_0) + \lambda(g \cdot f_0)^2(x_0) = b^2|\nabla f_2|^2(x_0) + \lambda a^2 f_0^2(x_0). \tag{2.3}$$

The last equality follows from (2.2) and the fact that f_0 attains its supremum at x_0. $a^2 + b^2 = 1$ implies

$$|\nabla f_0|^2(x) + \lambda f_0^2(x) = \lambda f_0^2(x_0) + b^2[|\nabla f_2|^2(x_0) - \lambda f_0^2(x_0)] \le \lambda \|f_0\|^2_\infty + b^2[\|\nabla f_2\|^2_\infty - \lambda \|f_0\|^2_\infty]. \tag{2.4}$$

Now we claim that the second term of the right hand side of (2.4) is non-positive.

In fact, if we consider the subspace $\tilde{E} = \{df \mid f \in E^0_\lambda\}$ of 1-forms, then it is easy to see that since $\lambda \ne 0$, $\tilde{E}$ is a subspace of dimension $k = \dim E^0_\lambda$. Also $\tilde{E}$ is invariant under G by the fact that d commutes with any $g \in G$. Hence by proposition 1,

$$\|\nabla f\|^2_\infty \le \frac{k}{V}\|\nabla f\|^2_2 = \frac{\lambda k}{V}\|f\|^2_2 \quad \text{for all} \quad f \in E^0_\lambda. \tag{2.5}$$

On the other hand, corollary 4 gives

$$\|f_0\|_\infty^2 = \frac{k}{V}. \tag{2.6}$$

Therefore

$$\|\nabla f_2\|_\infty^2 \le \frac{\lambda k}{V} = \lambda \|f_0\|_\infty^2 \tag{2.7}$$

which proves the theorem.

COROLLARY 6. *The first eigenvalue* λ_1 *for functions on a homogeneous manifold satisfies* $\pi^2/4d^2 \le \lambda_1$, *where* $d =$ *diameter of* M.

Proof. By theorem 5, we have

$$\frac{|\nabla f_0|}{\sqrt{\|f_0\|_\infty^2 - f_0^2}} \le \lambda^{1/2}. \tag{2.8}$$

Integrating along the shortest geodesic γ joining x_0 and N the zero set of f_0 yields

$$d \cdot \lambda^{1/2} \ge \int \frac{|\nabla f_0|}{\sqrt{\|f_0\|_\infty^2 - f_0^2}} \ge \sin^{-1}\left(\frac{f_0(x_0)}{\|f_0\|_\infty}\right) = \frac{\pi}{2}. \tag{2.9}$$

The corollary follows.

Remark. The gradient estimate in theorem 5 is the same as the one obtained in [9], where we had to assume M is non-negatively Ricci-curved. In general without the assumption $\mathrm{Ric}_M \ge 0$, the conclusion of theorem 5 is false. It is hence rather surprising that the homogeneity condition alone gives such strong gradient estimate.

If M is assumed to be non-negatively Ricci-curved and also if $\dim E_{\lambda_1}^0 \ge 2$, then by following the method in [9] one can derive a sharp lower bound for λ_1.

THEOREM 7. *Let* M *be a compact manifold (not necessarily homogeneous) with Ricci curvature bounded below by* $(n-1)K$. *Suppose* λ_1 *is the first non-zero eigenvalue for*

(i) $\Delta\varphi = -\lambda_1\varphi$ *when* $\partial M = \emptyset$

(ii) $\Delta\varphi = -\lambda_1\varphi$ *and* $\partial\varphi/\partial\nu = 0$ *when* $\partial M \ne \emptyset$,

where $\partial/\partial\nu$ *denotes the outward unit normal to* ∂M. *Assuming also* ∂M *is convexed.*

If $\dim E_{\lambda_1}^0 \ge 2$, *then* $\lambda_1 \ge \pi^2/d^2 + \min\{(n-1)K, 0\}$.

