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## Poincaré duality groups of dimension two

BENO ECKMANN and HEINZ MÜLLER

In this paper we prove that 2-dimensional Poincaré duality groups with positive first Betti number  $\beta_1$  are surface groups. As a corollary it follows that a connected Poincaré 2-complex with  $\beta_1 > 0$  is homotopy equivalent to a closed surface, and so is any finite connected Poincaré 2-complex.

### 1. Statement of algebraic results

1.1. A Poincaré duality group of dimension  $n$ , in short  $PD^n$ -group, is a group  $G$  acting on  $\mathbf{Z}$  such that there are natural duality isomorphisms

$$H^k(G; A) \cong H_{n-k}(G; \mathbf{Z} \otimes A) \tag{1}$$

for all integers  $k$  and all  $G$ -modules  $A$  (where  $G$  acts diagonally on  $\mathbf{Z} \otimes A$ ); the isomorphisms (1) can be given by the cap-product  $e \cap -$  with an element  $e \in H_n(G; \mathbf{Z})$  called fundamental class. If (1) holds, the “formal dimension”  $n$  (= cohomology dimension of  $G$ ) and the  $G$ -module  $\mathbf{Z}$  ( $\cong H^n(G; \mathbf{Z}G)$ ) are determined by  $G$ . A  $PD^n$ -group  $G$  is called orientable or non-orientable according to whether  $\mathbf{Z}$  is a trivial  $G$ -module or not.

The fundamental group  $\pi_1(M^n)$  of a closed connected aspherical  $n$ -dimensional manifold is a  $PD^n$ -group. In particular, if  $M^2$  is a closed surface of genus  $\geq 1$ , then  $\pi_1(M^2)$  is a  $PD^2$ -group. We will call such a group  $\pi_1(M^2)$  a “surface group”; it admits a finite presentation of well-known canonical type. It has been conjectured that these surface groups are the only  $PD^2$ -groups. We will show that this is so except in a very special case which remains open.

1.2. From general arguments [5], [2] it is known that  $PD^n$ -groups are of type (FP); this means that there exists a  $\mathbf{Z}G$ -projective resolution of the trivial  $G$ -module  $\mathbf{Z}$ , of finite length and finitely generated over  $\mathbf{Z}G$ . In particular, a  $PD^n$ -group  $G$  is finitely generated, and its Betti numbers  $\beta_i(G)$  and the Euler characteristic  $\chi(G) = \sum_{i=0}^n (-1)^i \beta_i$  are defined. Our main result is

**THEOREM 1.** *Let  $G$  be a  $PD^2$ -group with  $\beta_1(G) > 0$ . Then  $G$  is a surface group.*

The condition  $\beta_1(G) > 0$  means, in the orientable case, that  $\beta_1(G)$  is an even integer  $\geq 2$ ; in the non-orientable case, an integer  $\geq 1$ . Thus  $\beta_1(G) > 0$  is equivalent to  $\chi(G) \leq 0$  (since  $\chi(G) = 2 - \beta_1(G)$  in the orientable,  $1 - \beta_1(G)$  in the non-orientable case). If  $G$  is non-orientable, it contains an orientable  $\text{PD}^2$ -group  $G_1$  as subgroup of index 2. By the multiplicative property of the Euler characteristic (which holds for groups of type (FP), cf. [6]) one has  $\chi(G_1) = 2\chi(G)$ ; hence  $\beta_1(G) > 0$  if and only if  $\beta_1(G_1) > 0$ .

1.3. A group  $G$  is said to be of type (FF) if it admits a  $\mathbf{Z}G$ -free resolution of finite length and finitely generated over  $\mathbf{Z}G$ . Obviously surface groups are of type (FF). It is not known whether there exist groups of type (FP) which are not of type (FF).

**COROLLARY 1.** *A  $\text{PD}^2$ -group  $G$  of type (FF) is a surface group.*

*Proof.* We first assume  $G$  orientable. Then the method of proof used by J. Cohen [7] is valid for any (FF)-resolution and shows that the assumption  $\beta_1(G) = 0$  (i.e.  $H_1(G; \mathbf{Z}) = 0$ ) leads to a contradiction. Hence  $\beta_1(G) > 0$ , and the assertion follows from Theorem 1.

If  $G$  is non-orientable, let  $G_1$  be the orientable subgroup of index 2; it is also of type (FF), and thus  $\beta_1(G_1) > 0$ . The Euler characteristic argument above then shows that  $\beta_1(G) > 0$ .

