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Affine varieties dominated by $\mathbf{C}^2$ [1]

By R. V. Gurjar

Introduction

The two main results of this paper are Theorems 1 and 2 below.

THEOREM 1. *Let V be an affine, non-singular, variety over* $\mathbf{C}$, *which is topologically contractible. Then all the algebraic line bundles on V are trivial i.e.* Pic $V = (0)$.

THEOREM 2. *Let $f:\mathbf{C}^2 \to V$ be a proper morphism onto an affine, normal variety/*$\mathbf{C}$. *Then V is topologically contractible and* Pic $V = (0)$.

Actually, using the ideas in the recent proof of the cancellation Theorem for affine 2-space by T. Fujita, M. Miyanishi and T. Sugi, we can see that in Theorem 2, if V is non-singular, then $V \approx \mathbf{C}^2$. We will give a brief outline of this argument. The method of cancellation Theorem is geometric. In [20], it was proved that if V is non-singular in Theorem 2 and if the degree of the map f is not divisible by 120, then $V \approx \mathbf{C}^2$. The method in [20] is mostly topological. We will indicate this method also. Theorem 2 remains true for Stein-manifold V and analytic map f.

Recently the author has been able to prove the following generalisation of Theorem 2.

"Let $f:\mathbf{C}^n \to V$ be a proper morphism onto an affine, non-singular variety V. Then V is simply-connected, $H_i(V, \mathbf{Z})$ is finite for all $i > 0$ and Pic $V = (0)$."

The proof will be published elsewhere.

M. Kang proved in [8] that if V is the quotient variety of $\mathbf{C}^n$ by a finite group of automorphisms, then Pic $V = (0)$. We will indicate a topological proof of this by making use of some strong results of R. Oliver. See [14].

In §2, we will show some evidence for the validity of the conjecture that on a non-singular, affine surface which is topologically contractible, all vector bundles

[1] This work formed part of the author's Ph.D. thesis at the University of Chicago, March 1979.

are trivial. The analogous statement in the analytic category has been proved by Grauert for contractible Stein-manifolds of arbitrary dimension.

In Theorem 1, if $\dim V = 1$, then it is easy to see that $V \approx \mathbf{C}^1$. We will therefore assume that $\dim V > 1$. Using a result of D. Anderson [1], we get the following Corollary of Theorem 2.

COROLLARY 2 (§3). *In the situation of Theorem 2, all vector bundles on V are trivial.*

I am very grateful to Prof. M. P. Murthy and Prof. R. Narasimhan for the help and encouragement they gave me during this work.

§1. Picard groups of contractible varieties

We will begin with the preparation for Theorem 1. Recall that we are assuming $\dim V \geqslant 2$, and V is affine, non-singular, irreducible variety/$\mathbf{C}$. Furthermore, V is topologically contractible. By the theorem of resolution of singularities [7], there exists a projective algebraic variety X with the following properties.

(a) V is a Zariski-open subset of X, X is non-singular.

(b) $X - V = \bigcup_{i=1}^{r} C_i$, where C_i are closed, irreducible, non-singular subvarieties of codimension 1. For any abelian group G, let $H^*(Y, G)$ denote the singular cohomology groups of space Y with coefficients in G. Recall the well-known exponential sequence [see 9] $0 \to \mathbf{Z} \to O_X \to O_X^* \to 0$. Here O_X is the sheaf of germs of holomorphic functions on X and O_X^* denotes the sheaf of germs of invertible holomorphic functions. The associated long exact cohomology sequences gives

$$\cdots H^1(X, Z) \to H^1(X, O_X) \to H^1(X, O_X^*) \to H^2(X, Z). \cdots$$

Since V is simply-connected, so is X. Hence $H^1(X, Z) = (0)$. The group $H^1(X, O_X^*)$ is in one-to-one correspondence with the group of invertible sheaves on X. Since X is non-singular, each of the subvarieties $C_1, \ldots, C_r$ are locally principal, hence give rise to elements of $H^1(X, O_X^*)$.

