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Tangential homotopy equivalences

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§1. Introduction

Two (topological) manifolds M^n and N^n are called tangentially homotopy equivalent if there exists a homotopy equivalence $f\colon (N, \partial N) \to (M, \partial M)$ such that $f^*(\tau_M)$ is stably equivalent to τ_N. Let $\theta(m)$ denote the set of homeomorphisms types of manifolds which are tangentially homotopy equivalent to M. In this paper we study $\theta(M)$. In particular we give estimates of its size for suitable classes of manifolds.

Given any set S we use $|S|$ to denote its cardinality.

DEFINITION. A manifold M is said to satisfy our basic estimate if

$$|\theta(M)| \le \sum_{i>1} |H^{2^i-2}(\mathring{M}; \mathbf{Z}/2)|$$

where $\mathring{M} = M$-(open disc) if M is closed and $\mathring{M} = M$ otherwise.

Our first results give examples of classes of manifolds which satisfy our basic estimate. First we have

THEOREM A. *Let M^n be a closed manifold with $n \ge 5$ and $\pi_1 M = 0$. Then, if the group of stable isomorphism classes of vector bundles $K^\circ(M)$ is torsion free, M satisfies our basic estimate.*

Examples of manifolds to which Theorem A applies include simply-connected Lie groups ([Ho]), homogeneous spaces G/H where $H \subset G$ is a connected subgroup of maximal rank ([P]), and closed manifolds M^n such that $H^*(M; \mathbf{Z})$ is torsion free, $\pi_1 M = 0$.

We call a manifold, M^n, metastable if $c = \max\{i \mid \pi_i(M) = 0\}$, the connectivity of M^n, satisfies $c \ge (n+1)/3$. Then we have

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THEOREM B. *Let M^n be a closed metastable manifold with $n \geq 5$. Then M^n satisfies our basic estimate.*

It is not hard to see that if M is metastable than there is at most one $i > 1$ such that $H^{2^i-2}(\mathring{M}; \mathbf{Z}/2)$ is non-trivial. In fact, for certain n there is no such i, and we have

COROLLARY. *If M^n is a closed, metastable manifold with $n = 3 \cdot 2^i - \varepsilon$ for $\varepsilon = 3, 4, 5, 6, 7$, $i \geq 2$ then $|\theta(M)| = 1$.*

If $\pi_1(M^n) = 0$ and $n = 5$, Barden [Ba] proves $|\theta(M)| = 1$. If $\pi_1 M = 0$, $n = 6$, and $H_2(M; \mathbf{Z})$ is torsion free, Jupp [J] proves $|\theta(M)| = 1$. If M is 2-connected and $n = 7$, Wilkens [Wilk] has studied $\theta(M)$. Thus we shall often assume $n \geq 8$.

For highly connected manifolds it is possible to refine the estimate of $\theta(M)$. We have

THEOREM C (i). *Let M^{2n} be closed and $(n-1)$-connected with $n \geq 3$. Then $|\theta(M)| = 1$.*

(ii). *Let M^{2n+1} be closed and $(n-1)$-connected with $n \geq 3$. If $n = 2^i - 2$ assume that $H_n(M; \mathbf{Z})$ has no summands $\mathbf{Z}/2$ or $\mathbf{Z}/4$. Then $|\theta(M)| = 1$.*

A hypersurface is a manifold M^n which admits a locally flat codimension 1 embedding in S^{n+1}. For hypersurfaces we make the

CONJECTURE D. If two metastable hypersurfaces of dimension at least 5 are homotopy equivalent than they are homeomorphic.

Given a hypersurface M^n, let $\Sigma\,\theta(M^n) \subset \theta(M^n)$ be the subset realized by hypersurfaces. Conjecture D is then equivalent to $|\Sigma\,\theta(M^n)| = 1$ if M^n is metastable. If $M^n \subset S^{n+1}$ then $S^{n+1} = N_1 \cup_M N_2$ and $H^*(\mathring{M}) = H^*(N_1) \oplus H^*(N_2)$. We prove $|\Sigma\,\theta(M^n)| = 1$ if M^n is metastable and $H^q(N_1; \mathbf{Z}/2) = 0$ or $H^q(N_2; \mathbf{Z}/2) = 0$ for the relevant q of the form $2^i - 2$.

For specific manifolds the size of $\theta(M)$ depends on results in "classical" homotopy theory. Let ε_i be the following function ([BaM]),

$$\begin{aligned} \varepsilon_i &= 2 \quad \text{if} \quad i \equiv 0 \pmod 4 \\ &= 3 \quad \text{if} \quad i \equiv 1 \pmod 4 \\ &= 4 \quad \text{if} \quad i \equiv 2, 3 \pmod 4 \end{aligned}$$

THEOREM E. *Let M be a connected sum of r copies of $S^p \times S^q$, $1 \leq p \leq q$, $p + q \geq 5$.*

(i) *If* $q=2^t-2$, $1<p<q-2i+\varepsilon_i-1$ *and there exists an element of Arf invariant* 1 *in* $\pi_q^s(S^0)$, *then* $|\theta(M)|=2$.

(ii) *Otherwise,* $|\theta(M)|=1$.

By definition $\theta(M)$ is an invariant of the tangential homotopy type $\{M, \tau_M\}$. In general, however, it is not an invariant of the homotopy type itself. Indeed, we construct examples of homotopy equivalent manifolds M_1 and M_2 with $|\theta(M_1)|-|\theta(M_2)|$ arbitrarily large. See (7.9) and (7.10).

The proofs of the above results are based on the theory of (simply-connected) surgery. First we have $\theta(m)\subseteq\theta(\mathring{M})$ (equality if τ_M is stably fibre homotopically trivial), (4.12).

Let Q be a manifold representing a class $x\in\theta(\mathring{M})$. Then there is a normal map

$$f\colon (Q, \partial Q)\to(\mathring{M}, \partial\mathring{M}), \qquad \hat{f}\colon \nu_Q\to\nu_M$$

where f is a homotopy equivalence of pairs and $\hat{f}$ is a map of the topological normal bundles which cover f. The normal invariant of $(f, \hat{f})$,

$$N(f, \hat{f})\in[\mathring{M}, G/\text{TOP}]$$

lies in the image of $[\mathring{M}, G]\to[\mathring{M}, G/\text{TOP}]$, or equivalently in

$$\text{Cok } J(\mathring{M})=\text{cokernel }([\mathring{M}, \text{TOP}]\to[\mathring{M}, G])$$

Let $\varepsilon_t(\mathring{M})$ denote the set of tangential self-homotopy equivalences of $(\mathring{M}, \partial\mathring{M})$. There is an action

$$\varepsilon_t(\mathring{M})\times\text{Cok } J(\mathring{M})\to\text{Cok } J(\mathring{M})$$

given by $\alpha\cdot x=N(\alpha)+(\alpha^*)^{-1}(x)$; $N(\alpha)=N(\alpha, \hat{\alpha})$, where $\hat{\alpha}$ covers α.

If $\pi_1(\partial\mathring{M})=\pi_1(\mathring{M})$ and dim $\mathring{M}\geq 5$ then the theory of surgery gives a bijection

$$\theta(\mathring{M})\cong\text{Cok } J(\mathring{M})/\varepsilon_t(\mathring{M}).$$

This is proved in §2.

The space G (of stable self homotopy equivalences of the sphere) has finite homotopy groups, so Cok $J(\mathring{M})$ is a finite group. In §3 we use deep results about the map $G\to G/\text{TOP}$ to reduce the size of Cok $J(\mathring{M})$ as much as possible. Theorem A follows from this work.

Theorem B requires more work. It is not hard to find examples of metastable manifolds for which Cok $J(\mathring{M})$ is quite large. Thus to prove Theorem B we must construct sufficiently many tangential self-homotopy equivalences. We do this in §4 where to each $d \in \pi_n(\mathring{M})$ we associate a map $f_d \in \varepsilon_t(\mathring{M})$. Taking normal invariants we obtain a homomorphism $\pi_n(\mathring{M}) \to \text{Cok}\, J(\mathring{M})$ and the quotient group $V(\mathring{M})$ majorizes $\theta(\mathring{M})$, $|\theta(\mathring{M})| \leq |V(\mathring{M})|$. Theorem B is then derived from known results about the classical suspension $\Sigma^\infty\colon \pi_n(\mathring{M}) \to \pi_n^s(\mathring{M})$.

In §5 we use a formula of Barratt–Hanks and Thomeier's results about the first unstable stems in homotopy groups of spheres to prove Theorem C.

Section 6 is a discussion of Conjecture D and in §7 we calculate some examples, e.g. Theorem E.

The basic outline of the paper also works in the PL- and smooth categories. The PL and the topological cases are quite similar. But in the smooth case, G/O is such a complicated space that explicit calculations are usually impossible. One example though that the reader can work out from the enclosed theory is that $|\theta_{\text{diff}}(\mathring{M})| = 1$ if M is metastable with $\tilde{H}_*(\mathring{M}; \mathbf{Z}/2) = 0$. Also, see Theorem 5.10.

We would like to thank M. Barratt, M. Mahowald and R. J. Milgram for several useful conversations.

§2. Tangential normal maps

Let P^n be a manifold with boundary $\partial P^n \neq \emptyset$. A *tangential normal map* over P is a pair $(f, \hat{f})$.

$$f\colon (Q, \partial Q) \to (P, \partial P), \qquad \hat{f}\colon \nu_Q \to \nu_P \tag{2.1}$$

where Q is a manifold of the same dimension as P, f is any map of pairs, and $\hat{f}$ is a bundle map of stable normal bundles which covers f.

Let $\mathcal{S}^t(P)$ denote the set of tangential homotopy manifold structures of P: an element of $\mathcal{S}^t(P)$ is represented by a tangential normal map $(f, \hat{f})$ with f a homotopy equivalence of pairs. Two pairs $f_0\colon Q_0 \to P$ and $f_1\colon Q_1 \to P$ (with bundle maps $\hat{f}_0$ and $\hat{f}_1$) represent the same element in $\mathcal{S}^t(P)$ iff there exists a homeomorphism $h\colon Q_0 \to Q_1$ with differential $dh\colon \nu_{Q_0} \to \nu_{Q_1}$ such that $f_1 \circ h$ is homotopic as a map of pairs to f_0 and such that $\hat{f}_1 \circ dh$ is the same bundle map as $\hat{f}_0$.

Let $\varepsilon^t(P)$ denote the group of tangential normal maps $(\alpha, \hat{\alpha})$ with $\alpha\colon (P, \partial P) \to (P, \partial P)$ a homotopy equivalence of pairs. Clearly $\varepsilon^t(P)$ acts on $\mathcal{S}^t(P)$ via composition. The forgetful map $\mathcal{S}^t(P) \to \theta(P)$ induces a bijection of the orbit space $\mathcal{S}^t(P)/\varepsilon^t(P)$ and $\theta(P)$,

$$\mathcal{S}^t(P)/\varepsilon^t(P) \xrightarrow{\cong} \theta(P) \tag{2.2}$$

Surgery theory relates $\mathcal{S}^t(P)$ to the set $\Omega^0(P, \partial P)$ of tangential normal bordism classes of tangential normal maps over P. In addition, there is a well-known isomorphism (the normal invariant)

$$N^t: \Omega^0(P, \partial P) \to [P, \Omega^\infty S^\infty]$$

For our use in subsequent sections we briefly recall the definition of N^t and refer the reader to [B] for further details.

Let $(f, \hat{f})$ in (2.1) represent an element of $\Omega^0(P, \partial P)$ and let $c: (D^{n+k}, S^{n+k-1}) \to (T(\nu_Q), T(\nu_Q | \partial Q))$ be the natural collapse map. The S-dual of $T(\nu_p)/T(\nu_p | \partial P)$ is $P^+ (= P$ with a disjoint base point added) so the S-dual of the composite

$$S^{n+k} \to T(\nu_Q)/T(\nu_Q | \partial Q) \xrightarrow{T(f)} T(\nu_p)/T(\nu_p | \partial P)$$

is a stable (based) map $P^+ \to S^0$. Its adjoint is the element

$$N^t(f, \hat{f}) \in [P, \Omega^\infty S^\infty] \tag{2.3}$$

We let $\Omega^\infty S^\infty$ denote the component of $\Omega^\infty S^\infty$ consisting of maps of degree i (degree: $\pi_0(\Omega^\infty S^\infty) \xrightarrow{\cong} \mathbf{Z}$). Then

$$N^t(f, \hat{f}) \in [P, \Omega_i^\infty S^\infty]$$

iff $f: (Q, \partial Q) \to (P, \partial P)$ has degree i. In particular, for normal maps of degree ± 1, $N^t(f, \hat{f}) \in [P, G]$ where we follow the usual convention and write $G = \Omega_{-1}^\infty S^\infty \cup \Omega_1^\infty S$. Under composition G is an H-space.

If we vary (2.1) slightly by replacing ν_p with $\zeta = \nu_p \oplus \nu_{\hat{f}}$, where $\nu_{\hat{f}}$ is some fibre homotopy trivialized TOP-bundle, then there is a bijection between the resulting set of normal bordism classes, $\Omega_N^0(P, \partial P)$, and $[P, \Omega^\infty S^\infty/\mathrm{TOP}]$, where $\Omega^\infty S^\infty/\mathrm{TOP}$ fits into a fibration $\Omega^\infty S^\infty \to \Omega^\infty S^\infty/\mathrm{TOP} \to B\mathrm{TOP}$, cf. [BM].

Restricting further to bordism classes of pairs $(f, \hat{f})$ with $\deg(f) = \pm 1$, we get $[P, G/\mathrm{TOP}]$ instead of $[P, \Omega^\infty S^\infty/\mathrm{TOP}]$. The H-space structure on G/TOP coming from Whitney sum corresponds to multiplication of normal maps.

If we remove all normal bundle information from the definition of $\mathcal{S}^t(P)$ we get the ordinary set of homotopy manifold structures $\mathcal{S}(P)$.