1.4. We thus see that the case  $\beta_1(G) = 0$  not covered by Theorem 1 is equivalent to the existence of a  $\text{PD}^2$ -group  $G$  not of type (FF), but of course of type (FP). We further note that, by Theorem 1, the condition  $\beta_1(G) > 0$  not only implies type (FF) but also finite presentability.

1.5. A further corollary concerns the “Nielsen conjecture” for surface groups.

**COROLLARY 2.** *Let  $G$  be a torsion-free group containing a surface group  $G_1$  as a subgroup of finite index. Then  $G$  itself is a surface group.*

*Proof.* Any torsion-free group  $G$  containing a  $\text{PD}^2$ -group  $G_1$  as subgroup of finite index is itself a  $\text{PD}^2$ -group (cf. [1], [2]). Since  $\beta_1(G_1) > 0$ , i.e.,  $\chi(G_1) \leq 0$ , the multiplicative property of the Euler characteristic,  $\chi(G_1) = |G : G_1| \chi(G)$ , yields  $\chi(G) \leq 0$ . Hence  $\beta_1(G) > 0$ , and the assertion follows from Theorem 1.

1.6. The relative analogue of a  $\text{PD}^n$ -group is a  $\text{PD}^n$ -pair, cf. Bieri-Eckmann [3]. A group pair  $(G; S_0, S_1, \dots, S_m)$  consists of a group  $G$  and a family of subgroups  $\mathfrak{S} = (S_0, S_1, \dots, S_m)$ ,  $m \geq 0$ ; it is called a  $\text{PD}^n$ -pair if for some  $G$ -action on  $\mathbf{Z}$  there are duality isomorphisms between the cohomology of  $G$  and the relative homology of  $(G; \mathfrak{S})$ , analogous to (1) and also given by the cap product

$e \cap -$  with a fundamental class  $e \in H_n(G, \mathfrak{S}; \mathbf{Z})$ . The duality is, of course, of exactly the same form as that of compact manifolds-with-boundary. Examples of  $PD^2$ -pairs are obtained by taking for  $G$  the fundamental group of a closed surface with  $m + 1$  discs removed ( $m \geq 0$ , and  $m \geq 1$  if the surface is the sphere) together with the family of infinite cyclic subgroups generated by the circles bounding the discs. These  $PD^2$ -pairs of groups are called “geometric”.

**THEOREM 2.** *All  $PD^2$ -pairs of groups are geometric.*

This result is actually a consequence of Corollary 1. Indeed it is shown in [3] that it is implied by the assertion that one-relator  $PD^2$ -groups are surface groups. Since one-relator  $PD^2$ -groups are of type  $(FF)$ , Corollary 1 tells that this is the case.

However, Theorem 2 will be used in the proof of Theorem 1 and therefore requires a direct proof.

1.7. The proof of Theorem 2 will be given in Section 4, of Theorem 1 in Section 5. In Section 3 we describe the procedure of proof and list some auxiliary results, in particular the “decomposition theorems for group pairs” (H. Müller [10]). Section 2 deals with the topological aspect.

## 2. Topological application: Poincaré 2-complexes

2.1. A Poincaré  $n$ -complex is a CW-complex  $X$  dominated by a finite complex and fulfilling Poincaré duality for arbitrary local coefficients, with respect to a dualizing  $\pi_1(X)$ -module  $\mathbf{Z}$  and a formal dimension  $n$ . We will always assume here that it is *connected*.

The study of Poincaré complexes was initiated by Wall in the 60-s. In [15] Wall proved, in particular, that if  $X$  is a Poincaré 2-complex with  $\pi_1(X)$  finite, then  $X$  is homotopy equivalent to  $S^2$  or  $\mathbf{RP}^2$ ; if  $\pi_1(X)$  is infinite, then  $X$  is aspherical, i.e., it is an Eilenberg-Mac Lane complex  $K(G, 1)$  for  $G = \pi_1(X)$ . In the latter case the investigation is thus reduced to the study of finitely presented  $PD^2$ -groups. Later J. Cohen [7] showed that if  $X$  is a *finite* Poincaré 2-complex with  $\beta_1(X) = 0$  then the conclusion is the same as for  $\pi_1(X)$  finite; and that a Poincaré 2-complex  $X$  with  $\beta_1(X) = 1$  or 2 is homotopy equivalent to the appropriate closed surface.