Proof. First we show that there exists $\varphi \in E^0_{\lambda_1}$ such that

$$\sup \varphi = |\inf \varphi|. \tag{2.10}$$

Since $\dim E^0_{\lambda_1} \geq 2$, let φ_0 and φ_1 be two linearly independent eigenfunctions in $E^0_{\lambda_1}$. We may assume that

$$\sup \varphi_i > |\inf \varphi_i| \qquad i = 0, 1. \tag{2.11}$$

Consider the functions defined by

$$\varphi_t = (1-t)\varphi_0 - t\varphi_1 \qquad t \in [0, 1] \tag{2.12}$$

and

$$\Phi(t) = \sup \varphi_t + \inf \varphi_t. \tag{2.13}$$

Clearly $\varphi_t \in E^0_{\lambda_1}$ and $\Phi(t)$ is a continuous function in t. By (2.11), we know that

$$\Phi(0) = \sup \varphi_0 + \inf \varphi_0 > 0$$

and

$$\Phi(1) = \sup(-\varphi_1) + \inf(-\varphi_1) = -\inf \varphi_1 - \sup \varphi_1 < 0.$$

Therefore there exists $t \in [0, 1]$ such that

$$0 = \Phi(t) = \sup \varphi_t + \inf \varphi_t$$

which proves the claimed.

The rest of the proof follows the same way as in Theorems 10 and 12 of [9], with a slight modification as follows: Let γ be a shortest geodesic joining the supremum and infemum points of φ. Consider γ_1 and γ_2 as parts of γ joining the supremum point and the zero set, and joining the infemum point and the zero set respectively. Since γ has length no greater than d, either γ_1 or γ_2 has length no greater than $d/2$. Assume $l(\gamma_2) \leq d/2$. Integrating the gradient estimate along γ_2 and using the fact that $|\inf \varphi| = \sup \varphi$ the theorem follows.

COROLLARY 8. *Let M be a compact homogeneous manifold without boundary. Then*

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \min\{(n-1)K, 0\}.$$

Proof. In view of theorem 7, it suffices to show that $\dim E^0_{\lambda_1} \geq 2$. However if $E^0_{\lambda_1} = \langle f \rangle$, by proposition 1

$$f^2 = \text{const}$$

which contradicts the fact that f is the first eigenfunction.

Remark. Theorem 7 yields a sharp estimate for λ_1. If one considers $M = S^1(r) \times N$ where N has non-negative Ricci curvature. It is well known that the eigenvalues of M split into sums of eigenvalues of $S^1(r)$ and N. Hence for r sufficiently large

$$\lambda_1(M) = \lambda_1(S^1(r)) = \frac{1}{r^2}.$$

On the other hand $d^2(M) = d^2(S^1(r)) + d^2(N) = \pi^2 r^2 + d^2(N)$. Therefore

$$\lambda_1(M) \times d^2(M) = \pi^2 + \frac{d^2(N)}{r^2}$$

which tends to π^2 as $r \to \infty$. This shows the sharpness of theorem 7.

DEFINITION. $M = G/H$ is said to be a compact irreducible homogeneous manifold if G is a compact isometry group of M and the isotropy subgroup H acts irreducibly on the tangent space of M.

THEOREM 9. *Let M be a compact irreducible homogeneous Riemannian manifold. Suppose E^0_λ is an eigenspace of functions on M. If $f_0 \in E^0_\lambda$ is a zonal function of E^0_λ, then for all $x \in M$*

$$|\nabla f_0|^2(x) + \frac{\lambda}{n} f_0^2(x) \leq \frac{\lambda}{n} \|f_0\|^2_\infty.$$

Proof. It is known that [5] an irreducible homogeneous Riemannian manifold, M, can be isometrically minimally immersed into the standard sphere by any of its eigenspaces. In fact, the immersion $\Phi: M \to S^{k-1}(r) \subseteq \mathbb{R}^k$ is given by $\Phi = (\alpha\varphi_1, \ldots, \alpha\varphi_k)$ where $\{\varphi_i\}_{i=1}^k$ is an orthonormal basis of E_λ^0 and $\lambda = n/r^2$.