Let $f: Q \to P$ be a homotopy equivalence representing an element of $\mathcal{S}(P)$. Set $\zeta = (f^{-1})^*(\nu_Q)$ and let $\hat{f}: \nu_Q \to \zeta$ be the canonical map over f. The uniqueness theorem for Spivak normal bundles (see e.g. [B], ch. 1) implies a fibre homotopy equivalence $\nu_p \xrightarrow{t} \zeta$ such that

$$\begin{array}{ccc} S^{n+k} & \xrightarrow{c_Q} & T(\nu_Q)/T(\nu_Q|\partial Q) \\ \downarrow{\scriptstyle c_P} & & \downarrow{\scriptstyle T(\hat{f})} \\ T(\nu_p)/T(\nu_p|\partial P) & \xrightarrow{T(t)} & T(\zeta)/T(\zeta|\partial P) \end{array}$$

is commutative (k large). Here c_p, c_Q are the natural collapse maps. Thus $\zeta = \nu_p \oplus \nu_{\hat{f}}$ where $\nu_{\hat{f}}$ is homotopy trivialized. Moreover, equivalence classes of triples (ν_p, ζ, t) as above are classified by G/TOP (and by $\Omega^\infty S^\infty/\mathrm{TOP}$ if there is no condition on t). In particular (ν_p, ζ, t) determines an element in $[P, G/\mathrm{TOP}]$. This defines the usual normal invariant

$$N\colon \mathcal{S}(P) \to [P, G/\mathrm{TOP}].$$

If we start with a tangential homotopy equivalence $(f, \hat{f})$ we get $\zeta = \nu_p$ so our triple become (ν_p, ν_p, t) where $t\colon \nu_p \to \nu_p$ is a fibre homotopy equivalence. Such triples are classified by elements of $[P, G]$. It is direct to check from the definition of S-duality that we have recovered the element $N^t(f, \hat{f})$ from (2.3). In particular, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{S}^t(P) & \xrightarrow{N^t} & [P, SG] \\ \downarrow & & \downarrow \\ \mathcal{S}(P) & \xrightarrow{N} & [P, G/\mathrm{TOP}] \end{array} \tag{2.4}$$

If $\varepsilon(P)$ denotes the group of homotopy automorphisms of $(P, \partial P)$, then $\varepsilon(P)$ acts via composition on $\mathcal{S}(P)$. We wish to relate the geometric actions of $\varepsilon^t(P)$ on $\mathcal{S}^t(P)$ and $\varepsilon(P)$ on $\mathcal{S}(P)$ with the obvious action of $\varepsilon(P)$ on $[P, SG]$ and $[P, G/\mathrm{TOP}]$.

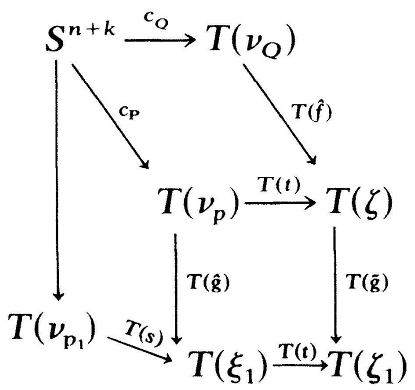
What we need is the following result (see also [Bru], Proposition 2.2)

LEMMA 2.5. *Let $f\colon (Q, \partial Q) \to (P, \partial P)$, $\hat{f}\colon \nu_Q \to \zeta$ be a normal map (of degree 1) and $g\colon (P, \partial P) \to (P_1, \partial P_1)$ a homotopy equivalence. Let $\tilde{g}\colon \zeta \to \zeta_1$, $\zeta_1 = (g^{-1}) * (\zeta)$ be the canonical map. Then*

$$N(g \circ f, \tilde{g} \circ \hat{f}) = (g^{-1})^* N(f, \hat{f}) + N(g)$$

where + refers to the group structure in $[P, G/\mathrm{TOP}]$ *induced from the Whitney sum operations in* G/TOP.

PROOF. Let $\xi_1 = (g^{-1})^*(\nu_p)$, $\zeta_1 = (g^{-1})^*(\zeta)$ and let $\hat{g} : \nu_p \to \xi_1$ be the canonical map which covers g. We have a commutative diagram in the S-category



where $t_1 = (g^{-1})^*(t)$. By definition, (ν_{p_1}, ξ_1, s) represents $N(g)$ and (ξ_1, ζ_1, t_1) represents $(g^{-1})^* N(f, \hat{f})$, so $(\nu_{p_1}, \zeta_1, t_1 \circ s)$ represents the sum. The outer part of the commutative diagram shows that $(\nu_{p_1}, \zeta_1, t_1 \circ s)$ also represents $N(g \circ f, \tilde{g} \circ \hat{f})$.

COROLLARY 2.6.

(i) If $(\alpha, \hat{\alpha}) \in \varepsilon^t(P)$ *and* $(f, \hat{f}) \in \mathcal{S}^t(P)$, *then*

$$N^t((\alpha, \hat{\alpha}) \circ (f, \hat{f})) = N^t(\alpha, \hat{\alpha}) - (\alpha^*)^{-1} N^t(f, \hat{f})$$

(ii) *If* $\alpha \in \varepsilon(P)$ *and* $f \in \mathcal{S}(P)$, *then*

$$N(\alpha \circ f) = N(\alpha) + (\alpha^*)^{-1} N(f).$$

Using 2.6 and surgery theory we can identify $\theta(P)$ with a more tractible object. We let $\mathrm{Cok}\, J(P) \subset [P, G/\mathrm{TOP}]$ be the cokernel of $[P, \mathrm{TOP}] \to [P, G]$. Furthermore we identify $[P, \mathrm{TOP}]$ with the group of bundle automorphisms of ν_p covering the identity and let $\varepsilon_t(P) \subset \varepsilon(P)$ be the cokernel of $[P, \mathrm{TOP}] \to \varepsilon^t(P)$.

THEOREM 2.7. *Let* $\alpha \in \varepsilon_t(P)$ *act on* $x \in \mathrm{Cok}\, J(P)$ *via the formula*

$$\alpha \cdot x = N(\alpha) + (\alpha^*)^{-1} x \qquad (2.7.1)$$

Then, if P *and* ∂P *are connected*; $\pi_1(\partial P) \to \pi_1(P)$ *is an isomorphism*; *and* $\dim P \geq 6$, *there is a bijection between* $\theta(P)$ *and the orbit space* $\mathrm{Cok}\, J(P)/\varepsilon_t(P)$.

PROOF. Standard surgery theory (cf. [W_3], ch. 4 and ch. 9) implies that $N^t: \mathcal{S}^t(P) \to [P, SG]$ is a bijection. Corollary 2.6 shows $\mathcal{S}^t(P)/\varepsilon^t(P) \to$ Cok $J(P)/\varepsilon_t(P)$ is a bijection, and 2.2 concludes the proof.

We next introduce a set midway between $\mathcal{S}^t(P)$ and $\theta(P)$. Let $\varepsilon_0(P) \subset \varepsilon(P)$ denote the normal subgroup of $\varepsilon(P)$ for which $\alpha \in \varepsilon_0(P)$ iff $\alpha | P$ is homotopic to the identity, *not necessarily as a map of pairs*. Note $\varepsilon_0(P) \subset \varepsilon_t(P)$.

Since $[P, \text{TOP}] \to \varepsilon^t(P) \to \varepsilon_t(P) \to 0$ is exact, we can define $\varepsilon^0(P)$ to make $[P, \text{TOP}] \to \varepsilon^0(P) \to \varepsilon_0(P) \to 0$ exact.

DEFINITION 2.8. $V(P)$ is the orbit space $\mathcal{S}^t(P)/\varepsilon^0(P)$.

Given $f: (Q, \partial Q) \to (P, \partial P), \hat{f}: \nu_Q \to \nu_p$, a tangential normal map, we write $\eta(f) \in V(P)$ for the image of $(f, \hat{f}) \in \mathcal{S}^t(P)$ in the orbit space. The image is easily seen to depend only on f, and hence $\eta(f)$ is defined for any homotopy equivalence of pairs $f: (Q, \partial Q) \to (P, \partial P)$ such that $f^*\nu_p$ is equivalent to ν_Q: we need not specify the bundle equivalence.

The set $V(P)$ arose aposteriori: it is what we spend most of the paper calculating. It does, however, have some geometric significance. Given $f_i: (Q_i, \partial Q_i) \to (P, \partial P)$, $i = 1, 2$, which are homotopy equivalences of pairs with $f_i^* \nu_p = \nu_{Q_i}$, then $\eta(f_1) = \eta(f_2)$ iff $f_2^{-1} \circ f_1: Q_1 \to Q_2$ is homotopic *not rel* ∂, to a homeomorphism.

We can summarize our results so far in

COROLLARY 2.9 (i). *The normal invariant defines a homomorphism* $N: \varepsilon_0(P) \to \text{Cok}\, J(P)$.

(ii) *If P and ∂P are connected, $\pi_1(\partial P) \to \pi_1(P)$ is an isomorphism and* dim $P \geq$ 6 *then $V(P)$ is the cokernel of N,* $V(P) = \text{Cok}\, J(P)/\varepsilon_0(P)$.

(iii) *The group $\varepsilon_t(P)$ acts on $V(P)$ via the formula in 2.7.1 and* $\theta(P) = V(P)/\varepsilon_t(P)$.

The set $V(P)$ is much easier to calculate than $\theta(P)$. With the assumptions of 2.9 (ii) it is a finite group and thus amenable to analysis one prime at a time. From 2.9 (ii) we also have that $V(P)$ is an invariant of the homotopy type of $(P, \partial P)$. In section 7 we give examples which show that $\theta(P)$ is *not* a homotopy invariant. See 7.5.

We close the section with a couple of remarks concerning $\varepsilon_t(P)$ and its action on $V(P)$. First,

LEMMA 2.10. *Let $f: (Q, \partial Q) \to (P, \partial P)$ be a homotopy equivalence of pairs. If $\alpha \in \varepsilon_t(P)$ then $f^{-1}\alpha f \in \varepsilon_t(Q)$ iff $\alpha^* N(f) \equiv N(f)$ modulo* Cok $J(P)$.

Proof. Given $g \in \varepsilon(Q)$, *then* $g \in \varepsilon_t(Q)$ precisely when $N(g) \in \text{Cok } J(Q)$. But we can compute $N(f^{-1}\alpha f)$ from 2.5,

$$N(f^{-1}\alpha f) = f^*N(\alpha) + N(f^{-1}) + f^*(\alpha^*)^{-1}N(f);$$

and $0 = N(id) = N(f) + (f^*)^{-1}N(f^{-1})$. Hence

$$N(f^{-1}\alpha f) = (f^*)N(\alpha) + f^*((\alpha^*)^{-1}N(f) - N(f))$$

Since $f^*: \text{Cok } J(P) \to \text{Cok } J(Q)$ and since $N(\alpha) \in \text{Cok } J(P)$, $N(f^{-1}\alpha f) \in \text{Cok } J(Q)$ iff $(\alpha^*)^{-1}N(f) - N(f) \in \text{Cok } J(P)$.

Remark 2.11. With the notation above, suppose that $\alpha^*N(f) - N(f) \in \text{Cok } J(P)$. It need not follow that

$$f^*(\alpha \cdot x) = (f^{-1}\alpha f) \cdot f^*(x)$$

where f^*: $V(P) \to V(Q)$, so f^* does not necessarily pass to a map of orbit spaces, f^*: $\theta(P) \to \theta(Q)$.

§3. The group Cok $J(P)$

We first study the p-primary part of Cok $J(P)$ at odd primes p. Recall the space J_p is the fibre of the map $\psi^q - 1: \text{BO}_{(p)} \to \text{BO}_{(p)}$, where q is a positive integer which projects to a generator of $(\mathbf{Z}/p^2)^*$. Also recall that Sullivan defined a map $G/\text{TOP} \to \text{BO}_{(p)}$ which is a p-local equivalence. The next result is well-known, see e.g. [MM_2] ch. 5 for a proof.

THEOREM 3.1. *For p an odd prime, the Sullivan orientation identifies* Cok $J(P)_{(p)}$ *with the image of* $[P, J_p]$ in $\text{KO}^0(P)_{(p)}$.

The well known structures of J_p and the map $J_p \to \text{BO}_{(p)}$ give rise to two obvious corollaries.

COROLLARY 3.2. *Let* $d_p(P)$ *be the smallest integer such that* $H^i(P; \mathbf{Z}/p) = 0$ *for all* $i > d_p(P)$. *Then for all primes p such that* $2(p-2) \geq d_p(P)$, Cok $J(P)_{(p)} = 0$.

Note if $n = \dim P$, and if $2p \geq n + 4$, Cok $J(P)_{(p)} = 0$.

COROLLARY 3.3. *If* $\text{KO}^0(P)$ (*or equivalently*, $\text{KU}^0(P)$) *has no p-torsion, p an odd prime, then* Cok $J(P)_{(p)} = 0$.

These corollaries apply to show that Cok $J(P)$ has no p-torsion in any of the following situations

(i) if $P_{(p)}$ is an H-space ([L])

(ii) if $P_{(p)}=(G/H)_{(p)}$, G connected Lie group and H a closed connected subgroup of maximal rank ([P])

(iii) if $H^{4i}(P;\mathbf{Z}_{(p)})$ is torsion free for all i.

We next turn our attention to the 2-primary component of Cok $J(P)$. Since G/TOP is a product of Eilenberg-Mac Lane spaces at 2 we have

$$\operatorname{Cok} J(P)_{(2)} \subseteq \prod_{i\geq 1} H^{4i}(P;\mathbf{Z}_{(2)})\times H^{4i-2}(P;\mathbf{Z}/2)$$

This is true even as groups. Indeed, let

$$k_{4n-2}\in H^{4n-2}(G/\mathrm{TOP};\mathbf{Z}/2),\ L_n\in H^{4n}(G/\mathrm{TOP};\mathbf{Z}_{(2)}) \tag{3.4}$$

be the cohomology classes constructed in [RS] and [MS] respectively. (An alternative set of classes $K_n\in H^{4n}(G/\mathrm{TOP};\mathbf{Z}_{(2)})$ was defined in [Mi] but these classes are not suitable for our purpose; cf. [M_2].)

The k_{4n-2} are primitive; the L_n are not. But $1+8\Sigma L_n$ is a genus, and we set

$$l_n=\frac{1}{8n}s_n(8L_1,8L_2,\ldots,8L_n)$$

where s_n denotes the Newton polynomial. Then l_n is a $\mathbf{Z}_{(2)}$ integral polynomial in $L_1,\ldots,L_n$ and defines a primitive cohomology class in $H^{4n}(G/\mathrm{TOP};\mathbf{Z}_{(2)})$. Moreover, the classes k_{4n-2} and l_n give rise to a map of H-spaces

$$G/\mathrm{TOP}\to\prod_{n\geq 1}K(\mathbf{Z}/2,4n-2)\times K(\mathbf{Z}_{(2)},4n) \tag{3.5}$$

which is a 2-local equivalence.