2.2. As a consequence of Theorem 1 we obtain

**COROLLARY 3.** *Let  $X$  be a Poincaré 2-complex with  $\beta_1(X) > 0$ . Then  $X$  is homotopy equivalent to a closed surface (of genus  $\geq 1$ ).*

Indeed, since  $\beta_1(X) > 0$  implies that  $\pi_1(X)$  is infinite,  $G = \pi_1(X)$  is a PD<sup>2</sup>-group with  $\beta_1(G) > 0$  and thus isomorphic to  $\pi_1(Y)$ , where  $Y$  is a closed surface of genus  $\geq 1$ . The isomorphism provides a homotopy equivalence between  $X = K(G, 1)$  and  $Y$ .

**COROLLARY 4.** *A finite Poincaré 2-complex  $X$  is homotopy equivalent to a closed surface.*

*Proof.* If  $\pi_1(X)$  is finite, one applies Wall's result mentioned above. If  $\pi_1(X) = G$  is infinite, then  $G$  is a PD<sup>2</sup>-group of type (FF), hence isomorphic to a surface group by Corollary 1. Thus  $X = K(G, 1)$  is homotopy equivalent to a closed surface.

2.3. Thus all Poincaré 2-complexes  $X$  are homotopy equivalent to closed surfaces, except possibly if (a)  $\pi_1(X)$  is infinite and  $\beta_1(X) = 0$ , and (b)  $X$  is not homotopy equivalent to a finite complex. Note that each of properties (a) and (b) implies the other. Except for finite presentability of  $G = \pi_1(X)$  this exceptional possibility is exactly the same as the case not covered by Theorem 1, cf. 1.4.

### 3. Splitting of groups and group pairs

3.1. A group  $G$  is said to *split over a subgroup  $H$*  if it is either ( $\alpha$ ) an amalgamated free product  $G = G_1 *_H G_2$ ,  $G_1 \neq H \neq G_2$  or ( $\beta$ ) an HNN-extension  $G = G_1 *_H$ . Cases where  $H$  is finitely generated or even finite will be of special importance.

If  $G$  is a PD<sup>2</sup>-group with  $\beta_1(G) > 0$  then  $G$  admits an infinite cyclic factor group (infinite cyclic groups will be denoted by  $C$  in the following, or by  $C(g)$  if we want to emphasize a generator  $g$ ). Since  $G$  is of type (FP), it is "almost finitely presented". By a theorem of Bieri–Strebel [4], any almost finitely presented group admitting a factor group  $C$  splits over a *finitely generated* group  $L$  (by a splitting ( $\beta$ )). Thus Theorem 1 is a consequence of

**THEOREM 1'.** *Let  $G$  be a PD<sup>2</sup>-group which splits over a finitely generated subgroup  $L$ . Then  $G$  is a surface group.*

If one confines attention to *finitely presented* PD<sup>2</sup>-groups only (e.g., in the context of Poincaré 2-complexes or of the Nielsen conjecture), the Bieri–Strebel argument can be replaced by a somewhat simpler one which is just a modification of Moldavanskii's method [9]; cf. Eckmann–Müller [8].

3.2. The proof of Theorem 1' will proceed as follows. By Strebel's theorem [13] the subgroup  $L$ , being of infinite index in  $G$ , is free. If the rank of  $L$  is  $> 1$ , the splitting can be changed so as to become a splitting of  $G$  over a subgroup of smaller rank. One is thus reduced to the case where  $L = C$  is infinite cyclic. Then the group pairs  $(G_1; C)$  and  $(G_2; C)$  in case  $(\alpha)$ , or  $(G_1; C, p^{-1}Cp)$  in case  $(\beta)$ , are  $PD^2$ -pairs; this follows from general results on  $PD^n$ -groups and -pairs (Bieri-Eckmann [3]). By our Theorem 2 these  $PD^2$ -pairs are geometric, which easily implies that  $G = G_1 *_C G_2$ , or  $G = G_1 *_C p$  respectively, is a surface group.

3.3. Both the reduction process above and the proof of Theorem 2 are based on "decomposition theorems for group pairs" (H. Müller [10]). For the convenience of the reader we summarize the appropriate definitions and those results which are needed.

In this context, a splitting of  $G$  is understood to be over a *finite* subgroup  $K$ . A group pair  $(G; S_1, S_2, \dots, S_m)$ ,  $m \geq 0$ , and a splitting  $(\alpha) G = G_1 *_K G_2$  or  $(\beta) G = G_1 *_K p$  are said to be *adapted to each other* if each  $S_j$ ,  $j = 1, \dots, m$  is conjugate to a subgroup of  $G_1$  or  $G_2$ . If for  $(G; S_1, S_2, \dots, S_m)$  such a splitting of  $G$  exists we simply say that the pair is adapted. If  $G$  is finitely generated, the pair  $(G; S_1, \dots, S_m)$  is adapted if and only if  $\bigcap_{j=1}^m N_j \neq 0$ , where  $N_j$  is the kernel of the restriction map  $\text{res}_j: H^1(G; \mathbf{Z}G) \rightarrow H^1(S_j; \mathbf{Z}G)$ . This is just a restatement of Swarup's relative version of Stallings' structure theorem for finitely generated groups with more than one end.