LEMMA 1.1. *The map $H^1(X, O^*) \to H^2(X, Z)$ is surjective.*

Remark. By the Hodge decomposition theorem [9] $2 \dim_{\mathbf{C}} H^1(X, O_X) = \dim_{\mathbf{C}} H^1(X, \mathbf{C})$. Hence in our case, Lemma 1.1 will prove that $H^1(X, O^*) \approx H^2(X, Z)$.

Proof of Lemma 1.1. We will in fact prove that the cohomology classes corresponding to the subvarieties $C_1, \ldots, C_r$ generate $H^2(X, Z)$ freely.

Let $C = X - V = \cup_{i=1}^{r} C_i$. There is a long exact sequence of cohomology groups corresponding to the pair (X, C).

$$\cdots H^2(X, C) \to H^2(X) \to H^2(C) \to H^3(X, C) \cdots$$

In this sequence, the cohomology groups are considered with arbitrary abelian coefficient group G.

If $\dim V = n$ (recall, $n \geq 2$), then we have [17, Theorem 6.2.17] $H^i(X, C) \approx H_{2n-i}(X - C)$. In particular, since V is contractible $H_{2n-i}\ (X - C) = (0)$ for $i < 2n$, and $H^{2n-i}(X, G) \approx H^{2n-i}(C, G)$ for $i \geq 2$. We will prove that $H_r(C, G) \approx H_r(X, G)$ for $r < 2n$ by induction on r.

For $r = 0$, we observe that C is connected (since $X - G$ is affine), hence $H_0(C, G) \approx H_0(X, G)$. By the relation between cohomology and homology groups, we get two exact sequences,

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Ext}^1\,(H_{r-1}(C, Z), G) & \to & H^r(C, G) & \to & \mathrm{Hom}\,(H_r(C, Z), G) & \to 0 \\ & \uparrow & & \uparrow & & \uparrow & \\ 0 \to & \mathrm{Ext}^1\,(H_{r-1}(X, Z), G) & \to & H^r(X, G) & \to & \mathrm{Hom}\,(H_r(X, Z), G) & \to 0. \end{array}$$

The vertical arrows are induced by the inclusion $C \subset X$. By induction, the first and the middle vertical arrow is an isomorphism for some $r < 2n - 1$. Hence $\mathrm{Hom}\,(H_r(X, Z), G) \to \mathrm{Hom}\,(H_r(C, Z), G)$ for $r < 2n - 1$. Since this is true for every coefficient group G, it follows that $H_r(X, Z) \approx H_r(C, Z)$ for $r < 2n - 2$. For $r = 2n - 1$, by Poincaré duality, $H_{2n-1}(X, Z) \approx H^1(X, Z)$. But since X is simply connected, $H^1(X, Z) = (0)$. Also $\dim_{\mathbf{R}} C = 2n - 2$, hence $H_{2n-1}(C, Z) = (0)$. Since $H_r(C, Z) \approx H_r(X, Z)$ for $r < 2n - 1$, it follows by the universal coefficient theorem that

$$H_r(C, G) \approx H_r(X, G) \quad \text{for} \quad r \leq 2n - 1.$$

In particular $H_{2n-2}(C, Z) \approx H_{2n-2}(X, Z)$. Now it is easy to see that the fundamental cycles of $C_1, \ldots, C_r$ generate $H_{2n-2}C, Z)$ freely. Hence by Poincare duality, the cohomology classes corresponding to $C_1, \ldots, C_r$ generate $H^2(X, Z)$ freely. It follows that the map

$$H^1(X, O^*) \to H^2(X, Z)$$

is surjective. By the remark following the statement of Lemma 1.1, we now know that $H^1(X, O^*) \approx H^2(X, Z)$. This means that the invertible sheaves corresponding to the divisors $C_1, \ldots, C_r$ generate the group of invertible sheaves $H^1(X, O^*)$ (even freely). By the well-known one-to-one correspondence between the algebraic line bundles on X and the analytic line bundles on X [16], we see that the algebraic line bundles corresponding to the divisors $C_1, \ldots, C_r$ generate Pic X. Clearly the restrictions of these line bundles $[C_1], \ldots, [C_r]$ to V are trivial because the supports of the divisors $C_1, \ldots, C_r$ do not meet V. All the algebraic line bundles on V are trivial because of the following.