Let $\pi\colon SG\to G/\mathrm{TOP}$ be the natural map. It is completely described at 2 by the classes $\pi^*(k_{4n-2})$ and $\pi^*(l_n)$ which were calculated in [BMM] and [M_2] respectively. From [BMM] we have

THEOREM 3.6. *$\pi^*(k_{4n-2})=0$ unless $n=2^i$ (in which case it is not 0).*

We need some preparational remarks before we can state the result for $\pi^*(l_n)$. Basic to our description is the following commutative diagram

$$\begin{array}{ccccc}
J^{\oplus} & \xrightarrow{\hat{A}} & SG_{(2)} & \xrightarrow{\hat{e}} & J^{\otimes} \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\
BSO^{\oplus}_{(2)} & \xrightarrow{A} & (G/O)_{(2)} & \xrightarrow{e} & BSO^{\otimes}_{(2)} \\
\downarrow{\scriptstyle \psi^3-1} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \psi^3/1} \\
BSO^{\oplus}_{(2)} & \xrightarrow{Id} & BSO^{\oplus}_{(2)} & \xrightarrow{\rho^3_{\mathbf{R}}} & BSO^{\otimes}_{(2)}
\end{array} \tag{3.7}$$

The columns are all fibrations of infinite loop spaces; $BSO^{\oplus}$ and $BSO^{\otimes}$ denote the space BSO with its two natural infinite loop space structures; the maps e, $\hat{e}$ and $\rho^3_{\mathbf{R}}$ and all vertical maps in 3.7 are infinite loop maps [MST]. The maps A and $\hat{A}$ are implied by the affirmed Adams conjecture, but they are not even H-maps. The composites $e \circ A$ and $\hat{e} \circ \hat{A}$ are however infinite loop maps since, for example, $e \circ A = \rho^3_{\mathbf{R}}$.

The common homotopy fibre of e and $\hat{e}$ is the space usually denoted Cok J, and since $\rho^3_{\mathbf{R}}$ is a 2-local equivalence we have homotopy equivalences

$$SG_{(2)} \cong J^{\oplus} \times \text{Cok } J$$

$$(G/O)_{(2)} \cong BSO^{\oplus}_{(2)} \times \text{Cok } J$$

Next, we need some notations and results from [A]. Given an arbitrary space X, we let $k: X[i, \infty] \to X$ be the fibration such that $k: \pi_j(X[i, \infty]) \to \pi_j(X)$ is an isomorphism for $j \geq i$ and $\pi_j(X[i, \infty]) = 0$ for $j < i$. In this notation $B^{8i}(BSO^{\oplus}) = BSO[8i+2, \infty]$ and $B^{2i}(BU^{\oplus}) = BU[2i+2, \infty]$.

Adams constructs 2-local cohomology classes

$$ch_{i,n} \in H^{2i+2n}(BU[2i, \infty]; \mathbf{Z}_{(2)})$$

with rational reduction $2^n k^*(ch_{i+n})$ and $\mathbf{Z}/2$ reduction $\chi(Sq^{2n})(u_{2i})$ where u_{2i} is the bottom cohomology class. They are stable in the sense that $ch_{i,n}$ and $ch_{i-1,n}$ are connected by the double suspension, $\sigma^2(ch_{i,n}) = ch_{i-1,n}$.

Complexication defines a map

$$C: BSO^{\oplus} \to BSU^{\oplus} = BU[4, \infty]$$

and we have

THEOREM 3.9. *The cohomology class* $\pi^*(l_n)$ *is the composition*

$$SG \to (G/O)_{(2)} \xrightarrow{e} BSO^{\otimes}_{(2)} \xrightarrow{(\rho^3_{\mathbf{R}})^{-1}} BSO^{\oplus}_{(2)} \xrightarrow{C} BSU \xrightarrow{ch_{2,2n-2}} K(\mathbf{Z}_{(2)}, 4n)$$

Proof. This is proved in $[\mathrm{M}_2]$ based on previous work in $[MM_1]$. The proof used information on the Bockstein spectral sequence of Cok J which was stated without proof in $[M_2]$, Lemma 3.5 (ii). Since the writeup of $[\mathrm{M}_2]$, J. P. May has published similar calculations on the Bockstein spectral sequence for B Cok J from which it is easy to deduce Lemma 3.5 of $[M_2]$. See [CLM], p. 191–203.

COROLLARY 3.10. *If either* $KO^0(P)_{(2)}$, $KSU^0(P)_{(2)}$ $KU^0(P)_{(2)}$ *or* $\bigoplus_{i\geq 1} H^{4i}(P; \mathbf{Z}_{(2)})$ *is torsion-free, then* Cok $J(P)_{(2)} \subset \bigoplus_{i\geq 2} H^{2^i-2}(P; \mathbf{Z}/2)$.

Proof. By 3.6 it is enough to show that $[P, SG] \to H^{4n}(P; \mathbf{Z}_{(2)})$ is trivial. The map factors through $KO^0(P)_{(2)}$ and $KSU^0(P)_{(2)}$ by 3.9. If $KU^0(P)_{(2)}$ is torsion-free, so is $KSU^0(P)_{(2)}$. Since $[P, SG]$ is a torsion group, if any of the listed groups is torsion-free we are done.

Remark 3.11. Theorem A of the introduction follows easily from 3.3 and 3.10.

The next theorem is one of the main ingredients of the proof of Theorem B of the introduction. The other ingredient is given in the next section.

THEOREM 3.12. *Let* ev: $S^2\Omega^2 SG \to SG$ *be the evaluation map and* f *the composite* $S^2\Omega^2(SG[3, \infty]) \to S^2\Omega^2 SG \to SG$; *then* $f^*\pi^*(l_n) = 0$.

We postpone the proof of 3.12 to discuss its applications. First, as the assignment $X \mapsto S^2\Omega^2 X[3, \infty]$ is a functor we have that Cok $J(P)_{(2)}$ is contained in the kernel of

$$\bigoplus_{i\geq 1} \tilde{H}^{2^i-2}(P; \mathbf{Z}/2) \oplus H^{4i}(P; \mathbf{Z}_{(2)}) \xrightarrow{f^*} \bigoplus_{i\geq 1} H^{4i}(S^2\Omega^2 P[3, \infty]; \mathbf{Z}_{(2)}) \tag{3.13}$$

To employ 3.13 usefully we observe

LEMMA 3.14. (i) *If* X *is the double suspension of a connected space,* $H^*(X; \mathbf{Z}_{(2)}) \xrightarrow{f^*} H^*(S^2\Omega^2 X[3, \infty]; \mathbf{Z}_{(2)})$ *is monic.*

(ii) *If* $\pi_1(X)=0$ *and* $\tilde{H}_i(X;\mathbf{Z}/2)=0$ *for* $i\le r$, *then* $H^i(X;\mathbf{Z}_{(2)})\xrightarrow{f^*} H^i(S^2\Omega^2X[3,\infty];\mathbf{Z}_{(2)})$ *is monic for* $i\le 2r$.

Proof. (i) Clearly $X[3,\infty]\to X$ is an equivalence, and if $X=S^2Y$, $S^2\Omega^2S^2Y\to S^2Y$ has a section: double suspend $Y\to\Omega^2S^2Y$. This proves (i).

(ii) By naturality we may assume X is a $2r$ dimensional CW complex. If $r=1$ the result is trivial to prove, so assume $r\ge 2$. If Y denotes the 2-localization of X, then Y is 2-connected, so $\Omega^2X[3,\infty]\to\Omega^2Y[3,\infty]$ is a 2-local equivalence. Hence it suffices to prove the result for Y. But Y is an r-connected, $2r$-complex, and hence a double suspension of a connected space by the Freudenthal suspension theorem. Lemma 3.14(i) applies.

COROLLARY 3.15. *Let* M *be an* n*-manifold whose connectivity is at least* $(n-1)/3$ *(e.g. metastable). Then* Cok $J(\mathring{M})_{(2)}\subset H^{2^i-2}(M;\mathbf{Z}/2)$ *for the unique i such that* $(n-1)/3<2^i-2<(2n+5)/3$.

Remark 3.16. If M is a 2-connected 7 or 8 manifold, 3.13 and 3.14 show Cok $J(\mathring{M})_{(2)}=0$.

Now both Cok $J(X)$ and $H^*(X)$ are defined and natural in the stable category. Our map Cok $J(X)_{(2)}\to\oplus H^{2^i-2}(X;\mathbf{Z}/2)\oplus H^{4i}(X;\mathbf{Z}_{(2)})$ is not stable. However, results of Madsen and Milgram [MM_1] give

COROLLARY 3.17. *If* $f\colon S^2X\to S^2Y$ *is a map. Then*

$$\begin{array}{ccc} \mathrm{Cok}\,J(Y)_{(2)} & \to & \oplus H^{2^i-2}(Y;\mathbf{Z}/2)\oplus H^{4i}(Y;\mathbf{Z}_{(2)}) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{Cok}\,J(X)_{(2)} & \to & \oplus H^{2^i-2}(X;\mathbf{Z}/2)\oplus H^{4i}(X;\mathbf{Z}_{(2)}) \end{array}$$

commutes.

Proof. This is just a reformulation of the fact that $B^2(G/\mathrm{TOP})_{(2)}$ is a product of Eilenberg-MacLane spaces.

Corollary 3.17 can profitably be applied to hypersurfaces. A hypersurface M is an n-manifold which can be embedded in S^{n+1} in a locally flat fashion. The sphere is then the union of two manifolds with boundary, W_1 and W_2. Moreover $\Sigma\mathring{M}\simeq\Sigma W_1\vee\Sigma W_2$ so we can use 3.17 and analyse the maps

$$\mathrm{Cok}\,J(W_i)_{(2)}\to\oplus H^{2^i-2}(W_i;\mathbf{Z}/2)\oplus H^{4i}(W_i;\mathbf{Z}_{(2)})$$

instead of the map for $\mathring{M}$.

As an example, RP^2 embeds in S^4, and hence $\Sigma^2 RP^2$ embeds in S^6. Let W_1 be a regular neighbourhood of $\Sigma^2 RP^2$; let W_2 be $S^6 - W_1$; and let $M = \partial W_1$. Then 3.17 and 3.14 imply Cok $J(\mathring{M})_{(2)} \subset \mathbf{Z}/2$ even though $\pi_1 M \neq 0$.

We conclude this section with

Proof of 3.12. The map

$$S^2\Omega^2 SG[3,\infty] \xrightarrow{f} SG \xrightarrow{\pi^*(l_n)} K(\mathbf{Z}_{(2)}, 4n)$$

can by 3.9 be identified with the double suspension of the composite

$$\Omega^2 SG[3,\infty] \to \Omega^2 J^{\otimes}[3,\infty] \to \Omega^2 BSO^{\otimes}_{(2)}[4,\infty] \to$$

$$\to \Omega^2 BSO^{\oplus}_{(2)}[4,\infty] \xrightarrow{\Omega^2 C} \Omega^2 BSU^{\oplus}_{(2)} \xrightarrow{\Omega^2 \mathrm{ch}_{2,2n-2}} \Omega^2 K(\mathbf{Z}_{(2)}, 4n)$$

followed by the evaluation ev: $S^2\Omega^2 K(\mathbf{Z}_{(2)}, 4n) \to K(\mathbf{Z}_{(2)}, 4n)$

When we make the identifications $\Omega^2 BSU^{\oplus} \cong BU^{\oplus}$ and $\Omega^2 BSO^{\oplus}[4,\infty] \cong SO/U$ promised us by Bott periodicity we have $\Omega^2 \mathrm{ch}_{2,2n-2} = \mathrm{ch}_{1,2n-2}$. Moreover,

$$\begin{array}{ccc} \Omega^2 BSO^{\oplus}[4,\infty] & \xrightarrow{\Omega^2 C} & \Omega^2(BSU^{\oplus}) \\ \| & & \| \\ SO/U & \xrightarrow{j} & BU \end{array}$$

commutes, where $SO/U \xrightarrow{j} BU \xrightarrow{r} BSO$ is a fibration.

Let $\varphi: (SO/U)_{(2)} \to (SO/U)_{(2)}$ be the map such that

$$\begin{array}{ccc} \Omega^2 BSO^{\oplus}_{(2)}[4,\infty] & \xrightarrow{\Omega^2(\psi^3-1)} & \Omega^2 BSO^{\oplus}_{(2)}[4,\infty] \\ \| & & \| \\ (SO/U)_{(2)} & \xrightarrow{\varphi} & (SO/U)_{(2)} \end{array}$$

commutes. Hence we have a fibration $\Omega^2 J^{\oplus}[3,\infty] \to (SO/U)_{(2)} \xrightarrow{\varphi} (SO/U)_{(2)}$.

The integral cohomology $H^*(SO/U; \mathbf{Z})$ is a polynomial algebra on generators, g_{4n-2}, in dimensions congruent to 2 modulo 4.

Moreover,

$$j^* s_{2n-1}(c_1, \ldots, c_{2n-1}) = 2g_{4n-2}$$

(see e.g. [DL]).

Hence

$$j^*(\mathrm{ch}_{1,2n-2}) = j^*(2^{2n-2}s_{2n-1}/(2n-1)!) = 2^{2n-1}n/(2n)!j^*(s_{2n-1})$$
$$= 2\alpha^{(n)-1}nuj^*(s_{2n-1}) = 2^{\alpha(n)}nug_{4n-2}$$

where $u \in \mathbf{Z}^*_{(2)}$ and $\alpha(n)$ is the number of ones in the dyadic expansion of n. We have here used that the 2-adic valuation of $(2n)!$ is $2n - \alpha(n)$.

We prove below that $2g_{4n-2} \in \mathrm{Image}(\varphi^*)$. It follows that $j^*(\mathrm{ch}_{1,2n-2}) \in \mathrm{Image}(\varphi^*)$. Using 3.7 it follows that the composition 3.19 is zero. This will prove the result.

Hence we need only understand $\varphi: (SO/U)_{(2)} \to (SO/U)_{(2)}$. Now

$$\begin{array}{ccc} (SO/U)_{(2)} \to & BU_{(2)} \cong & \Omega^2 BSU_{(2)} \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\Omega^2(\psi^3-1)} \\ (SO/U)_{(2)} \to & BU_{(2)} \cong & \Omega^2 BSU_{(2)} \end{array}$$

certainly commutes. Under the identification $BU_{(2)} \cong \Omega^2 BSU_{(2)}$, the map $\Omega^2(\psi^3 - 1)$ becomes $3(\psi^3 - 1): BU_{(2)} \to BU_{(2)}$. On primitive cohomology classes of dimension $2n$, $3(\psi^3 - 1)$ induces multiplication by $3(3^n - 1)$. Hence $\varphi^* g_{4n-2} = 3(3^{2n-1} - 1)g_{4n-2} = 2u_1 g_{4n-2}$, where $u_1 \in \mathbf{Z}^*_{(2)}$.

§4. The map $\varepsilon_0(P) \to \mathrm{Cok}\, J(P)$

Following Novikov [N] we next construct a homomorphism $\varphi: \pi_n^0(P, \partial P) \to \varepsilon^0(P)$, where $\pi_n^0(P, \partial P) \subset \pi_n(P, \partial P)$ are the elements of degree 0. Note, if $\partial P = S^{n-1}$ then $\pi_n^0(P, \partial P)$ is the image of $\pi_n(P)$ under the natural map $\pi_n(P) \to \pi_n(P, \partial P)$.