In the following we assume that  $(G; S_1, \dots, S_m)$  is an adapted pair and that  $G$  is finitely generated. With respect to the pair  $(G; S_1, \dots, S_m)$  a number  $n(T)$ , called *weight* of  $T$ , is associated with every subgroup  $T$  of  $G$ . The definition uses the restriction map

$$\text{res}: H^1(G; \mathbf{Z}G) \rightarrow H^1(T; \mathbf{Z}G).$$

For simplicity we only consider the case where  $T$  is finitely generated. We regard  $H^1(T; \mathbf{Z}T)$  as  $T$ -submodule of the (right)  $G$ -module  $H^1(T; \mathbf{Z}G)$  (the embedding is induced by the inclusion  $\mathbf{Z}T \rightarrow \mathbf{Z}G$ ). Since  $T$  is finitely generated, we have a decomposition (as abelian group)

$$H^1(T; \mathbf{Z}G) = \bigoplus_{x_i \in G/T} H^1(T; \mathbf{Z}T)x_i$$

(see, e.g., [2] Proposition 5.3).

**DEFINITION.** The weight  $n(T)$  is the minimal number of non-trivial components of  $\text{res}(c) \in \bigoplus_{x_i \in G/T} H^1(T; \mathbf{Z}T)x_i$  for all  $c \in \bigcap_{j=1}^m N_j$ ,  $c \neq 0$ .

3.4. For different values of  $n(T)$  various types of a *simultaneous* splitting of  $G$  and a graph-decomposition of  $T$  are obtained. We describe here only two special cases (Corollaire 2 and Corollaire 5 of [11]). In the statements the splitting  $G = G_1 * G_2$  or  $G = G_1 *_{e,p}$  written  $G * \langle p \rangle$ , is always meant to be adapted to the pair  $(G; S_1, \dots, S_m)$ .

**THEOREM A.** *Assume that  $T$  is torsion-free and  $n(T) = 1$ . Then we have one of the following cases*

- 1)  $G = G_1 * G_2, \quad T = T_1 * T_2, \quad T_1 \subset G_1, T_2 \subset G_2;$
- 2)  $G = G_1 * \langle p \rangle, \quad T = T_1 * pT_2p^{-1}, \quad T_1, T_2 \subset G_1;$
- 3)  $G = \langle p \rangle, \quad T = C(p), \quad S_1 = \dots = S_m = e \quad \text{or} \quad m = 0.$

**THEOREM B.** *Assume that  $G$  is torsion-free,  $T$  infinite cyclic and  $n(T) = 2$ . Then we have one of the following cases*

- 1)  $G = G_1 * G_2, \quad T = C(g_1 g_2), \quad e \neq g_i \in G_i, \quad i = 1, 2;$
- 2)  $G = G_1 * \langle p \rangle, \quad T = C(pg_1p^{-1} g_2), \quad e \neq g_1, g_2 \in G_1;$
- 3)  $G = \langle p \rangle, \quad T = C(p^2), \quad S_1 = \dots = S_m = e \quad \text{or} \quad m = 0.$

#### 4. Proof of Theorem 2

4.1. Let  $(G; S_0, S_1, \dots, S_m)$ ,  $m \geq 0$ , in short  $(G; \underline{S})$ , be a  $PD^2$ -pair.  $G$  acts on  $\mathbf{Z}$ , and there is a fundamental class  $e \in H_2(G, \underline{S}; \mathbf{Z})$  such that

$$e \cap - : H^k(G; A) \rightarrow H_{2-k}(G, \underline{S}; \mathbf{Z} \otimes A) \tag{2}$$

is an isomorphism for all  $k$  and  $A$ . The *geometric*  $PD^2$ -pairs (cf. 1.6) are as follows:

*Orientable case*

- (3)  $G$  is freely generated by  $t_1, \dots, t_m, x_1, y_1, \dots, x_g, y_g, \quad m + g > 0,$   
 $S_1, \dots, S_m$  are generated by conjugates to  $t_1, \dots, t_m$  and  $S_0$  is generated  
 by  $t_1 \cdots t_m \cdot \prod_{i=1}^g [x_i, y_i]$ .