LEMMA 1.2. *Any algebraic line bundle on V is the restriction of a line bundle on X.*

Proof. [10, Lemma 6.2]. Let L be any algebraic line bundle on V and let F be the associated sheaf of sections. By [3, Proposition 2] there exists a coherent algebraic sheaf G on X whose restriction to V is isomorphic to F. We will denote by O the structure sheaf of X (as an algebraic variety).

Consider the sheaf Hom (Hom $(G, O), O) = G^{**}$. Clearly $G^{**}|_V \approx F$ because $G|_V \approx F$ and F is locally free. It suffices to show that G^{**} is locally free. But the stalk at x of G^{**} is a reflexive, f.g. module of rank 1 over the regular local ring $O_{X,x}$. Consequently G^{**} is locally free and we are done.

This completes the proof of

THEOREM 1. *Let V be an affine, irreducible, non-singular variety/$\mathbf{C}$. Suppose V is topologically contractible. Then* Pic $V = (0)$. *Alternatively, the co-ordinate ring of V, $\Gamma(V)$, is a unique factorization domain.*

§2

In this section V is an affine, non-singular, irreducible surface/$\mathbf{C}$, which is topologically contractible. Let X be a non-singular, projective compactification of V such that all the components of $X - V$ are non-singular curves with transverse intersections. As usual, let $P_g(X) =$ geometric genus of $X = \dim_{\mathbf{C}} H^2(X, O_X)$ and $q(X) = \dim_{\mathbf{C}} H^1(X, O_X)$.

LEMMA 2.1. $q(X) = P_g(X) = 0$.

Proof. By Hodge decomposition theorem [9], $2q(X) = \dim_{\mathbf{C}} H^1(X, \mathbf{C})$.

Since X is simply connected, $q(X) = 0$. From Lemma 1.1, the map $H^1(X, O^*) \to H^2(X, Z)$ is an isomorphism. Also by Lemma 1.1, since $H^2(X, Z)$ is a free group generated by the cohomology classes of the components of $X - V$, the homorphism $H^2(X, Z) \to H^2(X, \mathbf{C})$ is an injection and the image in $H^2(X, \mathbf{C})$ generates $H^2(X, \mathbf{C})$ over $\mathbf{C}$ (again by the proof of Lemma 1.1).

On the other hand, the image of the composite map $H^1(X, O^*) \to H^2(X, Z) \to H^2(X, \mathbf{C})$ is contained in the component $H^1(X, \Omega^1)$ of $H^2(X, \mathbf{C})$ in the Hodge decomposition $H^2(X, \mathbf{C}) \approx H^2(X, O) \oplus H^1(X, \Omega^1) \oplus H^0(X, \Omega^2)$. It follows that $P_g(X) = (0)$.

Now X is a simply-connected, non-singular, projective surface with $q(X) = (0) = P_g(X)$. In a recent article I. Dolgačev asks the following.

QUESTION. Let X be a non-singular, irreducible, projective surface/$\mathbf{C}$ with $P_g(X) = 0$, which is simply-connected. Is it true that X is not of general type? (For definition of surface of general type, see [4]).

In our situation, X is a compactification of a contractible affine surface. Hence we ask the

QUESTION. Is a projective, algebraic compactification of a contractible, non-singular surface rational?

Let $A_0(X)$ be the group of 0-cycles of degree 0 modulo rational equivalence and Alb (X) be the Albanese variety of X. Let $SA_0(X) = \text{Kernel}\,[A_0(X) \to \text{Alb}\,(X)]$. Then, with the notation of this section (V not necessarily contractible), P. Murthy and R. Swan proved [10, Theorem 2] that if $SA_0(X)$ is finite, then all the vector bundles on V are direct sums of line bundles.