First we will show that for any $f \in E_\lambda^0$,

$$|\nabla f|^2 \leq \frac{\lambda}{n} \sum_{i=1}^{k} \varphi_i^2 = \frac{\lambda}{n} \cdot \frac{k}{V}. \tag{2.14}$$

We may assume that $f = \varphi_1$. By the fact that Φ is an isometry, we have

$$d\Phi(X) = 1 \tag{2.15}$$

for all unit vector $X \in T \times M$. This implies

$$\alpha^2 \sum_{i=1}^{k} (X\varphi_i)^2 = 1. \tag{2.16}$$

By choosing X appropriately, we conclude that

$$\frac{1}{\alpha^2} \geq (X\varphi_1)^2 = |\nabla f|^2. \tag{2.17}$$

On the other hand, since $\Phi(M) \subseteq S^{k-1}$, we have

$$\alpha^2 \sum_{i=1}^{k} \varphi_i^2 = r^2 = \frac{n}{\lambda}. \tag{2.18}$$

Hence combining with (2.17) gives

$$|\nabla f|^2 \leq \frac{\lambda}{n} \left(\sum_{i=1}^{k} \varphi_i^2 \right).$$

Now Theorem 9 follows from the proof of Theorem 5 where we substitute λ/n instead of λ.

COROLLARY 10. *Let M be an irreducible homogeneous Riemannian manifold. Then the first eigenvalue* λ_1 *for functions satisfies*

$$\frac{\pi^2}{4d^2} \leq \lambda_1.$$

Proof. Follow the proof of Corollary 6 but using Theorem 9 instead of Theorem 5.

Remark. If we integrate the inequality

$$|\nabla f_0|^2(x)+\frac{\lambda}{n}f_0^2(x)\le\frac{\lambda}{n}\|f_0\|_\infty^2$$

over M, we obtain

$$\lambda\|f_0\|_2^2+\frac{\lambda}{n}\|f_0\|_2^2\le\frac{\lambda}{n}\cdot V\cdot\|f_0\|_\infty^2. \tag{2.18}$$

Hence

$$(n+1)\|f_0\|_2^2\le V\|f_0\|_\infty^2 \tag{2.19}$$

for all zonal functions in any eigenspaces of M. Moreover equality holds if $M\simeq S^n(r)$. In fact, if $(n+1)\|f_0\|_2^2=V\|f_0\|_\infty^2$ then combining with proposition 2: $n+1=k$. However since $\Phi: M^n\to S^{k-1}(r)=S^n(r)$ is an isometric immersion, this implies that M^n is a constant curvature manifold with curvature $=1/r^2$. It is not hard to see that the only constant positive curvature irreducible homogeneous space which can be isometrically immersed in $S^n(r)$ via its eigenspace has to be $S^n(r)$ (see [6]).

§3. Higher eigenvalues for functions

In the following theorem we show that λ_m can be estimated from above in terms of λ_i, $i\le m-1$. In [10] and [2], the authors utilized the fact that the coordinate functions are harmonic and gave upper bounds for λ_m on domains and minimal submanifolds in $\mathbf{R}^n$. Since the coordinate functions of a minimal submanifold in S^n are eigenfunctions, Yang and Yau [11] found upper bounds for λ_m using similar philosophy as mentioned above. It turns out that a similar method carries through when M is homogeneous, which depends heavily on proposition 1.