*Non-orientable case*

- (4)  $G$  is freely generated by  $t_1, \dots, t_m, z_0, \dots, z_g, \quad m \geq 0, g \geq 0,$   
 $S_1, \dots, S_m$  are generated by conjugates to  $t_1, \dots, t_m$  and  $S_0$  is generated  
 by  $t_1 \cdots t_m \cdot \prod_{i=0}^g z_i^2$ .

4.2. By Theorem 4.2 and 9.3 of [3] we know that a PD<sup>2</sup>-pair  $(G; S_0, S_1, \dots, S_m)$  consists of a finitely generated free group  $G$  and a family  $\underline{S} = (S_0, S_1, \dots, S_m)$  of cyclic subgroups. Moreover, the fundamental class  $e \in H_2(G; \underline{S}; \mathbf{Z})$  determines fundamental classes  $e_i$  for the PD<sup>1</sup>-groups  $S_0, \dots, S_m$ , namely the components of  $\partial e \in H_1(\underline{S}; \mathbf{Z}) = \bigoplus_{i=0}^m H_1(S_i; \mathbf{Z})$ , where  $\partial$  is the connecting homomorphism in the exact homology sequence of  $G$  modulo  $\underline{S}$ . By [3], Theorem 2.1 one has the following commutative diagram

$$\begin{array}{ccc}
 0 \rightarrow H^1(G; \mathbf{Z}G) \xrightarrow{\{\text{res}_i\}} \bigoplus_{i=0}^m H^1(S_i; \mathbf{Z}G) \xrightarrow{\delta} H^2(G, \underline{S}; \mathbf{Z}G) \rightarrow 0 & & \\
 \cong \downarrow \{e_i \cap -\} & & \cong \downarrow (e \cap -) \\
 \bigoplus_{i=0}^m H_0(S_i; \mathbf{Z} \otimes \mathbf{Z}G) \xrightarrow{\text{cor}} H_0(G; \mathbf{Z} \otimes \mathbf{Z}G) & & (5) \\
 \cong \downarrow j & & \cong \downarrow \\
 \bigoplus_{i=0}^m (\mathbf{Z} \otimes_{S_i} \mathbf{Z}G) \xrightarrow{p} \mathbf{Z} & & 
 \end{array}$$

where the top row is exact and  $p(1 \otimes_{S_i} y) = 1 \cdot y$  for  $y \in G$ .

4.3. We now prove, by induction on the rank  $rk(G)$ , that  $(G; \underline{S})$  has a presentation (3) or (4) and thus is geometric.

If  $rk(G) = 1$  then  $\bigoplus_{i=0}^m (\mathbf{Z} \otimes_{S_i} \mathbf{Z}G)$  is free Abelian of rank 2, by (5). This is possible only if either  $m = 1$  and  $S_0 = S_1 = G$ ; or if  $m = 0$  and  $S_0 = C(a^2)$  where  $G = \langle a \rangle$ . Thus we either have a presentation (3) with  $m = 1, g = 0$ , or a presentation (4) with  $m = 0, g = 0$ .

If  $rk(G) \geq 2$  we put  $T = S_0$  and determine the weight  $n(T)$  with respect to the pair  $(G; S_1, \dots, S_m)$ , which is adapted by (5). We consider elements  $\text{res}_0(c)$ ,  $0 \neq c \in \bigcap_{j=1}^m N_j$  (i.e., elements  $(d, 0, \dots, 0) \in \text{im} \{\text{res}_i\}, d \neq 0$ ) and count the number of components of  $d$  in  $H^1(T; \mathbf{Z}G) = \bigoplus_{x_v \in G/T} H^1(T; \mathbf{Z}T)_{x_v}$ . From (5) we see that  $\text{im} \{\text{res}_i\} = \ker \delta = \ker pj\{e_i \cap -\}$ , and  $pj\{e_i \cap -\}$  restricted to any  $H^1(T; \mathbf{Z}T)_{x_v}$  is bijective. Thus the minimal number of components of elements  $d \neq 0$  is two, i.e., the weight of  $T = S_0$  is 2. By Theorem B we therefore have one of the two following cases:

- 1)  $G = G_1 * G_2; S_0 = C(g_1 g_2), e \neq g_i \in G_i, i = 1, 2$ , and the subgroups  $S_1, \dots, S_k$  are conjugate to subgroups of  $G_1$ , while  $S_{k+1}, \dots, S_m$  are conjugate to subgroups of  $G_2$ , for some  $k, 0 \leq k \leq m$ .
- 2)  $G = G_1 * \langle p \rangle; S_0 = C(p g_1 p^{-1} g_2), e \neq g_1, g_2 \in G_1$ , and  $S_1, \dots, S_m$  are conjugate to subgroups of  $G_1$ .