On the other hand, it is proved in [2] that if $P_g(X) = 0$ and X is not of general type then $SA_0(X) = (0)$. It follows that if either of the above questions has an affirmative answer, then from Theorem 1, we get the

COROLLARY. *Let V be an affine, non-singular, contractible surface/$\mathbf{C}$, then all the vector bundles on V are trivial.*

It is easy to see that if V is an affine, non-singular, irreducible curve, then Pic $V = (0)$ iff V is rational. Unfortunately, this is not true if dim $V = 2$ (even if all the vector bundles on V are trivial). In [5], Dolgačev constructs an example of a non-singular, projective surface X/C such that X is simply connected, $P_g(X) = 0$ and X is not of general type, and not rational Then Pic X is finitely generated. If the divisors $D_1, D_2, \ldots, D_n$ geate Pic X, then any affine open subset V of $X - \bigcup_{i=1}^{n}(\text{Supp}\, D_i)$ will have Pic $V = (0)$ and then by the results mentioned above, all the vector bundles on V are trivial but V is not rational.

§3

In this section we will prove Theorem 2. We begin with some general results.

LEMMA 3.1. *Let $\mathbf{C}^n \to V$ be a proper morphism of complex affine n-space onto an affine variety V/***C***. Then the fundamental group of V is finite. If d is the degree of the map f (i.e., the number of inverse images of a generic point of V, this number being clearly finite), then the order of the fundamental group of V divides d.*

Proof. Let $\tilde{V} \xrightarrow{\pi} V$ be the universal cover of V. Since $\mathbf{C}^n$ is simply connected, there exists a continuous map $\mathbf{C}^n \xrightarrow{\varphi} \tilde{V}$ such that $f = \pi \cdot \varphi$. After removing the singular locus of V and then applying the Purity of Branch locus [18], we see that there exists a proper subvariety, S, of V such that $\mathbf{C}^n - f^{-1}(S) \to V - S$ is a finite unramified covering. We have a commutative diagram

$$\begin{array}{ccc} \mathbf{C}^n - f^{-1}(S) & \xrightarrow{\varphi} & \tilde{V} - \pi^{-1}(S) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & V - S & \end{array}$$

Since $\tilde{V} - \pi^{-1}(S) \xrightarrow{\pi} V - S$ is also a covering map, it follows by general properties of covering spaces [17] that $\varphi : \mathbf{C}^n - f^{-1}(S) \to \tilde{V} - \pi^{-1}(S)$ is also a covering map. It follows that degree of $f|_{\mathbf{C}^n - f^{-1}(S)} = (\text{degree of } \varphi|_{\mathbf{C}^n - f^{-1}(S)}) \cdot (\text{degree } \pi|_{\tilde{V} - \pi^{-1}(S)})$.
From this the lemma follows.

Remark 1. In the analytic case, if there is an analytic proper map $\mathbf{C}^n \xrightarrow{f} V$ onto an analytic space V, then exactly the same proof shows that the fundamental group of V is finite and its order divides the degree of the map f. Also note that $\mathbf{C}^n$ can be replaced by any irreducible, simply-connected analytic space (and in the algebraic case by an irreducible, simply-connected algebraic variety), and the map f is proper with finite fibres.

LEMMA 3.2. *With the same notation as in Lemma* 3.1, Pic *V is finite provided V is normal.*

Proof. Let S be the singular locus of V. Let P be a point of $V - S$ and $\mathfrak{m}$ the maximal ideal of P in $R = \Gamma(V)$, the coordinate ring of V. Then $R_\mathfrak{m}$ is regular. Suppose $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ are the maximal ideals of $k[x_1, \ldots, x_n]$ lying above $\mathfrak{m}$. The integral closure of $R_\mathfrak{m}$ in the quotient field of $k[x_1, \ldots, x_n]$ is $\bigcap_{i=1}^r k[x_1, \ldots, x_n]_{\mathfrak{m}_i} = \bar{R}_\mathfrak{m}$ say. Then $\bar{R}_\mathfrak{m}$ is a finitely generated $R_\mathfrak{m}$-module and $\bar{R}_\mathfrak{m}$ is a regular ring. Hence the depth of $\bar{R}_\mathfrak{m}$ as an $R_\mathfrak{m}$-module is equal to

$n = \dim R_m$. Since R_m is regular, it follows that the projective dimension of $\bar{R}_m$ as R_m-module is 0. Hence $\bar{R}_m$ is a projective and consequently a free R_m-module.