THEOREM 11. *Let* $\Lambda=\sum_{i=1}^{m-1}\lambda_i$. *Then*

$$\lambda_m-\lambda_{m-1}\le\frac{2}{m}(\sqrt{\Lambda^2+m\Lambda\lambda_1}+\Lambda)+\lambda_1.$$

Proof. Let $\{\varphi_\alpha\}_{\alpha=1}^k$ be an orthonormal basis of the first eigenspace $E^0_{\lambda_1}$, and $\{\varphi_i\}_{i=0}^{m-1}$ be the set of first m^{th} orthonormal eigenfunctions (including constant function). Then

$$\Delta\varphi_\alpha = -\lambda_1\varphi_\alpha \quad \text{for} \quad 1 \le \alpha \le k$$

and

$$\Delta\varphi_i = -\lambda_i\varphi_i \quad \text{for} \quad 0 \le i \le n-1. \tag{3.1}$$

We define

$$u_{\alpha i} = \varphi_\alpha\varphi_i - \sum_{j=0}^{m-1} a_{\alpha ij}\varphi_j \tag{3.2}$$

where

$$a_{\alpha ij} = \int \varphi_\alpha\varphi_i\varphi_j = a_{\alpha ji}.$$

Clearly

$$\int u_{\alpha i}\varphi_j = 0 \quad \text{for all} \quad 0 \le j \le n-1. \tag{3.3}$$

Hence by the variational principle for λ_m, we have

$$\lambda_m \le \frac{\int |\nabla u_{\alpha i}|^2}{\int u_{\alpha i}^2} \quad \text{for all} \quad \alpha, i.$$

However

$$\int |\nabla u_{\alpha i}|^2 = -\int u_{\alpha i}\Delta u_{\alpha i} = -\int u_{\alpha i}\Big[-(\lambda_1+\lambda_i)\varphi_\alpha\varphi_i + 2\langle\nabla\varphi_\alpha, \nabla\varphi_i\rangle + \sum_{j=0}^{m-1} a_{\alpha ij}\lambda_j\varphi_j\Big] = (\lambda_1+\lambda_i)\int u_{\alpha i}^2 - 2\int u_{\alpha i}\langle\nabla\varphi_\alpha, \nabla\varphi_i\rangle. \tag{3.4}$$

Therefore

$$\lambda_m \le \lambda_1 + \lambda_i - \frac{2\int u_{\alpha i}\langle\nabla\varphi_\alpha, \nabla\varphi_i\rangle}{\int u_{\alpha i}^2} \tag{3.5}$$

which implies

$$\lambda_m - \lambda_1 - \lambda_{m-1} \le \frac{-2\sum_{i,\alpha}\int u_{\alpha i}\langle\nabla\varphi_\alpha, \nabla\varphi_i\rangle}{\sum_{i,\alpha}\int u_{2\ \alpha i}}. \tag{3.6}$$

But

$$\begin{aligned}
-2\sum_{i,\alpha}\int u_{\alpha i}\langle\nabla\varphi_\alpha, \nabla\varphi_i\rangle &= \sum_{i,\alpha}\int -2\varphi_\alpha\varphi_i\langle\nabla\varphi_\alpha, \nabla\varphi_i\rangle + \sum_{\alpha,i,j}\int 2a_{\alpha ij}\varphi_j\langle\nabla\varphi_\alpha, \nabla\varphi_i\rangle \\
&= \sum_{i,\alpha}\frac{1}{2}\int -\langle\nabla(\varphi_\alpha^2), \nabla(\varphi_i^2)\rangle + \sum_{\alpha,i,j}\int a_{\alpha ij}\langle\nabla\varphi_\alpha, \nabla(\varphi_i\varphi_j)\rangle (\text{since } a_{\alpha ij} = a_{\alpha ji}) \\
&= \sum_{\alpha,i,j}\int a_{\alpha ij}\lambda_1\varphi_\alpha\varphi_i\varphi_j \ (\text{since } \textstyle\sum \varphi_\alpha^2 = \text{constant}). \\
&= \lambda_1\sum_{\alpha,i,j} a_{\alpha ij}^2
\end{aligned} \tag{3.7}$$