Let $\partial D^n = S^{n-1} = D_+^{n-1} \cup D_-^{n-1}$. Embed D^n in P^n such that $\partial P \cap D^n = D_-^{n-1}$. If we pinch D_+^{n-1} to a point, we get a map

$$\rho: (P, \partial P) \to (P \vee D^n, \partial P \vee S^{n-1}).$$

Moreover, there is a bundle map covering

$$\hat{\rho}\colon \nu_p \to \nu_p \vee \varepsilon^k$$

where $\nu_p \vee \varepsilon^k$ is the obvious bundle over $P \vee D^n$ and $k = \dim \nu_p$.

Given $\delta \in \pi_n^0(P, \partial P)$ we also use δ to denote a representative $\delta\colon (D^n, S^{n-1}) \to (P, \partial P)$. There is a unique bundle map $\hat{\delta}\colon \varepsilon^k \to \nu_p$ covering δ. The normal map $\varphi(\delta)$ is defined to be the composite

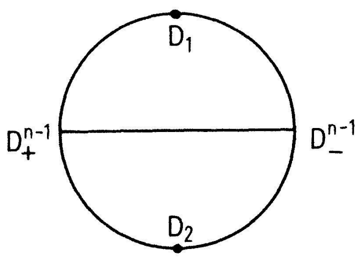
$$(P, \partial P) \xrightarrow{p} (P \vee D^n, \partial P \vee S^{n-1}) \xrightarrow{\mathrm{Id} \vee \hat{\delta}} (P, \partial P)$$

covered by the bundle map

$$\nu_p \xrightarrow{p} \nu_p \vee \varepsilon^k \xrightarrow{\mathrm{Id} \vee \hat{\delta}} \nu_p.$$

It is clear that $\varphi(\delta)$ is homotopic to the identity since there is an embedding $c\colon P \to P$ such that c is homotopic to the identity and $c(P)$ misses the disc we embedded in P. Hence we have a map $\varphi\colon \pi_n^0(P, \partial P) \to \varepsilon^0(P)$.

The following trick shows φ is a homomorphism. We divide D^n into two discs D_1 and D_2 as in the following picture



Now if $\delta_i \in \pi_n(P, \partial P) i = 1, 2$, we can assume without loss of generality that $\delta_i | D_i = *$. With this assumption

$$\begin{array}{ccccccccc}
P & \xrightarrow{\rho} & P \vee D^n & \xrightarrow{\mathrm{Id} \vee \delta_1} & P & \xrightarrow{\rho} & P \vee D^n & \xrightarrow{\mathrm{Id} \vee \delta_2} & P \\
\downarrow{\scriptstyle\rho} & & & & & & & & \downarrow{\scriptstyle\mathrm{Id}} \\
P \vee D^n & \xrightarrow{1 \vee f} & P^n \vee D^n \vee D^n & & \xrightarrow{\mathrm{Id} \vee \delta_1 \vee \delta_2} & & & & P
\end{array}$$

commutes, where $f\colon D^n \to D^n \vee D^n$ pinches $D_1 \cap D_2$ to a point., Thus

$$\varphi(\delta_2) \circ \varphi(\delta_1) = \varphi(\delta_1 + \delta_2)$$

as claimed.

Let $\Phi\colon \pi_n^0(P, \partial P) \to \varepsilon_0(P)$ denote φ composed with the homomorphism $\varepsilon^0(P) \to \varepsilon_0(P)$.

LEMMA 4.1. *If $P = \mathring{M}$, where M is closed, then Φ is onto.*

Proof. Let $f \in \varepsilon_0(\mathring{M})$. Corresponding to f there is a map $\bar{f}\colon M \to M$ since we may assume $f \mid \partial\mathring{M} = \mathrm{Id}$. Moreover, $f = \mathrm{Id}$ in $\varepsilon_0(\mathring{M})$ iff $\bar{f}$ is homotopic to the identity. But clearly $\bar{f}$ has the form $M \to M \vee S^n \to M$, where $\delta \in \pi_n(M)$ is constructed from the restriction to $\partial\mathring{M}$ of a homotopy $f_t\colon \mathring{M} \to \mathring{M}$ from f to Id. Hence f is equivalent in $\varepsilon_0(\mathring{M})$ to $\mathring{M} \xrightarrow{p} \mathring{M} \vee D^n \xrightarrow{\mathrm{Id} \vee \delta_1} \mathring{M}$ where δ_1 is an element of $\pi_n(\mathring{M})$ which hits δ (which can always be found since $\pi_n(\mathring{M}) \to \pi_n(M)$ is onto).

Recall from section 2 that the tangential normal invariant $N^t\colon \varepsilon^t(P) \to [P, G]$ induces a map $N^t\colon \varepsilon_t(P) \to \mathrm{Cok}\, J(P)$ which is a homomorphism on the subset $\varepsilon_0(P) \subset \varepsilon_t(P)$. Thus Lemma 4.1 and Corollary 2.9 gives

COROLLARY 4.2. *Suppose M is a closed, simply connected manifold of dimension at least 5. There is an exact sequence of abelian groups*

$$\pi_n(\mathring{M}) \xrightarrow{\psi} \mathrm{Cok}\, J(\mathring{M}) \to V(\mathring{M}) \to 0$$

where $\psi = N^t \circ \Phi$.

We proceed to give a convenient alternate description of ψ. Any $\delta \in \pi_n^0(P, \partial P)$ gives rise to a degree 0 tangential normal map $\delta\colon (D^n, S^{n-1}) \to (P, \partial P)$. From 2.3 we have a homomorphism

$$N^t\colon \pi_n^0(P, \partial P) \to [P, \Omega_0^\infty S^\infty]$$

where the addition in $[P, \Omega_0^\infty S^\infty]$ is induced from loop sum (denoted $*$).

The loop sum yields a transitive action of $[P, \Omega_0^\infty S^\infty]$ on $[P, SG]$, and we have:

LEMMA 4.4. *The diagram below is commutative.*

$$\begin{array}{ccc}
\pi_n^0(P, \partial P) \times \mathcal{S}^t(P) \xrightarrow{\varphi \times 1} \varepsilon^0(P) \times \mathcal{S}^t(P) & \longrightarrow & \mathcal{S}^t(P) \\
\Big\downarrow {\scriptstyle N^t \times N^t} & & \Big\downarrow {\scriptstyle N^t} \\
[P, \Omega_0^\infty S^\infty] \times [P, SG] & \longrightarrow & [P, SG]
\end{array}$$

Proof. Since elements in $\mathcal{S}^t(P)$ are represented by degree 1 maps, any element has a representative $f: Q \to P$ such that we can find embedded discs $D^n \subset Q, D^n \subset P$ with $\partial Q \cap D^n = D_-^{n-1}$; $\partial P \cap D^n = D_-^{n-1}$ such that $f|D^n$ is a homeomorphism. Then f commutes with the pinch maps and, for any $\delta \in \pi_n^0(P, \partial P)$ $\varphi(\delta)\cdot(f, \hat{f})$ is represented by $Q \xrightarrow{p} Q \vee D^n \xrightarrow{f\vee\delta} P$ with the obvious bundle map over it. Thus $N^t(\varphi(\delta)\cdot(f, \hat{f}))$ is represented by

$$S^{n+k} \to T(\nu_Q)/T(\nu_Q|\partial Q) \to T(\nu_Q)/T(\nu_Q|\partial Q) \bigvee T(\varepsilon^k)/T(\varepsilon^k|S^{n-1})$$

$$\xrightarrow{T(\hat{f})\vee T(\hat{\delta})} T(\nu_P)/T(\nu_P|\partial P) \bigvee T(\nu_P)/T(\nu_P|\partial P) \to T(\nu_P)/T(\nu_P|\partial P).$$

The S-dual of $T(\hat{\delta})$ represents $N^t(\delta)$ and the lemma follows since loop sum is adjoint to addition of stable maps.

COROLLARY 4.5. *The homomorphism* ψ *of* 4.2 *is the composite*

$$\pi_n(P) \to \pi_n^0(P, \partial P) \xrightarrow{N^t} [P, \Omega_0^\infty S^\infty] \xrightarrow{*[1]} 1[P, SG] \to \operatorname{Cok} J(P),$$

where $P = \mathring{M}$.

Remark. The bijection $*[1]$ is not necessarily a homomorphism. Nevertheless, it is induced by an equivalence of spaces, and hence induces a bijection from $[P, \Omega_0^\infty S^\infty]_{(p)}$ to $[P, SG]_{(p)}$. Hence we can prove ψ is onto the p-torsion in Cok $J(P)$ by proving N^t is onto the p-torsion in $[P, \Omega_0^\infty S^\infty]$.

We next recall the twisted suspension. Suppose (Y, B) is a pair of CW complexes and η is an oriented spherical fibration over Y with fibre S^{k-1}. The twisted suspension,

$$\Sigma_\eta: \pi_n(Y, B) \to \pi_{n+k}(T(\eta)/T(\eta|B)) \tag{4.6}$$

is defined as follows: $f: (D^n, S^{n-1}) \to (Y, B)$ is covered by a unique bundle map $\hat{f}: \varepsilon^k \to \eta$ and $\Sigma_\eta(f)$ is the induced map $T(\hat{f})$: $T(\varepsilon^k)/T(\varepsilon^k|S^{n-1}) \to T(\eta)/T(\eta|B)$ where we use the orientation to identify $T(\varepsilon^k)/T(\varepsilon^k|S^{n-1})$ with S^{n+k}.

In the special case $(Y, B) = (\mathring{M}, *)$, $* \in \partial\mathring{M}$, and $\eta = \nu_{\mathring{M}}$ we know that $T(\nu_{\mathring{M}})/T(\nu_{\mathring{M}}|*)$ is S-dual to $\mathring{M}$ and it is direct from the definitions to prove

Theorem 4.7. *The composition*

$$\pi_n(\mathring{M}, *) \xrightarrow{\Sigma_\eta} \pi_{n+k}(T(\nu_{\mathring{M}})/T(\nu_{\mathring{M}}|*)) \overset{D}{\cong} [\mathring{M}, \Omega_0^\infty S^\infty]$$

is equal to the tangential normal invariant N^t. Here D is the S-duality isomorphism.

(Note in 4.7 that $[\mathring{M}, \Omega_0^\infty S^\infty]$ denotes the homotopy set of based maps; however, as $\Omega_0^\infty S^\infty$ is an abelian H-space this is equal to the homotopy set of free maps).

In general it seems hard to calculate Σ_η and we shall only consider the case where Y is a suspension and B is a single point (the base point).

Let $Y = SX$ and consider the characteristic map for η, $X \to SG(k)$. Here $SG(k)$ is the space of oriented homotopy equivalences of S^{k-1} in the compact open topology, i.e. the structure monoid for η. Let $c: X \times S^{k-1} \to S^{k-1}$ be the adjointed map and let

$$h: S(X \wedge S^{k-1}) \to S^k$$

be its Hopf construction: $h(t, x, s) = (c(x, s), t)$ where t is the suspension coordinate, $x \in X$, and $s \in S^{k-1}$.

It is well-known that the cofibre of h is the Thom space of η so we get

$$T(\eta)/T(\eta|*) \cong S^{k+1}X.$$

Hence the twisted suspension in this case is a map

$$\Sigma_\eta: \pi_n(SX, *) \to \pi_{n+k}(S^{k+1}X, *)$$

but Σ_η is not always the ordinary suspension. Of course if η is trivial, Σ_η is just the Freudenthal suspension and in general Barratt and Hanks [H] have calculated Σ_η in terms of more classical operations in homotopy theory (cf. §5). For the moment however we will be satisfied with the following simple result.

LEMMA 4.8. *The composition*

$$\pi_{n-1}(X, *) \xrightarrow{\Sigma} \pi_n(SX, *) \xrightarrow{\Sigma_\eta} \pi_{n+k}(S^{k+1}X, *)$$

is the $(k+1)^{st}$ *suspension.*

Proof. Let $f: S^{n-1} \to X$ represent an arbitrary element of $\pi_{n-1}(X, *)$ and let $X \to SG(k)$ be the characteristic map for η. Their composite is the characteristic map for $\eta' = (\Sigma f)^*(\eta)$, so we have a commutative ladder of cofibrations

$$\begin{array}{ccccccc}
S(X \wedge S^{k-1}) & \xrightarrow{h} & S^k & \to & T(\eta) & \to & S^2(X \wedge S^{k-1}) \\
\uparrow{\scriptstyle \Sigma(f\wedge 1)} & & \uparrow{\scriptstyle Id} & & \uparrow & & \uparrow{\scriptstyle \Sigma^2(f\wedge 1)} \\
S(S^{n-1} \wedge S^{k-1}) & \xrightarrow{h'} & S^k & \to & T(\eta') & \to & S^2(S^{n-1} \wedge S^{k-1})
\end{array}$$

But the right hand vertical map is $\Sigma_\eta(\Sigma f)$ by definition.

We can interpret the composition in 4.3 as the map induced by the inclusion $X \to \Omega^{k+1}S^{k+1}X$ and we will pass to the limit $QX = \Omega^\infty S^\infty X$. We first consider the case where X itself is a suspension, say $X = SY$. The study of $X \to QX$ in homotopy becomes equivalent with the study of $\Omega SY \to QY$. We have (see also Williams [Will$_2$])

THEOREM 4.9. *Suppose $X = SY$ is $(q-1)$-connected. Then*

$$\pi_m(X) \oplus \mathbf{Z}[\tfrac{1}{2}] \to \pi_m^s(X) \oplus \mathbf{Z}[\tfrac{1}{2}]$$

is onto for $m \leq 3q-2$.

Proof. There are well-known "models" for $\Omega^k S^k Y$, $1 \leq k \leq \infty$ (see e.g. [May]). In particular there is a map

$$Y \cup (S^{k-1} \times_T Y \times Y) \to \Omega^k S^k Y$$

inducing isomorphism on homotopy in dimensions less than $3q-3$. (In the domain, we have made the identifications $(w, y, *) = (w, *, y) = y$). Thus in the same range we have a diagram of cofibrations

$$\begin{array}{ccccc}
Y & \to & \Omega SY & \xrightarrow{h_2} & Y \wedge Y \\
\| & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} \\
Y & \to & \Omega^\infty S^\infty Y & \xrightarrow{h_2} & S^\infty \times_T Y \wedge Y/RP^\infty
\end{array}$$

Calculations with the Serre spectral sequence show that the homotopy fibres of the two h_2 agree through dimension $3q-3$. Thus it suffices to show that i' induces a surjection in homotopy in the stated range.

First note by Freudenthal's suspension theorem that $\pi_*(Y\wedge Y)$ and $\pi_*(S^\infty\times_T Y\wedge Y/RP^\infty)$ are stable groups in our range. Thus it is enough to show that

$$Q(Y\wedge Y)\to Q(S^\infty\times_T Y\wedge Y/RP^\infty)$$

has a section in the p-local category when p is odd. The section is given as follows. The cofibration

$$RP^\infty\to S^\infty\times_T Y\wedge Y\to S^\infty\times_T Y\wedge Y/RP^\infty$$

stably splits to give a map from $Q(S^\infty\times_T Y\wedge Y/RP^\infty)$ to $Q(S^\infty\times_T Y\wedge Y)$. The transfer gives a map $Q(S^\infty\times_T Y\wedge Y)\to Q(S^\infty\times Y\wedge Y)\simeq Q(Y\wedge Y)$.