This shows that the coherent algebraic sheaf $\pi_*(O_{\mathbf{C}^n})$ restricted to $V-S$ is locally free of rank d. Now let L be any line bundle on V. By projection formula $\pi_*(\pi^*L) \approx L \otimes_{O_V} \pi_*(O_{\mathbf{C}^n})$ [6]. But π^*L is a trivial line bundle on $\mathbf{C}^n$, because all line bundles are trivial on $\mathbf{C}^n$. Therefore

$$\pi_*(\pi^*L) \approx \pi_*(O_{\mathbf{C}^n}) \approx L \otimes_{0_V} \pi_*(O_{\mathbf{C}^n}).$$

Since $\pi_*(O_{\mathbf{C}^n})|_{V-S}$ is locally free, by taking d^{th} exterior,

$$\bigwedge^d (\pi_*(O_{\mathbf{C}^n})|_{V-S}) \approx L^d|_{V-S} \otimes_{O_{V-S}} \bigwedge^d (\pi_*(O_{\mathbf{C}^n})_{V-S}).$$

Since the sheaf on the left hand side is locally free of rank 1, we see that $L^d|_{V-S}$ is trivial. From this it follows that L^d is trivial on V because of the following.

LEMMA 3.3. *Let V be a normal, affine, irreducible variety and L a line bundle on V. If restricted to the regular points of V, L is trivial, then L is trivial on V.*

Proof. Let $\{U_i\}$ be a covering of V such that $L|_U$is trivial for all i. Let $f_{ij} : (L|_{U_i})_{U_i \cap U_j} \to (L|_{U_j})_{U_i \cap U_j}$ be transition functions for L. Here f_{ij} are invertible regular functions in $U_i \cap U_j$.

We know that $L|_{V-S}$ is trivial. $\{U_i \cap (V-S)\}$ is a covering of $V-S$. There exist regular invertible functions $g_i \in \Gamma(U_i - S)$ such that $f_{ij} = g_i/g_j$ in $(U_i - S) \cap (U_j - S)$. But since V is normal, the function g_i will be regular on the whole of U_i. Also, since on a normal variety the zeros of a regular function form a subvariety of pure codimension 1, it follows that g_i is a unit in $\Gamma(U_i)$. Clearly $f_{ij} = g_i/g_i$ in $U_i \cap U_j$. This means L is a trivial bundle.

Remark. If there is a proper, analytic map $\mathbf{C}^2 \xrightarrow{f} V$ onto V and V is normal then an exactly similar proof will show that the group of analytic line bundles on V is finite.

Suppose now we have a proper morphism $\mathbf{C}^n \xrightarrow{f} V$, onto an affine variety/$\mathbf{C}$. We know by Lemma 3.1 that the fundamental group of V is finite. If $\tilde{V}$ is the universal covering space of V, then it is possible to show that $\tilde{V}$ has the structure of an affine variety such that the covering map $\tilde{V} \to V$ is a morphism. Also the map φ in the proof of Lemma 3.1, $\varphi : \mathbf{C}^n \to \tilde{V}$ is also a morphism. To prove that V is contractible, it suffices to show that $\tilde{V}$ is contractible, in view of the following lemma in J. Milnor's Morse Theory [11]. For completeness' sake we will reproduce the simple proof.