Also

$$\begin{aligned}
\sum_{\alpha,i}\int u_{\alpha i}^2 &= \sum_{\alpha,i}\int \varphi_\alpha^2\varphi_i^2 - 2\sum_{\alpha,i,j}\int \varphi_\alpha\varphi_i a_{\alpha ij}\varphi_j + \sum_{\alpha,i,j,l}\int a_{\alpha ij}a_{\alpha il}\varphi_j\varphi_l \\
&= \sum_{\alpha}\varphi_\alpha^2\sum_{i=0}^{m-1}\int\varphi_i^2 - 2\sum_{\alpha,i,j}a_{\alpha ij}^2 + \sum_{\alpha,i,j}a_{\alpha ij}^2 = \frac{km}{V} - \sum_{\alpha,i,j}a_{\alpha ij}^2.
\end{aligned} \tag{3.8}$$

Hence, if we let $A = \sum_{\alpha,i,j} a_{\alpha ij}^2$ then

$$\lambda_m - \lambda_{m-1} - \lambda_1 \le \frac{\lambda_1 A}{\frac{km}{V} - A}. \quad (3.9)$$

On the other hand

$$\begin{aligned}
\sum_{\alpha,i}\left|\int u_{\alpha i}\langle\nabla\varphi_\alpha, \nabla\varphi_i\rangle\right| &\le \sum_{\alpha,i}\left(\int u_{\alpha i}^2\right)^{1/2}\left(\int\langle\nabla\varphi_\alpha, \nabla\varphi_i\rangle^2\right)^{1/2} \\
&\le \left(\sum_{\alpha,i}\int u_{\alpha i}^2\right)^{1/2}\left(\sum_{\alpha,i}\int\langle\nabla\varphi_\alpha, \nabla\varphi_i\rangle^2\right)^{1/2} \\
&= \left(\sum_{\alpha,i}\int u_{\alpha i}^2\right)^{1/2}\left(\sum_{i}\int\left(\sum_{\alpha}|\nabla\varphi_\alpha|^2\right)|\nabla\varphi_i|^2\right)^{1/2} \\
&= \left(\sum_{\alpha,i}\int u_{\alpha i}^2\right)^{1/2}\left(\frac{\lambda_1 k}{V}\sum_{i}\lambda_i\right)^{1/2}.
\end{aligned} \tag{3.10}$$

The last equality follows from Proposition 1 and the fact that $\int|\nabla\varphi_\alpha|^2=\lambda_1$. Substituting this into (3.6), we get

$$\lambda_m-\lambda_{m-1}-\lambda_1\leq\frac{-2\sum_{\alpha,i}\int u_{\alpha i}\langle\nabla\varphi_\alpha,\nabla\varphi_i\rangle}{\left[\sum_{\alpha,i}\left|\int u_{\alpha i}\langle\nabla\varphi_\alpha,\nabla\varphi_i\rangle\right|\right]^2}\times\frac{k\lambda_1\Lambda}{V}$$

$$\leq\frac{2k\lambda_1\Lambda}{V}\left(\sum_{\alpha,i}\left|\int u_{\alpha i}\langle\nabla\varphi_\alpha,\nabla\varphi_i\rangle\right|\right)^{-1}\leq\frac{2k\lambda_1\Lambda}{V}\left(\frac{2}{\lambda_1 A}\right)\qquad\text{(by 3.7)}$$

$$=\frac{4\Lambda k}{VA}.\tag{3.11}$$

Combining (3.11) with (3.9) yields

$$\lambda_m-\lambda_{m-1}-\lambda_1\leq\min\left\{\frac{\lambda_1 VA}{km-VA},\frac{4\Lambda k}{VA}\right\}.\tag{3.12}$$