Since hypothesis and assertion are invariant under conjugation we may assume that  $S_1, \dots, S_m$  are actually subgroups of  $G_1$  or  $G_2$  respectively.

Case 1). We can write  $G$  as  $G = (G_1 * C(g_2)) *_{C(g_2)} G_2$ . The subgroups  $S_0 = C(g_1 g_2)$  and  $S_1, \dots, S_k$  are in  $G_1 * C(g_2)$ , and the  $S_{k+1}, \dots, S_m$  in  $G_2$ . If  $G_2 \neq C(g_2)$ , Theorem 8.1 of [3] tells that  $(G_2; C(g_2), S_{k+1}, \dots, S_m)$  is a  $PD^2$ -pair. We claim that this is also true if  $G_2 = C(g_2)$ ; namely, that pair is then  $(C(g_2); C(g_2), C(g_2))$ .

To prove this we note that quite generally, in Case 1), diagram (5) implies that  $\text{res}: H^1(G; \mathbf{Z}G) \rightarrow \bigoplus_{i=k+1}^m H^1(S_i; \mathbf{Z}G)$  is surjective, and so is  $\text{res}: H^1(G_2; \mathbf{Z}G_2) \rightarrow \bigoplus_{i=k+1}^m H^1(S_i; \mathbf{Z}G_2)$ . If  $G_2 = C(g_2)$ , then  $H^1(G_2; \mathbf{Z}G_2) = \mathbf{Z}$ , so this is possible only if  $k = m$ , or  $k = m - 1$  and  $S_m = G_2 = C(g_2)$ . Assume  $k = m$ ; then all subgroups  $S_1, \dots, S_m$  are in  $G_1$ , hence  $H^1(G, \underline{S}; \mathbf{Z}) \neq 0$ , since  $G = G_1 * C(g_2) = G_1 * C(g_1 g_2) = G_1 * S_0$ . However, for a  $PD^2$ -pair  $H^1(G, \underline{S}; \mathbf{Z}G) = 0$ , so  $k = m$  is not possible and we are left with  $k = m - 1$  and  $(G_2; C(g_2), S_{k+1}, \dots, S_m) = (C(g_2); C(g_2), C(g_2))$ , which is a  $PD^2$ -pair.

Thus  $(G_2; C(g_2), S_{k+1}, \dots, S_m)$  is a  $PD^2$ -pair, and so is  $(G_1; C(g_1), S_1, \dots, S_k)$ . By induction hypothesis they have presentations of the type (3) or (4). It follows immediately that  $(G; \underline{S})$  has a presentation (3) or (4): This is obvious if both above pairs have a presentation (3), or both a presentation (4). Otherwise one gets a presentation (4), i.e. non-orientable, by using transformations of the form

$$a^2[b, c] = \bar{a}^2 \bar{b}^2 \bar{c}^2; \quad \bar{a} = a^2 b c a^{-1}, \quad \bar{b} = a c^{-1} b^{-1} a^{-1} c a^{-1}, \quad \bar{c} = a c^{-1} \quad (6)$$

Case 2). Write  $G$  as  $G = (G_1 * C(a)) *_{C(ag_2^{-1}), p} G_2$  with  $p^{-1}(ag_2^{-1})p = g_1$ . The subgroups  $S_0 = C(a)$  and  $S_1, \dots, S_m$  are in  $G_1 * C(a)$ . By [3], Theorem 8.3,  $(G_1 * C(a); C(a), S_1, \dots, S_m, C(ag_2^{-1}), C(g_1))$  is a  $PD^2$ -pair. By the method used in Case 1) it follows that  $(G_1; S_1, \dots, S_m, C(g_1), C(g_2))$  is a  $PD^2$ -pair; the induction hypothesis tells that it has a presentation of the type (3) or (4). We may assume that this presentation is as follows.