LEMMA 3.4. *Suppose V is a $C-W$ complex of finite dimension whose universal covering space $\tilde{V}$ is contractible. Then the fundamental group of V has no element of finite order.*

Proof. Let $\pi_i(V)$ (resp. $\pi_i(\tilde{V})$) denote the homotopy groups of V (resp. $\tilde{V}$) w.r.t. a fixed base point. For $i>1$, one knows $\pi_i(\tilde{V})\approx\pi_i(V)$. If $\tilde{V}$ is contractible, $\pi_i(\tilde{V})=(0)$ for all i. Also the cohomology group $H^k(V,Z)$ can be identified with the cohomology group $H^k(\pi_1(V),Z)$. If $\pi_1(V)$ contains a non-trivial cyclic subgroup G, then for a suitable intermediate covering $\hat{V}$ of V, we have $\pi_1(\hat{V})=G$. Therefore $H^k(G,Z)=H^k(\hat{V},Z)=(0)$ for $k>m+1$, where m is the dimension of V. But a cyclic group has non-trivial cohomology in arbitrarily high dimensions, a contradiction.

Thus, with the notation of Lemma 3.1, to show that V is contractible, we can assume that V is simply connected. Therefore, $H_1(V,Z)=(0)$. Assume now that V is normal. To prove that V is contractible, it suffices to show that $H_i(V,Z)=(0)$ for all $i>0$ because of the following theorem of Whitehead [17; Cor. 7.6.24].

THEOREM (Whitehead). *Let V be a path connected topological space. If $\pi_i(V)=(0)$ for all i, then V is contractible.*

But by Hurewicz Theorem [17, Theorem 7.5.5], $\pi_i(V)=(0)$ for all i if and only if $\pi_1(V)=(1)$ and $H_i(V,Z)=(0)$ for $i>0$. Any affine variety is a Stein space [13]. For a Stein space of dimension n, R. Narasimhan has shown that $H_i(V,Z)=(0)$ for $i>n$ and $H_n(V,Z)$ is a torsion free group [13]. We can prove

THEOREM 2. *Let $\mathbf{C}^n \xrightarrow{f} V$ be a proper morphism onto an affine, algebraic, normal variety/$\mathbf{C}$. If V is simply connected, then $H^2(V,Z)=(0)$ and* Pic $V=(0)$. *In particular, for $n=2$, V is contractible and* Pic $V=(0)$.

Proof. Since $H_1(V,Z)=(0)$, $H^2(V,Z)\approx\operatorname{Hom}(H_2(V,Z),Z)$. From the exponential sequence, we get $H^1(V,O^*)\approx H^2(V,Z)$. Here O^* is the sheaf of germs of invertible holomorphic functions. To show that $H^2(V,Z)=(0)$, it suffices to show that $H^1(V,O^*)=(0)$, i.e. all analytic line bundles on V are analytically trivial.

We have already seen by the remark following Lemma 3.3 that the group of analytic line bundles (and also the group of algebraic line bundles) is finite. To complete the proof of Theorem 2, we need

LEMMA 3.5. *Let V be any irreducible algebraic variety/$\mathbf{C}$ (resp. irreducible analytic space). If L is an algebraic (resp. analytic) line bundle on V which is not trivial but some power of which is trivial, then V has a non-trival unramified covering.*

Proof. We will indicate a proof for the algebraic case; the proof in the analytic case is very similar. We can assume that L is not trivial but L^p is trivial for some prime p. Let $\{U_i\}$ be a covering of V by affine open sets such that $L|_{U_i}$ is trivial for all i and let $\varphi_{ij}:(L|_{U_i})_{U_i\cap U_j}\to(L|_{U_j})_{U_i\cap U_j}$ be the transition functions. The φ_{ij} are regular, nowhere zero functions in $U_i\cap U_j$.

Since L^p is trivial, there exist regular nowhere-zero functions g_i on U_i such that $(g_i/g_j)=\varphi_{ij}^p$ on U_i. We can find elements α_i in a suitable extension of the quotient field of $\Gamma(U_i)$ such that $\alpha_i^p=g_i$. Let $\tilde{U}_i$ be the affine variety whose co-ordinate ring is $\Gamma(U_i)[\alpha_i]$. Then there exists a natural morphism $\tilde{U}_i\to U_i$ which is unramified (since g_i are units in $\Gamma(U_i)$). We can patch up the $\tilde{U}_i$ to obtain a variety $\tilde{V}$ and a morphism $\tilde{V}\to V$ which is unramified and using the fact that L is not trivial, one sees that $\tilde{V}$ is irreducible.