Observe that as a function of VA, $\lambda_1 VA/km-VA$ is an increasing function on $[0,\text{km}]$ and approaches ∞ as VA tends to km. Also $4\Lambda k/VA$ is a decreasing function on $[0,km]$ and approaches ∞ as VA tends to 0. Hence the minimum between the two functions is bounded by their common value taken at

$$VA=\frac{\sqrt{16\Lambda k^2(\Delta+m\lambda_1)}-4\Lambda k}{2\lambda_1}.$$

Therefore

$$\lambda_m-\lambda_{m-1}-\lambda_1\leq\frac{2}{m}\left(\sqrt{\Lambda^2+m\Lambda\lambda_1}+\Lambda\right).$$

§4. First eigenvalues for differential forms

The celebrated Hodge theorem tells us that the p^{th} Betti number is given by the dimension of the space of harmonic p-forms. Clearly the Laplace-Beltrami operator $\Delta=\delta d+d\delta$ depends heavily on the metric. Yet the notion of Betti numbers are purely topological. This phenomenon explains why the study of the first eigenvalues for differential forms is much more difficult than for functions.

The only known result in estimating lower bounds for λ_1^p is due to Gallot and Meyer [4]. They had to assume that the curvature operator is bounded below by a positive number. However this assumption automatically implied the vanishing of the Betti numbers, which is the famous vanishing theorem of S. Bochner.

The objective of this section is to establish a lower bound for λ_1^1 on homogeneous manifolds. In most cases, we have to impose additional assumption about the geometry in order to avoid the topological difficulty mentioned above.

THEOREM 12. *Let E_λ^p be the eigenspace of p-forms with eigenvalue λ. Then there exists $\omega_0 \in E_\lambda^p$, such that*

$$1 \le d\left[\min\left\{k, \binom{n}{p}\right\} \times (\lambda - (n-p)pK_p)\right]^{1/2} + \inf|\omega_0|\left(\frac{V}{k} \times \min\left\{k, \binom{n}{p}\right\}\right)^{1/2}$$

where $k = \dim E_\lambda^p$ and

$$K_p = \begin{cases} (n-1)^{-1} \times (\textit{lower bound of Ricci curvature}), \textit{ if } p = 1 \\ \textit{lower bound of the curvature operator, if } p > 1. \end{cases}$$

Proof. Consider an orthonormal basis $\{\omega_i\}_{i=1}^k$ for the eigenspace E_λ^p. A formula of Bochner gives

$$|\nabla\omega_i|^2 = \lambda|\omega_i|^2 + \tfrac{1}{2}\Delta|\omega_i|^2 - F(\omega_i). \tag{4.1}$$

Summing over all i and using (1.5) of Proposition 1 yields

$$\sum_i |\nabla\omega_i|^2 = \lambda\sum_i |\omega_i|^2 - \sum_i F(\omega_i). \tag{4.2}$$

However it is known that [4]

$$F(\omega_i) \ge p(n-p)K_p\,|\omega_i|^2. \tag{4.3}$$

Hence

$$\sum|\nabla\omega_i|^2 \le (\lambda - p(n-p)K_p)\sum|\omega_i|^2 = (\lambda - p(n-p)K_p)\frac{k}{V}. \tag{4.4}$$

However Lemma 9 of [8] implies

$$\big|\nabla|\omega_i|\big|^2 \le |\nabla\omega_i|^2. \tag{4.5}$$

Therefore

$$\|\nabla |\omega_1|\|_\infty \leq [\lambda - (n-p)pK_p]^{1/2}\left(\frac{k}{V}\right)^{1/2}. \tag{4.6}$$

By Theorem 12 of [8], we can choose $\omega_0 \in E_\lambda^p$ to satisfy

$$\min\left\{k, \binom{n}{p}\right\} \|\omega_0\|_\infty^2 \geq \frac{k}{V} \|\omega_0\|_2^2. \tag{4.7}$$

Letting $\omega_0 = \omega_1$ and integrating (4.6) along the shortest geodesic γ joining $\inf|\omega_0|$ and $\sup|\omega_0| = \|\omega_0\|_\infty$ yields

$$\begin{aligned} d[\lambda - p(n-p)K_p]^{1/2}\left(\frac{k}{V}\right)^{1/2} &\geq d\,\|\nabla |\omega_0|\|_\infty \geq \int_\gamma \|\nabla |\omega_0|\| \\ &\geq \|\omega_0\|_\infty - \inf|\omega_0| \geq \left(\frac{k}{V \times \min\left\{k, \binom{n}{p}\right\}}\right)^{1/2} - \inf|\omega_0| \end{aligned}$$

This proves the theorem.