$G_1$  is freely generated by  $t_0, t_1, \dots, t_m$  and some  $x_i, y_i$  (orientable case (3)) or some  $z_i$  (non-orientable case (4)); and  $S_i$  is conjugate to  $C(t_i)$ ,  $i = 1, \dots, m$ ,  $C(g_1)$  to  $C(t_0)$ , i.e.,  $g_1$  is conjugate to  $t_0$  or  $t_0^{-1}$ ; and  $g_2 = t_0 \cdots t_m r$  where  $r = \prod [x_j, y_j]$  or  $\prod z_j^2$  respectively.  $S_0$  is generated by  $p g_1 p^{-1} t_0 \cdots t_m r$ . By changing  $p$  if necessary we may assume  $g_1 = t_0^{\pm 1}$ . Using transformations of the form

$$p t p^{-1} t = \bar{p}^2 \bar{t}^2; \quad \bar{p} = p t p^{-1} t^{-1} p^{-1}, \quad \bar{t} = p t \quad (7)$$

and of the form (6), we get a presentation (3) or (4) for the pair  $(G; S_0, S_1, \dots, S_m)$ .

The passage from the two geometric pairs  $(G_1; \dots)$  and  $(G_2; \dots)$  to  $(G; \underline{S})$  in Case 1), or from  $(G_1; \dots)$  to  $(G; \underline{S})$  in Case 2) can, of course, be replaced by a geometric procedure on the corresponding surfaces-with-boundary.

## 5. Proof of Theorem 1'

5.1. We recall that surface groups have canonical presentations

$$G = \left\langle x_1, y_1, \dots, x_g, y_g \mid \prod_{j=1}^g [x_j, y_j] = 1 \right\rangle, \quad g \geq 1 \quad (8)$$

in the orientable, and

$$G = \left\langle z_0, \dots, z_g \mid \prod_{j=0}^g z_j^2 = 1 \right\rangle, \quad g \geq 1 \quad (9)$$

in the non-orientable case.

Let  $G$  be a  $PD^2$ -group which splits over a finitely generated group  $L$  as  $(\alpha)$   $G = G_1 *_L G_2$ ,  $G_1 \neq L \neq G_2$  or  $(\beta)$   $G = G_1 *_L p$ . Since  $L$  has infinite index in  $G$  it is free [13].

If  $rk(L) = 1$ ,  $L = C$ , we consider the pairs  $(G_1; C)$  and  $(G_2; C)$  in case  $(\alpha)$ , or  $(G_1; C, p^{-1}Cp)$  in case  $(\beta)$ . By [3], Theorem 8.1 and 8.3 these pairs are  $PD^2$ -pairs and hence geometric; they have presentations (3) or (4), and by amalgamation or HNN-extension these yield presentations of the form (8) or (9) (by using, if necessary, transformations (6) and (7)). Thus  $G$  is a surface group.

Of course, the appropriate surface can also be obtained geometrically from the surfaces-with-boundary corresponding to the group pairs.

5.2. If  $rk(L) \geq 2$ , we will obtain from Theorem A a new splitting of  $G$  over a subgroup  $M$  with  $rk(M) < rk(L)$ . This reduces the problem to the case  $rk(L) = 1$  above.

$(\alpha)$  Assume first that  $G = G_1 *_L G_2$ . We consider the Mayer-Vietoris sequence

$$\begin{aligned} \dots \rightarrow 0 \rightarrow H^1(G_1; \mathbf{Z}G) \oplus H^1(G_2; \mathbf{Z}G) \xrightarrow{(\text{res}_1, -\text{res}_2)} \\ H^1(L; \mathbf{Z}G) \xrightarrow{\delta} H^2(G; \mathbf{Z}G) \rightarrow \dots \end{aligned}$$

and show the following:

(10) If the weight of  $L$  with respect to both  $(G_1; \emptyset)$  and  $(G_2; \emptyset)$  is greater

than one, then  $H^1(L; \mathbf{Z}L) \cap \text{im}(\text{res}_1, -\text{res}_2) = 0$ . (Here we consider  $H^1(L; \mathbf{Z}L)$  as submodule of  $H^1(L; \mathbf{Z}G)$ .)

*Proof.* Let  $C_L$  denote  $H^1(L; \mathbf{Z}L)$  and  $C_i = H^1(G_i; \mathbf{Z}G_i)$ ,  $i = 1, 2$ . Choose sets  $\{x_i; i \in I\}$  and  $\{y_j; j \in J\}$  of representatives of the (right) cosets  $\in G_1/L$  and  $G_2/L$  (both sets containing  $e$ ). We then have the following sets of representatives:

$$\Sigma_1 = \{e\} \cup \{y_{j_1} x_{i_2} \cdots; y_{j_i} \neq e \neq x_{i_i}\} \quad \text{for } G/G_1;$$

$$\Sigma_2 = \{e\} \cup \{x_{i_1} y_{j_2} \cdots; y_{j_i} \neq e \neq x_{i_i}\} \quad \text{for } G/G_2;$$