THEOREM 4.10. *If M is a closed simply connected manifold such that $M_{(p)}$ is c-connected for $c\geq(n+1)/3$, then if $n\geq 5$*

$$V(\mathring{M})_{(p)}=\{0\} \text{ for } p \text{ an odd prime.}$$

Proof. Theorem 4.7, Lemma 4.8, Corollary 4.5 and Corollary 4.2 reduce the problem to showing that $\pi_{n-1}(X)\to\pi^s_{n-1}(X)$ is onto, when $\mathring{M}_{(p)}=\Sigma X$. Since $\mathring{M}_{(p)}=\Sigma^2 Y$, Theorem 4.9 applies to $X=\Sigma Y$; $m=n-1$; $q=c$.

Remark 4.11. Theorem B now follows easily from 4.10 and 3.15.

We next examine the inclusions $\theta(M)\subset\theta(\mathring{M})$ for closed manifolds M.

THEOREM 4.12. *Let M be a closed, simply-connected manifold of dimension at least 5. Then, if the normal bundle of M is fibre homotopically trivial, $\theta(M)=\theta(\mathring{M})$.*

Proof. It is easy to see that $\theta(M)=\theta(\mathring{M})$ iff given any element in $[\mathring{M}, SG]$ it comes from an element in $[M, SG]$ on which the surgery obstruction is zero.

Since the normal bundle of M is fibre homotopically trivial, the top cell of M stably splits off, so $[M, SG]\to[\mathring{M}, SG]$ is onto. If M is odd dimensional there is no surgery obstruction so we are done. If the dimension of M is $4r$, the Hirzebruch signature formula shows that the obstruction is again zero.

If the dimension of M is $4r-2$, Sullivan's formula for the surgery obstruction (e.g. [BMM], (2.6)) and the fact that M has vanishing Wu classes, shows that the surgery obstruction is zero unless the $(4r-2)$-Kervaire class $k_{4r-2}\in$

$H^{4r-2}(G/\mathrm{TOP}; \mathbf{Z}/2)$, pulls back non-zero to M under the map $M \to SG$. By 3.6 this can happen only if $r = 2^i$.

So suppose we have our map $M \to SG$ pulling k_{4r-2} back non-zero. If there is a map $S^{4r-2} \to SG$ pulling k_{4r-2} back non-zero, then it is easy to change our map and get a new map $M \to SG$ pulling k_{4r-2} back to zero and still giving our element in $[\mathring{M}, SG]$. We finish the proof by showing

Claim. There exists an element of Arf invariant 1 in $\pi_n^s(S^0)$ iff there exists a manifold M^n with fibre homotopically trivial normal bundle and a map $M \to SG$ pulling k_n back non-zero.

Proof of Claim. It follows easily from work of Brown [Bro] that there is an n-sphere of Arf invariant 1 iff there is a framed n-manifold of Arf invariant 1. Hence there is an element of Arf invariant 1 iff k_n evaluates non-zero on the image of $\pi_n^s(M)$ in $H_n(SG; \mathbf{Z}/2)$.

If we have an element of Arf invariant 1 in $\pi_n^s(S^0)$, $M = S^n$ will do. For the converse, suppose we have M and a map $M \to SG$. Then we get a map $\Sigma^s M \to \Sigma^s SG$ which pulls back the s-fold suspension of k_n non-zero. But, since M has fibre homotopically trivial normal bundle, for s large enough we have a map $S^{n+s} \to \Sigma^s M$ such that the composite $S^{n+s} \to \Sigma^s SG$ pulls k_n back non-zero.

Remark. The result does not require $\pi_1 M = \{0\}$. One can use the formulas in [*TW*] or [W_4] with the proof above.

Here is an example to show that the inequality can be strict. Let $M = HP^2 \times S^{30}$. By Corollaries 3.3 and 3.10, $|\theta(\mathring{M})| \le 2$ and the exotic candidate is given by the map

$$\mathring{M} \to M \to S^{30} \xrightarrow{k_{30}} SG.$$

We get a tangential homotopy equivalence $f\colon \mathring{N} \to \mathring{M}$ and an almost tangential homotopy equivalence $f\colon N \to M$. Using Sullivan's formula for the surgery obstruction, we see that $N(f)\colon M \to G/\mathrm{TOP}$ must pull k_{38} back non-zero. But k_{38} comes from $H^{38}(\mathrm{BTOP}; \mathbf{Z}/2)$ ([BMM]) so N and M are not tangentially homotopy equivalent at all. Hence $|\theta(\mathring{M})| = 2$, but $|\theta(M)| = 1$.

In principle, Theorem 4.7 can also be applied to reach conclusions about $V(\mathring{M})_{(2)}$ although the calculations become much harder. In particular one would have to compute the composite

$$k_i : \pi_n(\mathring{M}) \xrightarrow{N_t} [\mathring{M}, \Omega_0^\infty S^\infty] \to [\mathring{M}, SG] \xrightarrow{k_{2^i-2}} H^{2^i-2}(\mathring{M}; \mathbb{Z}/2)$$

If ν is an r-dimensional bundle, stably equivalent to the normal bundle of $\mathring{M}$, this problem is equivalent, via 4.7 to computing

$$\Sigma_\nu\colon \pi_n(\mathring{M}) \to \pi_{n+r}(T(\nu)/T(\nu \mid *))$$

$$\hat{k}_i\colon \pi_{n+r}(T(\nu)/T(\nu \mid *)) \to \pi^s_{n+r}(T(\nu)/T(\nu \mid *)) \xrightarrow{D} [\mathring{M}, \Omega_0^\infty S^\infty]$$

$$\to [M, SG] \to H^{2^i-2}(\mathring{M}; \mathbf{Z}/2) \to H_{n-2^i+2}(\mathring{M}; \mathbf{Z}/2)$$

since $k_i = \hat{k}_\iota \circ \Sigma_\nu$.

Under favourable conditions we can extend the domain of definition of $\hat{k}_i$ (and k_ι) and prove naturality results: this will aid our calculations.

Let X be a complex and ν an r-dimensional bundle over X. We assume $T(\nu)/T(\nu \mid *)$ has an $(n+r)$-dual, that is, there exists a complex K and a (stable) duality map (see e.g. [B])

$$\theta\colon T(\nu)/T(\nu \mid *) \wedge K \to S^{n+r}$$

(K certainly exists as a stable object–we require an honest complex). Define $\hat{k}_i$ to be the composition

$$\begin{aligned} \hat{k}_i\colon \pi_{n+r}(T(\nu)/T(\nu \mid *)) &\to \pi^s_{n+r}(T(\nu)/T(\nu \mid *)) \to [K, \Omega_0^\infty S^\infty] \\ &\to [K, SG] \to H^{2^i-2}(K; \mathbf{Z}/2) \to H_{n+r-2^i+2}(T(\nu)/T(\nu \mid *); \mathbf{Z}/2) \\ &\xrightarrow{\text{Thom}} H_{n-2^i+2}(X; \mathbf{Z}/2) \end{aligned} \tag{4.13}$$

and $k_i = \hat{k}_i \circ \Sigma_\nu$.

Let $f\colon Y \to X$ be a map and let ξ and ν be spherical fibrations over Y and X respectively. Let $\hat{f}\colon \xi \to \nu$ be a map of spherical fibrations covering f. Then

$$\begin{array}{ccc} \pi_n(Y) & \xrightarrow{\Sigma_\xi} & \pi_{n+r}(T(\xi)/T(\xi \mid *)) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle T(\hat{f})_*} \\ \pi_n(X) & \xrightarrow{\Sigma_\nu} & \pi_{n+r}(T(\nu)/T(\nu \mid *)) \end{array}$$

commutes.

If $T(\nu)/T(\nu \mid *)$ and $T(\xi)/T(\xi \mid *)$ have $(n+r)$-duals K and L respectively, there is a stable map $K \to L$ dual to $T(\hat{f})$.

LEMMA 4.14. *If the stable map $K \to L$ is actually a map of complexes, then*

$$\begin{array}{ccc} \pi_{n+r}(T(\xi)/T(\xi \mid *)) & \xrightarrow{\hat{k}_i} & H_{n-(2^i-2)}(Y; \mathbf{Z}/2) \\ \Big\downarrow T(\hat{f})_* & & \Big\downarrow f_* \\ \pi_{n+r}(T(\nu)/T(\nu \mid *)) & \xrightarrow{\hat{k}_i} & H_{n-(2^i-2)}(X; \mathbf{Z}/2) \end{array}$$

commutes.

The conditions of 4.14 are satisfied in the situations of interest to us because of

LEMMA 4.15. *If $n \geq 2d(X) - c(X) - 1$, where $c(X)$ is the connectivity of X and $d(X)$ is the homotopy dimension of X, then k_i and $\hat{k}_i$ are defined for any spherical fibration ν. If, in addition $n \geq 2d(Y) - c(Y)$ then the hypotheses of Lemma 4.14 are satisfied.*

Proof. If $X = e^l \cup \cdots \cup e^{l+s}$, then $c(X) = l - 1$, $d(X) = l + s$. $T(\nu)/T(\nu \mid *) = e^{l+r} \cup \cdots \cup e^{l+s+r}$, and, as an object in the stable category,

$$K = e^{n-(l+s)} \cup \cdots \cup e^{n-l}.$$

If $2(n-(l+s)) - 1 \geq n - l - 1$ the Freudenthal suspension theorem guarantees an honest complex K. Moreover, any stable map from K to L is realized by an honest map.

Note for $X = \mathring{M}^n$ that $n \geq 2d(X) - c(X) - 1$ when M^n is metastable. Also, to define $\hat{k}_i$ (and k_i) we really only need ν to be a 2-local spherical fibration. Hence 4.14 and 4.15 apply to $X_{(2)}$ and $Y_{(2)}$.

COROLLARY 4.16. *Let $X = S^p$ and let ν be an r-dimensional trivial spherical fibration. Then*

$$\hat{k}_i \colon \pi_{n+r}(T(\nu)/T(\nu \mid *)) \to H_{n-(2^i-2)}(S^p; \mathbf{Z}/2)$$

is onto iff

i) $n = p + 2^i - 2$;
ii) *there is an element of Arf invariant* 1 *in* $\pi_q^s(S^0)$ *where* $q = 2^i - 2$;
iii) $p + r \geq q - 2i + \varepsilon_i$ *where* $\varepsilon_i = 2$ *if* $i \equiv 0(4)$, $\varepsilon_i = 3$ *if* $i \equiv 1(4)$ *and* $\varepsilon_i = 4$ *if* $i \equiv 2, 3(4)$.

Proof. Since $T(\nu)/T(\nu\mid *)=S^{p+r}$ we wish to calculate the map

$$\pi_{n+r}(S^{p+r})\xrightarrow{\cong}\pi^s_{n+r}(S^{p+r})\xrightarrow{\cong}[S^{n-p},\Omega_0^\infty S^\infty]\to[S^{n-p},SG]\to$$

$$\to H^{2^i-2}(S^{n-p};\mathbf{Z}/2)\xrightarrow{\cong}H_{n+r-(2^i-2)}(S^{p+r};\mathbf{Z}/2)\xrightarrow{\cong}H_{n-(2^i-2)}(S^p;\mathbf{Z}/2)$$

Conditions i) and ii) are equivalent to the assertion that the composite from $\pi^s_{m+r}(S^{p+r})$ is onto. Barratt and Mahowald [BaM] have proved that iii) is equivalent to the statement that there exists an element of Arf invariant 1 in $\pi_{n+r}(S^{p+r})$.

COROLLARY 4.17. *If ν is a spherical fibration over S^p,*

$$k_i:\pi_n(S^p)\to H_{n-(2^i-2)}(S^p;\mathbf{Z}/2)$$

is onto iff

i) $n=p+(2^i-2)$;
ii) *there is an element of Arf invariant* 1 *in* $\pi^s_q(S^0)$ *where* $q=2^i-2$;
iii) $p\geq q-2i+\varepsilon_i$.

Proof. If ν is trivial the result follows from 4.16 with $r=0$. If ν is not trivial, Corollary 5.2 below reduces the result to the trivial case.

§5. The Barratt–Hanks formula and highly connected manifolds

We let η denote an $(r-1)$-dimensional spherical fibration over a suspension, ΣX, with X connected. It is classified by a map $c: X\to SG(r)$: let $\hat\eta_1: X\times S^{r-1}\to S^{r-1}$ denote the adjoint of c. Define inductively

$$\hat\eta_i: X\times\cdots\times X\times S^{r-1}\to S^{r-1}$$

by $\hat\eta_i=\hat\eta_{i-1}\circ(\mathrm{Id}_{X\times\cdots\times X}\times\hat\eta_1)$.

For any map $f: A_1\times\cdots\times A_k\to B$, the Hopf construction gives a map $J(f):\Sigma(A_1\wedge\cdots\wedge A_k)\to\Sigma B$. In particular we have

$$J(\hat\eta_i):\Sigma(X\wedge\cdots\wedge X\wedge S^{r-1})\to S^r.$$

As we saw in §4, the Thom space of η can be identified with $S^r\cup_{J(\hat\eta 1)}$ cone $(\Sigma(X\wedge S^{r-1}))$, so

$$T(\eta)/T(\eta\mid *)\cong\Sigma^{r+1}X.$$

THEOREM 5.1 (Barratt-Hanks [H]). *The twisted suspension* (4.6)

$$\Sigma_\eta\colon \pi_N(\Sigma X) \to \pi_{N+r}(T(\eta)/T(\eta \mid *)) = \pi_{N+r}(\Sigma^r(\Sigma X))$$

for a connected CW complex X is given by the formula

$$\Sigma_\eta(\gamma) = \Sigma^r(\gamma) + \sum_{i=2}^{\infty} (\mathrm{Id}_{\Sigma X} \wedge J(\hat{\eta}_{i-1})) \circ \Sigma^r h_i(\gamma)$$

where $\gamma \in \pi_N(\Sigma X)$; Σ^r *is the ordinary r-fold suspension; and* $h_i(\gamma) \in \pi_N(\Sigma X^{[i]})$ *is the i'th Hopf invariant, where* $X^{[i]} = X \wedge \cdots \wedge X$.

Remark. The sum is finite since $h_i(\gamma) = 0$ for $N \leq ic(X) + 1$.

For the rest of this section we assume r is large compared with the dimension of X so that η in 5.1 is a stable spherical fibration. In the range of dimensions we consider $\pi_{N+r}(\Sigma^{r+1} X)$ will be the stable group $\pi_N^s(\Sigma X)$ and $\Sigma^r = \Sigma^\infty$.