COROLLARY 13. *Let M be a compact homogeneous manifold with $\chi(M) \neq 0$. Then the first eigenvalue for 1-forms λ_1^1 satisfies*

$$\lambda_1^1 \geq \frac{1}{nd^2} + (n-1)K.$$

Proof. Since $K_p = K$, theorem 11 gives

$$1 \leq d[n(\lambda_1^1 - (n-1)K]^{1/2} + \inf|\omega_0| \times (V^{1/2})$$

However $\chi(M) \neq 0$, implies ω_0 has to vanish somewhere, hence $\inf|\omega_0| = 0$. The corollary follows.

COROLLARY 14. *Let M be a compact homogeneous manifold. Then the first eigenvalue λ_1 for functions satisfies*

$$\lambda_1 \geq \frac{1}{nd^2} + (n-1)K.$$

Proof. Let $E^0_{\lambda_1}$ be the first eigenspace for functions. Since $\tilde{E}=\{df \mid f\in E^0_{\lambda_1}\}$ is a G-invariant subspace of eigen 1-forms, Theorem 11 applies. Moreover, at the supremum point of f, $df=0$, hence $\inf|\omega_0|=0$.

COROLLARY 15. *Let M be a compact homogeneous manifold. Suppose*

$$\frac{1}{d^2}>-\binom{n}{p}p(n-p)K_p.$$

*Then the p*th *Betti number b_p satisfies*

$$b_p\le\binom{n}{p}.$$

Proof. If $b_p>\binom{n}{p}$, then since the dimension of p-tensors on an n-dimensional vector space is $\binom{n}{p}$, at a fixed point $x_0\in M$, there exists $\omega_0\in E^p_0$ which vanishes at x_0. By theorem 11, we have

$$1\le d\left[-\binom{n}{p}p(n-p)K_p\right]^{1/2}$$

which is a contradiction to the assumption.

Remark. Corollary 15 actually shows that if the dimension of the first eigenspace for p-forms is greater than $\binom{n}{p}$, then

$$\lambda^p_1\ge\frac{1}{d^2\binom{n}{p}}+(n-p)pK_p.$$

COROLLARY 16. *Let M be a compact irreducible homogeneous manifold. If M is not parallelizable, then*

$$\lambda^1_1\ge\frac{1}{nd^2}+(n-1)K.$$

Proof. It suffices to show that if the $\dim E^1_{\lambda_1}\le n$ then there exists $\omega\in E^1_{\lambda_1}$ such that $\omega=0$ at some point. If not, say for all $\omega\in E^1_{\lambda_1}$, ω never vanish, we want to find a contradiction. Let us first fix a point $x\in M$. By the irreducibility condition of H_x, $\{h^*\omega(x)\}$ spans $T^*_xM=$ cotangent space of M at x, for any fixed $\omega\in E^1_{\lambda_1}$. On the other hand, since $h^*\omega$ $(h\in H_x)$ is also an eigen 1-form and $\dim(T^*_xM)=n$, dim $E^1_{\lambda_1}$ must be at least n. Therefore dim $E^1_{\lambda_1} = n$. However by the assumption that all $\omega \in E^1_{\lambda_1}$ do not vanish, this implies that there exist n linearly independent sections of the cotangent bundle of M. Hence M is parallelizable which is a contradiction.

Combining with the remark above, this proves the corollary.

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