The proof of Theorem 2 is therefore complete. We note that a statement analogous to Theorem 2 in the case of a proper analytic map $\mathbf{C}^n\to V$ can be proved by a similar method. We state it as

COROLLARY 1. *In the proof of Theorem 2. Let $\mathbf{C}^n\xrightarrow{f}V$ be a proper, analytic map onto a Stein space V which is normal. Then if V is simply connected, $H^2(V,Z)=(0)$ and all the analytic line bundles on V are trivial.*

In particular, if $n=2$, then V is contractible and the group of analytic line bundles on V is trivial.

D. Anderson showed in his University of Chicago thesis [1; Theorem 4.17], that in the situation of Theorem 2, any vector bundle is the direct sum of a trivial bundle and a line bundle. From this and our Theorem 2, we get

COROLLARY 2 TO THEOREM 2. *If $\mathbf{C}^2\to V$ is a proper morphism onto an affine, normal variety/$\mathbf{C}$, then all the vector bundles on V are trivial.*

Now let G be a finite group of automorphisms of $\mathbf{C}^n$. Then $\mathbf{C}^n/G=V$ has the structure of an affine, normal irreducible variety/$\mathbf{C}$ such that the natural map $\mathbf{C}^n\to V$ is a morphism. Recently R. Oliver has proved the following wonderful result [14]: "Let G be a compact Lie group acting on $\mathbf{R}^n$, then the quotient space $\mathbf{R}^n/G$ is contractible".

Using this result and our Lemma 3.1 and 3.5, we obtain

COROLLARY 3. *Let G be a finite group of algebraic (or analytic) automorphisms of $\mathbf{C}^n$ and let $V=\mathbf{C}^n/G$. Then any line bundle on (resp. any analytic line bundle on) V is trivial.*

In view of R. Oliver's result and our Theorem 2, we can ask the following.

QUESTION. Suppose $\mathbf{C}^n \to V$ is a proper morphism (or proper analytic map) onto a normal, affine variety V (resp. onto a normal Stein space V). Is V contractible?

Remark 1. The condition of properness for the map f in Theorem 2 is essential as shown by the following example. We consider the complex affine smooth surfaces, V, given by $-xy+z^2=1$ in $\mathbf{C}^3$ (by change of coordinates one can see that this is isomorphic to the complex 2-sphere). If we put $x=u(2+uv)$, $y=v$, $z=1+uv$, then the coordinate ring, $\mathbf{C}[x, y, z]/(xy+z^2-1)$, of this surface is isomorphic to $\mathbf{C}[u(2+uv), v, uv]$. Since this is a subring of $\mathbf{C}[u, v]$, we get a morphism $\mathbf{C}^2 \xrightarrow{f} V$, which is not proper. Clearly the ring $\mathbf{C}[u(2+uv), v, uv]$ is not a UFD. It is easy to see that the Picard group of V is infinite cyclic. The map f is generically one-to-one, hence V is simply-connected by Lemma 3.6. But V is not contractible.

Remark 2. In Theorem 2, if the degree of f is a prime number p, then we can see that Pic is trivial (for arbitrary n). To see this, let I be any invertible ideal in the coordinate ring, $\Gamma(V)$, of V. Let $A=\Gamma(\mathbf{C}^n)$. Then I^p will be a principal ideal in $\Gamma(V)$. $IA=\alpha A$. Let $I^p=(b)$. Then $I^pA=bA$. But since all units in A are constants, we can assume that $\alpha^p=b$. This forces either I to be trivial or $Q(A)/Q(\Gamma(V))$ to be Galois extension. In the latter case we use Corollary 3 above.