$$\Sigma_L = \Sigma_1 \cup \Sigma_2 \quad \text{for } G/L.$$

Hence we get decompositions

$$H^1(G_i; \mathbf{Z}G) = \bigoplus_{z \in \Sigma_i} C_i z, \quad i = 1, 2;$$

$$H^1(L; \mathbf{Z}G) = \bigoplus_{z \in \Sigma_L} C_L z.$$

The “length” of a summand  $C_i z$  or  $C_L z$  is defined as the number of representatives  $x_i, y_i \neq e$  occurring in  $z$ . Consider now  $0 \neq (c_1, c_2) \in H^1(G_1; \mathbf{Z}G) \oplus H^1(G_2; \mathbf{Z}G)$ . We want to show that  $\text{res}_1(c_1) - \text{res}_2(c_2) \notin C_L$ . For this we consider a non-trivial component  $d$  of  $(c_1, c_2)$  lying in a summand (of the above decompositions) of maximal length; say  $d = cz_1$  in  $C_1 z_1$  of length  $l$ . Let  $\text{res}_1(c)$  be  $\sum_{i \in I} b_i x_i$ ,  $b_i \in C_L$ . Because the weight of  $L$  with respect to  $(G_1; \emptyset)$  is greater than one, there is at least one  $i_0$  with  $x_{i_0} \neq e$ ,  $b_{i_0} \neq 0$ . So  $\text{res}_1(cz_1)$  contains the summand  $b_{i_0} x_{i_0} z_1$  in  $C_L x_{i_0} z_1$  of length  $l+1$ , and because of the maximality of  $l$  there is no other contribution in  $\text{res}_1(c_1) - \text{res}_2(c_2)$  to the component  $C_L x_{i_0} z_1$ . So indeed  $\text{res}_1(c_1) - \text{res}_2(c_2) \notin C_L$ , which proves (10).

By assumption,  $H^2(G; \mathbf{Z}G)$  is free abelian of rank one and  $L$  has infinitely many ends. Therefore the restriction of  $\delta$  to  $H^1(L; \mathbf{Z}L)$  cannot be injective. Because of the exactness of the Mayer—Vietoris sequence,  $H^1(L; \mathbf{Z}L) \cap \text{im}(\text{res}_1, -\text{res}_2) \neq 0$ . By (10),  $L$  has weight *one* with respect to  $(G_1; \emptyset)$  or  $(G_2; \emptyset)$ , say  $(G_1; \emptyset)$ . (Note that  $L$  cannot have weight 0, since  $\text{res}_1$  and  $\text{res}_2$  are injective.) By Theorem A, we have one of the following two cases:

$$1) \quad G_1 = H_1 * H_2, \quad L = L_1 * L_2, \quad e \neq L_i \subset H_i, \quad i = 1, 2;$$

$$2) \quad G_1 = H_1 * \langle t \rangle, \quad L = L_1 * tL_2t^{-1}, \quad e \neq L_1, L_2 \subset H_1.$$

In Case 1), we have  $G = H_1 *_{L_1} (H_2 *_{L_2} G_2)$ . If  $L_1 \neq H_1$ ,  $G$  splits over  $L_1$ ; if  $L_1 = H_1$ , then  $L_2 \neq H_2$  and  $G = H_2 *_{L_2} G_2$  splits over  $L_2$ .

In Case 2),  $G = (H_1 *_{L_1} G_2) *_{L_2, t^{-1}}$  splits over  $L_2$ .

So in both cases we have a splitting of  $G$  over a group  $M$  with  $rk(M) < rk(L)$ .

( $\beta$ ) The case  $G = G_1 *_{L_1, p}$  is treated similarly. If  $L$  is not cyclic, one can show that (by changing the notation if necessary)  $n(L) = 1$  with respect to  $(G_1; p^{-1}Lp)$ ; to prove that the pair is adapted and to compute the weight one proceeds by methods analogous to those in the proof of (10). By Theorem A we have again the cases 1) or 2) above, where moreover  $p^{-1}Lp$  is conjugate to a subgroup of  $H_1$ . By changing the stable letter  $p$  we can get  $p^{-1}Lp \subset H_1$ .

In Case 1),  $G = (H_1 *_{L_1, p}) *_{L_2} H_2$  splits over  $L_2$  if  $L_2 \neq H_2$ ; or else over  $L_1$ .

In Case 2),  $G = (H_1 *_{L_1, p}) *_{L_2, t^{-1}}$  splits over  $L_2$ . This completes the proof of Theorem 1'.

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