COROLLARY 5.2. *Let* $\Sigma X = S^k$ *and suppose* $N \leq 3k - 3$. *Then the image of* Σ_η *is the same as the image of* Σ^r, *unless* $N = 2k - 1$, $k = 2, 4$ *or* 8, *and* $\eta\colon S^k \to BG$ *is not divisible by* 2 (*when it is not*).

Proof. Given $\gamma \in \pi_N(\Sigma X)$ with $N \leq 3(c(X)+1)$, the Freudenthal suspension theorem shows that $h_2(\gamma) = \Sigma x$ for $x \in \pi_{N-1}(X^{[2]})$. Also $h_i(\gamma) = 0$ for $i > 2$.

If the map

$$\mathrm{Id}_{\Sigma X} \wedge J(\hat{\eta}_1)\colon \Sigma^{r+1}(X \wedge X) \to \Sigma^{r+1} X$$

is the $(r+1)$-fold suspension of a map $f\colon X \wedge X \to X$ we have $\Sigma_\eta(\gamma) = \Sigma^r(\gamma) + \Sigma^{r+1}(f \circ x)$.

Hence Image $\Sigma_\eta \subseteq$ Image Σ^r, and Lemma 4.8 proves the reverse inclusion. In our case $X = S^{k-1}$, and $\mathrm{Id}_{\Sigma X} \wedge J(\hat{\eta}_1) \in \pi_{2k-2}^s(S^{k-1})$.

But Thomeir [T] has shown that

$$\pi_{2k-2}(S^{k-1}) \to \pi_{2k-2}^s(S^{k-1}) \quad \text{is onto,} \quad k-1 \neq 1, 3, 7.$$

The remaining cases are done by hand using 4.8.

We also want a version of 5.2 for $\Sigma X = S^k \cup_{ps} e^{k+1}$. If p is odd, 4.8 and 4.9 give enough for us so we concentrate on the case $p = 2$. The following lemma will be useful in the sequel.

LEMMA 5.3. (i) *The stablization map*

$\pi_{2k-2}(S^{k-2}) \to \pi_k^s$ *is onto if* $k \neq 1, 2, 3, 7$.

(ii) *The map is split unless* $k = 2^i - 2$; $k > 6$; *and there exists an element,* θ_i, *in* π_k^s *such that* θ_i *has* Arf *invariant* 1 *and* $2\theta_i = 0$.

(iii) *In this exceptional case,* $\pi_k^s \cong G \oplus \mathbf{Z}/2\mathbf{Z}$ *where* θ_i *generates* $\mathbf{Z}/2$. *There is a map* $G \to \pi_{2k-2}(S^{k-2})$ *such that* $G \to \pi_{2k-2}(S^{k-2}) \to \pi_k^s \cong G \oplus \mathbf{Z}/2\mathbf{Z}$ *is the obvious inclusion. There is an element* $x \in \pi_{2k-2}(S^{k-2})$ *which stabilizes to be* θ_i, *and we have that*

x *has order* 32, Σx *has order* 16, $\Sigma^2 x$ *has order* 8, $2\Sigma^3 x = [\iota, \iota]$.

Proof. The theorem is essentially due to Thomeier [T]. The reader can also check Mahowald's [M], especially tables 4.2 and 4.3.

THEOREM 5.4. *Let* M^{2n} *be an* $(n-1)$*-connected closed manifold of dimension* $2n \geq 6$. *Then* $|\theta(M)| = 1$.

Proof. We have $|\theta(M)| \leq |\theta(\mathring{M})| \leq |V(\mathring{M})|$, cf. 4.2. The manifold $\mathring{M}$ is a wedge of n spheres, so $[\mathring{M}, \Omega_0^\infty S^\infty] = \oplus [S^n, \Omega_0^\infty S^\infty]$, and Cok $J(\mathring{M}) = 0$ unless $n = 2^i - 2$. In the exceptional case, 4.7, 5.2 and 5.3 shows that $\pi_{2n}(\mathring{M}) \to$ Cok $J(\mathring{M})$ is onto, so $V(\mathring{M}) = 0$.

LEMMA 5.5. *Let* $\Sigma X = S^k \cup_{2^s} e^{k+1}$ *where* $k \geq 4$ *and* $k \neq 8$. *If* $k - 1 = 2^i - 2$ *is an exceptional case for Lemma* 5.3, *assume* $s \geq 4$. *Then, if* $N \leq 3k - 6$, *the image of* Σ_η *is the same as the image of* $\Sigma^r (= \Sigma^\infty)$.

Proof. As in the proof of 5.2, $h_2(\gamma) = \Sigma x$ for $x \in \pi_{N-1}(X^{[2]})$ and $h_i(\gamma) = 0$ for $i > 2$.

Now $X = \Sigma^{k-2}(S^1 \cup_{2^s} e^2) = \Sigma^{k-2} Y$. By Lemma 5.6 below, $J(\hat{\eta}_1)$: $\Sigma^r X \to S^r$ is $\Sigma^{r-(k-2)} f$ for a map f: $S^{2k-3} \cup_{2^s} e^{2k-2} \to S^{k-2}$. Then $\mathrm{Id}_{\Sigma X} \wedge J(\hat{\eta}_1)$ is the $(r+1)$-fold suspension of $1_Y \wedge f$ and, as before, we are done.

LEMMA 5.6. *The stabilization map*

$$[S^{2k-3} \cup_{2^s} e^{2k-2}, S^{k-2}] \to \{S^{2k-3} \cup_{2^s} e^{2k-2}, S^{k-2}\}$$

is onto unless $k \leq 4$; *or* $k = 8$; *or* $k - 1 = 2^i - 2$ *is an exceptional case of Lemma* 5.3 *and* $s \leq 3$.

Proof. Given a stable map γ: $S^{2k-3} \cup_{2^s} e^{2k-2} \to S^{k-2}$, we can restrict to S^{2k-3} and get a stable map α: $S^{2k-3} \to S^{k-2}$ of order at most 2^s.

By 5.3 we can find an honest map $a\colon S^{2k-4}\to S^{k-3}$ which suspends to α with the order of Σa at most 2^s. It is now easy to extend Σa to a map $b\colon S^{2k-3}\cup_{2^s} e^{2k-2}\to S^{k-2}$. Let β denote the corresponding stable map.

The β-γ can be obtained as a composite

$$\delta\colon S^{2k-3}\cup_{2^s} e^{2k-2}\to S^{2k-2}\to S^{k-2}.$$

By 5.3 again, δ comes from an honest map $d\colon S^{2k-2}\to S^{k-2}$. It is now easy to get a map $f\colon S^{2k-3}\cup_{2^s} e^{2k-2}\to S^{k-2}$ which suspends to γ.

LEMMA 5.7. *The stabilization map*

$$\pi_{2k}(S^{k-1}\cup_{2^s} e^k)\to \pi_{2k+1}(S^k\cup_{2^s} e^{k+1})$$

is onto unless $k\le 3$; *or* $k=7$; *or* $k=2^i-2$ *is an exceptional case of lemma* 5.3 *and* $s\le 3$.

Proof. Given a stable map $\gamma\colon S^{2k+1}\to S^k\cup_{2^s} e^{k+1}$ we get a stable map $\alpha\colon S^{2k+1}\to S^{k+1}$. By Lemma 5.3 this comes from a map $a\colon S^{2k-2}\to S^{k-2}$ such that Σa has order at most 2^s. Hence $S^{2k-1}\xrightarrow{\Sigma a} S^{k-1}\xrightarrow{2^s} S^{k-1}$ is null homotopic; i.e.

$$\begin{array}{ccc} S^{2k-1} & \xrightarrow{\Sigma a} & S^{k-1} \\ \cap & & \downarrow{\scriptstyle 2^s} \\ D^{2k} & \longrightarrow & S^{k-1} \end{array}$$

commutes.

Passing to cofibres gives a map $b\colon S^{2k}\to S^{k-1}\cup_{2^s} e^k$: let β denote the corresponding stable map.

The map $\beta-\gamma$ factors as a composite $S^{2k+1}\xrightarrow{\delta} S^k\to S^k\cup_{2^s} e^{k+1}$. By Lemma 5.3, δ comes from an honest map $d\colon S^{2k}\to S^{k-1}$ and it is now easy to finish.

Quite similar arguments give

LEMMA 5.8. *If* $k=2^i-2$ *is an exceptional case of* 5.3, *the stabilization map*

$$\pi_{2k+1}(S^k\cup_{2^s} e^{k+1})\to \pi^s_{2k+1}(S^k\cup_{2^s} e^{k+1})$$

is onto unless $s=1$ *or* 2.

COROLLARY 5.9. *The twisted suspension map*

$$\Sigma_\eta: \pi_{2k+1}(S^k \cup_{2^s} e^{k+1}) \to \pi^s_{2k+1}(S^k \cup_{2^s} e^{k+1})$$

is onto unless $k \leq 3$; *or* $k = 7$; *or* $k = 2^i - 2$ *is an exceptional case of Lemma* 5.3 *and* $s \leq 2$.

Proof. If $k-1 = 2^i - 2$ is an exceptional case of Lemma 5.3, then Lemma 5.7 and Lemma 4.8 combine to prove the result. Otherwise 5.5, 5.7 and 5.8 prove the result.

THEOREM 5.10. *Let* M^{2n+1} *be an* $(n-1)$*-connected manifold. Assume* $n \geq 2$ *and, if* $n = 2^i - 2$ *is an exceptional case of Lemma* 5.3 *assume* $H_n(M; \mathbf{Z})$ *has no* $\mathbf{Z}/2\mathbf{Z}$ *or* $\mathbf{Z}/4\mathbf{Z}$ *summands. Then* $|\theta(M)| = 1$.

Proof. If $n = 2$, Barden [Ba] gives the result. If $n = 3$, Corollary 3.2 and Remark 3.16 prove the result if $H_3(M; \mathbf{Z})$ has no 3-torsion. Wilkens [Wilk] proves $M = M_1 \# M_2$ where $H_3(M_1; \mathbf{Z})$ has no 3-torsion and $H_3(M_2; \mathbf{Z})$ is all 3-torsion. Then M_2 is triangulable [KS] and Wilkens proves $|\theta(M_2)| = 1$. Also, $|\theta(M_1)| = 1$ and since $\theta(M_1 \# M_2) \subseteq \theta(\mathring{M}_1) \times \theta(\mathring{M}_2)$ by Browder's splitting theorem, see e.g. [W_3], 12.1, the result follows (Note that Wilkens' different PL manifolds are topologically the same.)

If $n > 3$, M^{2n+1} is metastable, so Theorem B of §1 applies to prove the result unless n or $n+1$ is $2^i - 2$. The space $\mathring{M}$ is homotopy equivalent to a wedge of spheres and Moore spaces. Now Theorem 4.7; Lemmas 4.14, 4.15; Corollaries 5.2 and 5.9; and Lemma 5.3 prove that $V(\mathring{M}) = 0$. And hence the result.

Theorems 5.4 and 5.10 have counterparts in the smooth category. For example we have

THEOREM 5.11. *Let* $f: N^{2n+1} \to M^{2n+1}$ *be a homotopy equivalence between smooth* $(n-1)$*-connected manifolds. Suppose* $f|\mathring{N}$ *is covered by an orthogonal bundle map* $\nu_{\mathring{N}} \to \nu_{\mathring{M}}$. *Then* $f|\mathring{N}$ *is homotopic to a diffeomorphism unless* $n = 1, 3$ *or* 7, *or* $n = 2^i - 2$ *is an exceptional case of* 5.3 *and* $H_n(M; \mathbf{Z})$ *has a* $\mathbf{Z}/2\mathbf{Z}$ *or a* $\mathbf{Z}/4\mathbf{Z}$ *summand.*

(As above, $N^t: \pi_{2n+1}(\mathring{M}) \to [\mathring{M}, \Omega_0^\infty S^\infty]$ is surjective, and one can recopy sections 2 and 4 to the smooth category to show that $|\theta(\mathring{M})| \leq |\mathrm{Cok}\, N^t|$).

Of course 5.11 is contained implicitly in [W_2] but seeing that Wall's invariants are tangential homotopy invariants is non-trivial. See [Ar] for an early attempt in this direction.

Remark 5.12. If $k=2^i-2$ is an exceptional case of 5.3 and $s=1$ or 2 then 5.8 fails. Indeed, we prove below that the stabilization map

$$\Sigma^\infty\colon \pi_{2k+1}(S^k\cup_{2s}e^{k+1})\to\pi^s_{2k+1}(S^k\cup_{2s}e^{k+1}),\qquad s=1,2.$$

has cokernel $\mathbf{Z}/2$. Thus, in 5.10 if one removes the cohomological conditions in the exceptional case, $V(\mathring{M})\neq 0$. (Note: $k>6$ from 5.3).

The proof that Cok $\Sigma^\infty=\mathbf{Z}/2$ is similar to the proof of 4.9 in that it use the approximation to $\Omega^\infty S^\infty(X)$. First, one checks by cohomological methods that in dimensions $\leq 2k+1$, $S^\infty\times_T S^k\wedge S^k/RP^\infty$ is homotopy equivalent to the fibre F in

$$F\to K(\mathbf{Z},2k)\xrightarrow{2Sq^2}K(\mathbf{Z}/4,2k+2)$$

and (in the same range) that

$$S^\infty\times_T L_1\wedge L_1=K(\mathbf{Z}/2,2k)\times K(\mathbf{Z}/2,2k+1)=F_1$$

$$S^\infty\times_T L_2\wedge L_2=F_2.$$

Here $L_s=S^k\cup_{2s}e^{k+1}$ and F_2 is the fibre in

$$F_2\to K(\mathbf{Z}/2,2k)\xrightarrow{2Sq^2}K(\mathbf{Z}/4,2k+2)$$

Moreover, the natural inclusion of $S^\infty\times_T S^k\wedge S^k/RP^\infty$ in $S^\infty\times_T L_s\wedge L_s/RP^\infty$ can be identified (in our range) with the natural map from F to F_s. It follows that

$$\begin{aligned}\pi_{2k+1}(S^\infty\times_T L_s\wedge L_s/RP^\infty)&=\mathbf{Z}/2, &&s=1,\\ &=\mathbf{Z}/4, &&s=2,\end{aligned}$$

and in both cases

$$\pi_{2k+1}(S^\infty\times_T S^k\wedge S^k/RP^\infty)\to\pi_{2k+1}(S^\infty\times_T L_s\wedge L_s/RP^\infty)$$

is surjective.