§4

In this section, we will briefly outline the proof of

THEOREM 2′. *Let $f: \mathbf{C}^2 \to V$ be a proper morphism onto an affine, non-singular surface/$\mathbf{C}$. Then $V \approx \mathbf{C}^2$.*

Proof. All the important ideas in this proof are due to T. Fujita, M. Miyanishi and T. Sugi See [21, 22]. Let $\Gamma(V)$ denote the coordinate ring of V. Theorem 2 says that Pic $\Gamma(V)=(0)$. Since $\Gamma(V)$ is a subring of $\Gamma(\mathbf{C}^2)$, all units in $\Gamma(V)$ are constants. Also since the morphism f is dominating, the Kodaira dimension $K(V)$ of V is $-\infty$. Now the Main Theorem of Fujita, Miyanishi, Sugi which enables them to prove the cancellation Theorem for affine 2-space is, "Let V be an affine, irreducible non-singular surface such that $\Gamma(V)$ is a UFD, all units in $\Gamma(V)$ are constants and $K(V)=-\infty$, then $V \approx \mathbf{C}^{2\prime}$. The proof of Theorem 2′ is thus complete.

Remark 1. The proof of cancellation Theorem works for arbitrary perfect field k. One can prove the following characteristic $p>0$ analogue of Theorem 2′.

"Let $f: A^2 \to V$ be a proper morphism onto an affine, non-singular surface V. If the degree of the function-field extension $[k(A^2):k(V)]$ is not divisible by p, then $V \approx A^2$ (here A^2 denotes the affine 2-space over the perfect field k)". The proof of this more general result is by induction on $[k(A^2):k(V)]$ by making repeated use of the Main Theorem of Fujita, Miyanishi and Sugi.

Remark 2. We want to indicate briefly a topological proof of Theorem 2′ which was given in [20]. Theorem 2 implies that V is contractible. C. P. Ramanujam defines the "fundamental group at infinity of V′. For this, we embed V in a projective, non-singular surface X such that $X-V$ is a divisor with normal crossings. If $X-V=\cup_{i=1}^n D_i$, where D_i are the non-singular components of $X-V$, then we take a "nice" tabular neighbourhood T of $D=\cup_{i=1}^n D_i$ in X. Then for "small" neighbourhoods T, the fundamental group of $T-D$ is independent of X and T, which we call the "fundamental group at infinity of V". One sees easily that $\mathbf{C}^2$ has trivial fundamental group at infinity. Then using the fact that f is proper and using an argument similar to Lemma 3.1, we prove that the fundamental group at infinity of V is finite.

Next we observe that the cohomology classes of the curves D_i generate $H^2(X, Z)$ freely. By Poincare duality, it follows that the intersection matrix $(D_i \cdot D_j)$ is unimodular. This forces, that the fundamental group at infinity of V is perfect i.e. equal to it's commutator. But the boundary of T, δT, is a 3-dimensional compact orientable manifold whose fundamental group is the fundamental group at infinity of V.

We have now $H_1(\delta T, Z)=(0)$ and $\pi_1(\delta T)$ finite. It follows that the universal covering space of δT is a homotopy 3-sphere on which $\pi_1(\delta T)$ acts fixed point freely. This implies that the group $\pi_1(\delta T)$ has periodic cohomology with period 4. Now the list of finite groups with periodic cohomology of period 4, given by J. Milnor in [12], shows that either $\pi_1(\delta T)=(0)$ or it is the group $SL(2, 5)$, of order 120. In proving that $\pi_1(\delta T)$ is finite, we actually observe that the order of $\pi_1(\delta T)$ divides degree of the map f (i.e. $[k(\mathbf{C}^2):k(V)]$).

Finally we invoke the beautiful result of C. P. Ramanujam [15] "If V is a contractible, affine, non-singular surface which is simply-connected at infinity, then $V \approx \mathbf{C}^2$". Using all these results, we now get the following slightly weaker statement than Theorem 2′.

"Let $f: \mathbf{C}^2 \to V$ be a proper, morphism onto a non-singular affine surface V. If 120 does not divide, the degree of f, then $V \approx \mathbf{C}^2$". For details, see [20].

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