As in the proof of 4.9 we have exact sequences

$$\begin{array}{ccccccc}\pi_{2k+1}(S^k)&\to&\pi^s_{2k+1}(S^k)&\to&\pi_{2k+1}(S^\infty\times_T S^k\wedge S^k/RP^\infty)&\xrightarrow{\partial}&\pi_{2k}(S^k)\\ \downarrow&&\downarrow&&\downarrow{\scriptstyle j}&&\downarrow\\ \pi_{2k+1}(L_s)&\to&\pi^s_{2k+1}(L_s)&\to&\pi_{2k+1}(S^\infty\times_T L_s\wedge L_s/RP^\infty)&\xrightarrow{\partial_s}&\pi_{2k}(L_s)\end{array}\qquad(5.13)$$

With the notation of 5.3 (iii) the generator of $\pi_{2k+1}(S^\infty \times_T S^k \wedge S^k/RP^\infty)$ maps to $2\Sigma^2 x$ in $\pi_{2k}(S^k)$, so for $s=1$, $\partial_s=0$ in 5.13. If $s=2$, j is an isomorphism, and $2\Sigma^2 x$ maps non-zero to $\pi_{2k}(L_s)$. Since $4\Sigma^2 x$ maps to zero, $\operatorname{Ker}\partial_s = \mathbf{Z}/2$ also in this case.

Remark. Lemma 5.8 also follows from these considerations.

§6. Hypersurfaces

In this section we study hypersurfaces of dimension at least 5, that is closed manifolds which admit a locally flat, co-dimension one embedding in a sphere. In fact, the entire section is a discussion of the

CONJECTURE 6.1. If two metastable hypersurfaces are homotopy equivalent then they are homeomorphic.

We begin with an observation from Morgan [Mo] which restricts the possible normal invariants.

LEMMA 6.2. *Let $f: M \to N$ be a homotopy equivalence between hypersurfaces. Its normal invariant $\eta(f) \in V(\mathring{M})$ is contained in the image of*

$$\hat{\Sigma}: \pi_{n+1}(\Sigma\mathring{M}) \to \pi_n^s(\mathring{M}) \xrightarrow{\cong} [\mathring{M}, \Omega_0^\infty S^\infty] \xrightarrow{\cong} [\mathring{M}, SG] \to V(\mathring{M}).$$

Proof. Let $\hat{f}: \nu_{\mathring{M}} \to \nu_{\mathring{N}}$ cover $f: \mathring{M} \to \mathring{N}$ where $\nu_{\mathring{M}}, \nu_{\mathring{N}}$ are the 1-dimensional trivial normal bundles. By definition, the normal invariant of $(f, \hat{f})$ is the S-dual of the composite

$$S^{n+1} \to T(\nu_{\mathring{N}})/T(\nu_{\mathring{N}} \mid \partial\mathring{N}) \to T(\nu_{\mathring{M}})/T(\nu_{\mathring{M}} \mid \partial\mathring{M})$$

Now, $T(\nu_{\mathring{M}})/T(\nu_{\mathring{M}} \mid \partial\mathring{N}) \simeq T(\nu_{\mathring{N}})/T(\nu_{\mathring{N}} \mid *) \vee S^{n+1}$ and $T(\nu_{\mathring{N}})/T(\nu_{\mathring{N}} \mid *) = \Sigma\mathring{N}$ with similar results for $T(\nu_{\mathring{M}}/T(\nu_{\mathring{M}} \mid \partial\mathring{M})$.

A hypersurface $M^n \subset S^{n+1}$ divides S^{n+1} into two parts, denoted N_1 and N_2. Let $K_i \subset N_i$ be the spine of N_i. It is a finite cell complex and $K_i \to N_i$ is a (simple) homotopy equivalence. Note that $c(M) = \min(c(K_1), c(K_2))$, where $c(\ )$ denotes connectivity. If M is metastable then

$$c(K_i) \geq 2d(K_i) - n + 1 > 1,\ 2(n+1) \geq 3(d(K_i) + 1), \tag{6.3}$$

where $d(K_i)$ = dimension of K_i.

Recall that a *trivial thickening* of a finite complex K is a simple homotopy equivalence $j: K \to N$ where $N \subseteq S^{n+1}$ is a codimension zero submanifold with boundary. The following standard result ([W_1]) will be used many times below.

THEOREM 6.4. *Let $j: K \to N$, $N \subseteq S^{n+1}$ be a trivial thickening of K and assume $n \geq 5$ and $c(K) \geq 2d(K) - n + 1$. Given any homotopy equivalence $f: K \to K$, there exists a homeomorphism $F: (N, \partial N) \to (N, \partial N)$ such that $F \circ j \simeq j \circ f$.*

Note in particular for a metastable hypersurface M^n, $S^{n+1} - M^n = N_1 \cup N_2$, that each self-homotopy equivalence of N_i can be realized by a homeomorphism up to homotopy.

We now fix a small disc $D_i^{n+1} \subset N_i$ with $D_1^{n+1} \cap M = D_2^{n+1} \cap M = D^n$ and we write $\mathring{N}_i = N_i - \mathring{D}_i$. We have

$$\begin{aligned} \Sigma \mathring{M} &\simeq \Sigma \mathring{N}_1 \vee \Sigma \mathring{N}_2 \\ \Sigma \mathring{M} &\simeq \mathring{N}_1/\mathring{M} \vee \mathring{N}_2 \mathring{M} \end{aligned} \tag{6.5}$$

The first homotopy equivalence in 6.5 is the sum of the inclusions, the second is the sum of the natural collapse maps $\mathring{N}_i/\mathring{M} \to \Sigma \mathring{M}$.

We combine the map in 6.2 with the collapse maps to get

$$\lambda_i: \pi_{n+1}(\mathring{N}_i/\mathring{M}) \to \mathring{V}(\mathring{M})$$

The next result is a corollary to work in [Will_1].

THEOREM 6.6. *With the assumptions in 6.3, for every element $\alpha_i \in \text{Image}(\lambda_i)$ there exists a homotopy equivalence $f_i: (N_i, M) \to (N_i, M)$ such that $f_i | N$ is homotopic to the identity and $\eta(f_i | \mathring{M}) = \alpha_i$.*

Proof. Let Emb (N, M) denote the set of concordance classes of Poincaré embeddings of (N, M) in S^{n+1} (see [Will_1]). Let $\varepsilon(N, M)$ denote the group of homotopy classes of (simple) homotopy equivalences of pairs. There is an obvious action of $\varepsilon(N, M)$ on Emb (N, M) and by acting on our given embedding we get a map $F: \varepsilon(N, M) \to \text{Emb}(N, M)$.

To each Poincaré embedding of (N, M) in S^{n+1} we get an element in the set of degree 1 classes in $\pi_{n+1}(N/M)$. This set is isomorphic to $\pi_{n+1}(\mathring{N}/\mathring{M})$ and a chase through the definitions involved show that

$$\begin{array}{ccc} \varepsilon(N, M) & \to & \varepsilon(M) \\ \downarrow & & \\ \text{Emb}(N, M) & & \Big\downarrow \\ \downarrow & & \\ \pi_{n+1}(\mathring{N}/\mathring{M}) & \to & \pi_{n+1}(\Sigma \mathring{M}) \end{array}$$

commutes, where the right hand vertical map is the unstable normal invariant from the proof of 6.2.

In [Will$_1$] it is shown that $\mathrm{Emb}(N, M) \to \pi_{n+1}(\mathring{N}/\mathring{M})$ is onto under our hypothesis. Hence, it suffices to show that F is onto.

A Poincaré embedding, T, consists of a map $g: M \to C$ such that $N \cup_M C$ is homotopy equivalent to S^{n+1}. By the splitting theorem ([W$_3$], 12.1) and the uniqueness of the trivial thickening of K we have a homotopy equivalence of triads

$$h: (S^{n+1}; N, S^{n+1} - \mathrm{Int}(N), M) \to (S^{n+1}; N, C, M)$$

and we may assume $h|N$ is homotopic to 1_N using 6.4. If $f = h|(N, M)$, then $F(f)$ is our Poincaré embedding T.

For a hypersurface M^n we write $\Sigma\theta(M^n)$ for the subset of $\theta(M^n)$ realized by hypersurfaces. Let $\Sigma V(\mathring{M})$ be the image of $\hat{\Sigma}$ in 6.2. Then $\Sigma\theta(M^n) \subseteq \Sigma V(\mathring{M})/\varepsilon(M)$ where $\varepsilon(M)$ is the group of homotopy automorphisms of M.

COROLLARY 6.7. *Suppose $M^n \subset S^{n+1}$ is a metastable hypersurface with $S^{n+1} - M^n = N_1 \cup N_2$. Suppose $n \geq 5$ and that there exists an integer q, necessarily unique of the form $2^t - 2$ with $c(M) < q \leq d(\mathring{M})$. If either $H^q(N_1; \mathbf{Z}/2)$ or $H^q(N_2; \mathbf{Z}/2)$ is trivial, then every hypersurface homotopy equivalent to M^n is homeomorphic to M^n.*

Remark. If there is no such q, Theorem B implies Conjecture 6.1.

Proof. Consider the diagram

$$\begin{array}{ccccccc}
\pi_{n+1}(\Sigma\mathring{M} & \xrightarrow{\Sigma} & \pi_n^s(\mathring{M}) & \xrightarrow{\cong} & [\mathring{M}, SG] \longrightarrow & & H^q(\mathring{M}) \to V(\mathring{M}) \\
\uparrow b & & \uparrow \cong & & \uparrow \cong & & \uparrow \cong \\
\bigoplus_{i=1}^{2} \pi_{n+1}(\mathring{N}_i/\mathring{M}) & \to & \bigoplus_{i=1}^{2} \pi_{n+1}^s(\mathring{N}_i/\mathring{M}) & \to & \bigoplus_{i=1}^{2} [\mathring{N}_i, SG] \to & & \bigoplus_{i=1}^{2} H^q(\mathring{N}_i)
\end{array}$$

It is classical that Image $(\Sigma \circ b) =$ Image (Σ). Thus in general if $\alpha \in \pi_{n+1}(\Sigma\mathring{M})$ goes to an element of the form $(x, 0)$ or $(0, y)$ in $H^q(\mathring{M}) = H^q(\mathring{N}_1) \oplus H^q(\mathring{N}_2)$ then it is easy to use 6.6 to find a self equivalence $f: M \to M$ with $\eta(f)$ being the image of α in $V(\mathring{M})$. With our assumptions $H^q(\mathring{M}) = H^q(\mathring{N}_1)$ or $H^q(\mathring{M}) = H^q(\mathring{N}_2)$ so $\Sigma V(\mathring{M})/\varepsilon(M) = 0$.

Remark 6.9. It is the twisting formula 2.5 which prevents us from proving 6.1 in general: even if each normal invariant of the form $(x, 0)$ or $(0, y)$ comes from a self-homotopy equivalence we cannot prove that (x, y) does. Note, if each automorphism of $H^q(\breve{M};\mathbf{Z}/2)$ is induced from a homeomorphism of M, then we can undo the twisting and $\Sigma V(\mathring{M})/\varepsilon(M)=0$ in these cases.

Remark. An example, shown to us by R. Schultz, shows that some connectivity is necessary in 6.1. From 7.1 we see there is a tangential homotopy equivalence $f: M \to S^2 \times S^6$ such that M is not homeomorphic to $S^2 \times S^6$. From Browder's embedding theorem [B$_1$] and some easy homotopy theory, M embeds in R^{11} with trivial normal bundle. Hence $S^2 \times M$ is a hypersurface in R^{11} and Schultz [S] shows how to see that $S^2 \times M$ is not homeomorphic to $S^2 \times S^2 \times S^6$. So $|\Sigma\theta(S^2 \times M)| \geq 2$, and in fact $|\Sigma\theta(S^2 \times M)| = 2$.

§7. Examples

In this section we calculate $\theta(M)$ for certain M. We give examples to show that $\theta(M)$ is not a homotopy invariant and that $\theta(M)$ may be arbitrarily large even for metastable hypersurfaces.

All manifolds will have fibre homotopically trivial normal bundles so $\theta(M) = \theta(\mathring{M})$ by 4.12.

EXAMPLE 7.1. $M = S^p \times S^q$, $2 \leq p \leq q$, $n = p+q \geq 5$. Then $|\theta(M)| = 1$ unless there exists an element of Arf invariant 1 in $\pi_q^s(S^0)$, $q = 2^i - 2$, and $p+1 < q - 2i + \varepsilon_i$. If $|\theta(M)| \neq 1$ then $|\theta(M)| = 2$.

Proof. It follows from 4.14 and 4.17 that $V(\mathring{M}) = 0$ unless there is an element of Arf invariant 1 in $\pi_q^s(S^0)$ and $p < q - 2i + \varepsilon_i$. In this case $V(\mathring{M}) = \mathbf{Z}/2$.

If $p+1 < q - 2i + \varepsilon_i$, then $\pi_{n+1}(\Sigma\mathring{M}) \to V(\mathring{M})$ is trivial (again by 4.17) so 6.2 gives $\theta(\mathring{M}) = V(\mathring{M})$.

Finally, if $p+1 = q - 2i + \varepsilon_i$, $\pi_{n+1}(\Sigma\mathring{M})$ maps onto $V(\mathring{M})$ and as M satisfies the hypothesis of 6.6, $|\theta(\mathring{M})| = 1$.

Note in 7.1 above, if $|\theta(M)| = 2$ and $f: N \to M$ is a tangential homotopy equivalence, then N is homeomorphic to M iff the q'th Kervaire class of f is trivial (written $K_q(f) = 0$).

We can sharpen 7.1 to

EXAMPLE 7.2. If M is any closed manifold homotopy equivalent to $S^p \times S^q$, $2 \leq p \leq q$, $p+q \leq 5$ then $\theta(M) = \theta(S^p \times S^q)$.

Proof. Since $V(\mathring{M})$ is a homotopy invariant by 2.9, the result is clear if $V(S^p \times S^q)^0) = 0$. Hence we may assume $V(\mathring{M}) = \mathbf{Z}/2$.

Suppose $|\theta(M)| = 1$, or, equivalently, there is a tangential self-equivalence

$f: M \to M$ with $\eta(f) \neq 0$. If $g: M \to S^p \times S^q$ is a homotopy equivalence, then 2.5 shows $\eta(gfg^{-1}) \neq 0$, so $|\theta(S^p \times S^q)| = 1$. Hence if $|\theta(S^p \times S^q)| = 2, |\theta(M)| = 2$.

In the remaining case, let $h: S^p \times S^q \to S^p \times S^q$ denote the exotic self-equivalence. By 2.10, $g^{-1}hg$ is tangential, and again we have $\eta(g^{-1}hg) \neq 0$, so $|\theta(M)| = 1$.

For simply connected M_1 and M_2 we have

$$\theta(M_1 \# M_2) \subseteqq \theta(M_1) \times \theta(M_2) \tag{7.3}$$

by the splitting theorem in [W$_3$], §12.1. Nevertheless we have

EXAMPLE 7.4. Let M denote the connected sum of r copies of $S^p \times S^q$, $2 \leq p \leq q$, $p+q \geqq 5$. Then $|\theta(M)| = |\theta(S^p \times S^q)|$.

Proof. If $|\theta(S^p \times S^q)| = 1$, 7.3 shows $|\theta(M)| = 1$, so we assume $|\theta(S^p \times S^q)| = 2$. Then, from 7.1 we recall that $V((S^p \times S^q)^0) = \mathbf{Z}/2$ and $\pi_{n+1}(\Sigma(S^p \times S^q)^0) \to V((S^p \times S^q)^0)$ is trivial. Since $(S^p \times S^q)^0$ is a tangential retract of $\mathring{M}$, Lemma 4.14 shows $V(\mathring{M}) = H^q(M; \mathbf{Z}/2)$ and $\pi_{n+1}(\Sigma \mathring{M}) \to V(\mathring{M})$ is trivial. Lemma 6.2 shows $\varepsilon(\mathring{M}) \xrightarrow{\eta} V(\mathring{M})$ is trivial so there is a $1-1$ correspondence between $\theta(M)$ and the orbit space $H^q(M; \mathbf{Z}/2)/\varepsilon(\mathring{M})$ where $h \in \varepsilon(\mathring{M})$ acts on $H^q(M; \mathbf{Z}/2)$ via x goes to $h^*(x)$.

Now M is the boundary of a trivial thickening of $K = \vee_1^r S^q (M = \partial(\#_1^r D^{p+1} \times S^q))$ and 6.4 shows that $\varepsilon(\mathring{M})$ maps onto $Gl(r; \mathbf{Z}/2)$ $(r = \dim H^q(M; \mathbf{Z}/2))$.

Hence $|\theta(M)| = 2$ and there are precisely two orbits: the zero vector and any non-zero vector.

EXAMPLE 7.5. Let M be a manifold homotpy equivalent to a connected sum of r copies of $S^p \times S^q$ where $2 \leq p \leq q$, $p+q \geq 5$ and $r \geq 2$. Assume M is not stably parallelizable. Then

$$|\theta(M)| = 1 \quad \text{if} \quad |\theta(S^p \times S^q)| = 1 \tag{i}$$

$$|\theta(M)| = 3 \quad \text{if} \quad |\theta(S^p \times S^q)| = 2 \tag{ii}$$

Proof. From 7.3, $|\theta(M)| = 1$ if $|\theta(S^p \times S^q)| = 1$, so we assume $|\theta(S^p \times S^q)| = 2$. Then $V(\mathring{M}) = H^q(\mathring{M}; \mathbf{Z}/2)$. Let N be the connected sum of r copies of $S^p \times S^q$. Then $\varepsilon(N) = \varepsilon_t(N)$ since N is stably parallelizable. Recall from the proof of 7.4 that $\eta: \varepsilon(N) \to V(\mathring{N})$ is trivial and that the natural map $\varepsilon(N) \to \operatorname{Aut}(H^q(N, \mathbf{Z}/2))$ defines a surjection onto $Gl(r; \mathbf{Z}/2)$.

Choose a specific homotopy equivalence $f: M \to N$. For technical reasons we want to assume $\eta(f)=0$. This is no loss of generality. Indeed, if $\eta(f)\neq 0$ choose a tangential homotopy equivalence $g: \mathring{M}_1 \to \mathring{M}$ with $\eta(g)=f^*(\eta(f))$. Then $\eta(f\circ g)=0$ and $\theta(\mathring{M}_1)\cong\theta(\mathring{M})$. By 4.12, $\theta(M_1)=\theta(M)$.

The equivalence $f: M\to N$ (with $\eta(f)=0$) induces via conjugation a map $c_f: \varepsilon_t(\mathring{M}) \to \varepsilon(N)$ and a map $(f^*)^{-1}: V(\mathring{M}) \to V(\mathring{N})$. The sets $\varepsilon_t(\mathring{M})$ and $\varepsilon_t(\mathring{N})$ act on $V(\mathring{M})$ and $V(\mathring{N})$, with orbits $\theta(\mathring{M})$ and $\theta(\mathring{N})$, cf. 2.9. From 2.5 we have

$$(f^*)^{-1}(\alpha\cdot x)=c_f(\alpha)\cdot(f^*)^{-1}(x), \tag{7.6}$$

$\alpha\in\varepsilon_t(\mathring{M})$, $x\in V(\mathring{M})$. Thus,

$$V(\mathring{M})/\varepsilon_t(\mathring{M})\cong V(\mathring{N})/\mathrm{Im}\,(c_f)\cong H^q(\mathring{N};\mathbf{Z}/2)/\varepsilon$$

where $\varepsilon\subset \mathrm{Gl}\,(r;\mathbf{Z}/2)$ is the image of

$$\bar{c}_f: \varepsilon_t(\mathring{M})\to \varepsilon(\mathring{N})\twoheadrightarrow \mathrm{Gl}\,(r;\mathbf{Z}/2)$$

Of course, $\varepsilon(\mathring{M})$ maps onto $\mathrm{Gl}\,(r;\mathbf{Z}/2)$ so 2.10 supplies the only restraint.

Since M is not stably parallelizable, $N(f)$ must be non-zero in $[\mathring{N}, G/\mathrm{TOP}]$. Since $H^*(\mathring{N};\mathbf{Z})$ is torsion free,

$$[\mathring{N}, G/\mathrm{TOP}]\subset[\mathring{N}, G/\mathrm{TOP}]\otimes\mathbf{Z}_{(2)}=H^q(\mathring{N};\mathbf{Z}/2)\oplus H^p(\mathring{N};R)$$

where $R=\mathbf{Z}/2$ if $p\equiv 2 \pmod 4$ and $R=\mathbf{Z}_{(2)}$ if $p\equiv 0 \pmod 4$.

The component of $N(f)$ in $H^q(\mathring{N};\mathbf{Z}/2)$ is $\eta(f)=0$, so $N(f)$ is a non-zero element of $H^p(\mathring{N};R)$. If $p\equiv 2 \pmod 4$, let $\delta=N(f)$. If $p\equiv 0\pmod 4$, let $\delta_1\in H^p(\mathring{N};\mathbf{Z}_{(2)})$ be the unique indivisible element with $s\delta_1=N(f)$ for some positive integer s. Let δ be the $\mathbf{Z}/2$-reduction of δ_1 and consider the homorphism $\rho: H^q(\mathring{N};\mathbf{Z}/2)\to\mathbf{Z}/2$ given by $\rho(x)=\langle x\cup\delta,[N]\rangle$.

The elements $\alpha\in Gl(r;\mathbf{Z}/2)$ which correspond to elements of $\varepsilon_t(\mathring{M})$ must satisfy $\rho(\alpha^*(X))=\rho(x)$. Thus there are at least 3 orbits under the action of $\varepsilon_t(\mathring{M})$ on $H^q(M;\mathbf{Z}/2)=V(\mathring{M})$ if $r\geq 2$:

$$\{0\};\{x\mid x\neq 0, \rho(x)=0\};\quad\text{and}\quad \{x\mid\rho(x)\neq 0\}$$

We leave to the reader the task of constructing the equivalences of N necessary to show that the above three sets do indeed form the orbits.

The "detection" result in the situation of 7.5 is the following: If $f_i: M_i\to$

M, $i=1,2$ are tangential homotopy equivalence then M_1 is homeomorphic to M_2 iff either

(i) $K_q(f_1)=K_q(f_2)=0$

or

(ii) $K_q(f_1)\neq 0$, $K_q(f_2)\neq 0$ and $\rho(K_q(f_1))=\rho(K_q(f_2))$.

Remark 7.7. Cappell's splitting theorem [C], Theorem 3 can be used to show $|\theta(M)|=1$ for any M the homotopy type of a connected sum of $S^1\times S^q$'s, $q\geq 4$.

Remark 7.8. The reader can easily show that for M^n the homotopy type of a connected sum of $S^p\times S^q$s, $|\theta(M)|=1$, 2 or 3 and even produce a detection result ($n\geq 5$). The only point is that, for a fixed n, there is at most one pair (p, q) such that $p+q=n$ and $|\theta(S^p\times S^q)|=2$.

To avoid leaving the impression that $|\theta(M)|$ must be small, we now construct a set of metastable hypersurfaces with arbitrary $|\theta(M)|$.

Let K_r be a wedge of r different Moore spaces $S^{18}\cup_{2^i} e^{19} i=1,2,\dots,r$ and let K_0 be a point. Up to homotopy, K_r embeds in S^{50} and we let M_r^{49} denote the boundary of the corresponding trivial thickening.

EXAMPLE 7.9. The manifold M_r^{49} is a metastable hypersurface and $|\theta(M_r)|=r+1$.

Proof. By construction there is a map $\rho\colon M_r=M\to K_r=K$. Let L denote a wedge of r 19-spheres and let $f\colon M\to L$ denote ρ followed by the collapse map. Note that

$$f_*\colon H_{19}(M;\mathbf{Z}/2)\to H_{19}(L;\mathbf{Z}/2)$$

is an isomorphism. Lemma 4.14 shows that

$$\begin{array}{ccc} \pi_{50}(\Sigma\mathring{M}) & \xrightarrow{\hat{k}_{50}} & \dot{H}_{19}(M;\mathbf{Z}/2\mathbf{Z}) \\ \downarrow{\scriptstyle(\Sigma f)_*} & & \downarrow{\scriptstyle f_*} \\ \pi_{50}(\Sigma L) & \xrightarrow{\hat{k}_{50}} & H_{19}(L;\mathbf{Z}/2\mathbf{Z}) \end{array}$$

commutes. Barratt and Mahowald (4.16) have shown the bottom $\hat{k}_{50}$ to be trivial: hence so is the top $\hat{k}_{50}$.

Therefore $V(\mathring{M})\cong H^{30}(\mathring{M};\mathbf{Z}/2)$ and $\theta(\mathring{M})$ is just the orbit space $H^{30}(\mathring{M};\mathbf{Z}/2)/\varepsilon(\mathring{M})$ with $h\in\varepsilon(\mathring{M})$ acting via x goes to $h^*(x)$. Once we compute the image of $\varepsilon(\mathring{M})$ in $\mathrm{Gl}(r;\mathbf{Z}/2)$ we are done.

Now $H_{18}(K_r; \mathbf{Z}) \cong \oplus_{i=1}^{r} \mathbf{Z}/2^i\mathbf{Z}$ and hence so is $H_{18}(M_r; \mathbf{Z})$. This decomposition gives rise to a natural filtration on $H_{19}(M; \mathbf{Z}/2)$: $F_aH_{19}(M; \mathbf{Z}/2)$ is the kernel of the a'th Bockstein from $H_{19}(M; \mathbf{Z}/2)$. We see

$$F_{a+1}/F_a \cong \mathbf{Z}/2 \quad \text{for} \quad 0 \le a \le r,$$

so we can choose a basis $x_1, \ldots, x_r \in H_{19}(M; \mathbf{Z}/2)$ such that x_{a+1} generates F_{a+1}/F_a. Any homotopy equivalence $g: M \to M$ gives rise to a lower triangular matrix

$$g_*: H_{19}(M; \mathbf{Z}/2) \to H_{19}(M; \mathbf{Z}/2)$$

with respect to the basis $\{x_1, \ldots, x_r\}$.

Using 6.4 it is easy to show that the image of $\varepsilon(\mathring{M})$ in $Gl(r; \mathbf{Z}/2)$ is the lower triangular matrices. Since $H^{30}(M; \mathbf{Z}/2)/\varepsilon(\mathring{M}) \cong H_{19}(M; \mathbf{Z}/2)/\varepsilon(\mathring{M})$ and since $|H_{19}(M; \mathbf{Z}/2)/\{\text{Lower triangular matrices}\}| = r+1$, we are done.

To formulate a "detection" result let $f: N \to M_r$ denote a tangential homotopy equivalence. From $N(f) \in [M_r, G/\mathrm{TOP}]$ we have a natural projection to $H^{30}(M; \mathbf{Z}/2)$. Since $H^{30}(M; \mathbf{Z}/2)$ is naturally isomorphic to $H_{19}(M; \mathbf{Z}/2)$ by Poincaré duality we consider the image of $N(g)$ in $H_{19}(M; \mathbf{Z}/2)$. Define $\mu(g)$ to be the image of $N(g)$ in the associated graded to the filtration on $H_{19}(M; \mathbf{Z}/2)$. Then, if $g_i: M_i \to M_r$ are tangential homotopy equivalences, $i = 1, 2$, M_1 is homeomorphic to M_2 iff $\mu(g_1) = \mu(g_2)$.

Let us conclude by considering manifolds which are homotopy equivalent to M_r but not necessarily stably parallelizable. Now $[M_r, G/\mathrm{TOP}] \cong H^{18}(M; \mathbf{Z}/2) \oplus H^{30}(M; \mathbf{Z}/2)$: given a homotopy equivalence $g: N \to M_r$ let $\overline{N(g)} \in H^{18}(M; \mathbf{Z}/2)$ denote the image of $N(g)$. The filtration on $H_{19}(M; \mathbf{Z}/2)$ gives rise to a filtration on $H^{18}(M; \mathbf{Z}/2)$. We say that the homotopy equivalence $g: N \to M_r$ has filtration s, if $\overline{N(g)}$ is in the s'th filtration but not the $(s-1)$'th.

EXAMPLE 7.10. Let $g: N \to M_r$ be a homotopy equivalence of filtration s. Then $|\theta(N)| = r + s + 1$.

Proof. First not that $\overline{N(g)} = N(g) \oplus x$ with $x \in H^{30}(M; \mathbf{Z})$. Actually, as in the proof of 7.5 we can assume that $\overline{N(g)} = N(g) \oplus 0$. But then c_g maps $\varepsilon_t(\mathring{N})$ into $\varepsilon_t(\mathring{M}_r)$ and the orbits correspond. Thus

$$\theta(N) \cong H^{30}(N; \mathbf{Z}/2)/\varepsilon_t(\mathring{N}) \cong H_{19}(N; \mathbf{Z}/2)/\varepsilon_t(\mathring{N})$$

and so we need only compute the image of $\varepsilon_t(\mathring{N})$ in $Gl(r; \mathbf{Z}/2\mathbf{Z})$. It is possible to choose our basis $\{x_1, \ldots, x_r\}$ for $H_{19}(N; \mathbf{Z}/2)$ such that $h \in \varepsilon_t(\mathring{N})$ iff $h_* \in Gl(r; \mathbf{Z}/2)$ is

i) lower triangular
ii) $h_*(x_{r+1-s}) = x_{r+1-s}$ (if $s = 0$ this is no condition)

(When g changes we will have to change the basis but we can always do so.)

Now $\vec{a} = \sum_{i=1}^r a_i x_i$ and $\vec{b} = \sum_{i=1}^r b_i x_i$ are in the same orbit iff the filtration of $\vec{a}$ is the filtration of $\vec{b}$ (say l), (so $a_l = b_l = 1$, $a_{l+1} = \ldots a_r = b_{l+1} = \ldots = b_r = 0$) and $a_{r+1-s} = b_{r+1-s}$, If $l \leq r+1-s$ this last is no condition so there are $1+(r-s)+2s$ orbits.

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