Factorisation is not unique for higher dimensional knots.

Comment. Math. Helvetici **55** (1980) 583–592 Birkhäuser Verlag, Basel

Factorisation is not unique for higher dimensional knots

EVA BAYER*

0. Introduction

An n-knot will be a smooth oriented submanifold K of the $(n+2)$-sphere S^{n+2}, where K is homeomorphic to S^n. A knot is irreducible if it cannot be written as a connected sum of two non-trivial knots. Schubert has shown that every 1-knot can be written uniquely as a connected sum of finitely many irreducible knots (see [S] or [K1, Section 1]). For $n>2$, Sosinskii has proved that it is still possible to factorise every n-knot into finitely many irreducible knots (cf. [So Theorem 5. 1] or [K1, Section 2]) but Kearton has shown that this factorisation is not necessarily unique for $n=3$ [K]. In the present note we shall prove the non uniqueness of the factorisation for $(2q-1)$-knots, $q\geqq 3$ and for $(2q)$-knots, $q\geqq 4$.

I would like to thank C. Weber for advising me to consider Levine duality in the even dimensional case. I also thank M. Kervaire for useful conversations.

1. Factorisation is not necessarily unique for $(2q-1)$-knots, $q\geqq 3$

Let $q\geqq 3$ be an integer.

DEFINITION. A *Seifert matrix* A is a square matrix of integers such that $\det(A+(-1)^q A^t)=\pm 1$, where A^t is the transpose of A.

Let A be a non-singular Seifert matrix (that is, $\det(A)\neq 0$). We shall say that A is *irreducible* if A is not S-equivalent to the orthogonal sum of two non-singular Seifert matrices. (See [Le] for the definition of S-equivalence. In the examples that we shall construct, the Seifert matrices will be unimodular, and unimodular Seifert matrices are S-equivalent if and only if they are integrally congruent (see [T, Proposition 4.3])). We shall use the notation

$$S=A+(-1)^q A^t, \qquad z=S^{-1}A$$

* Supported by a grant from the Alexander von Humboldt Foundation.

LEMMA 1. *Let A_1 and A_2 be Seifert matrices with $z_1 = z_2$. Then*

$$\begin{pmatrix} A_1 & 0 \\ 0 & -A_1 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} A_2 & 0 \\ 0 & -A_2 \end{pmatrix}$$

are integrally congruent.

LEMMA 2. *There exist irreducible Seifert matrices A_1 and A_2 such that*

a) $z_1 = z_2$
b) *A_1 and A_2 are not S-equivalent*
A_1 and $-A_2$ are not S-equivalent.

(We shall give explicit examples of such Seifert matrices after the proof of this lemma.)

The above two lemmas give the desired result. Indeed, let A_1 and A_2 be Seifert matrices as in Lemma 2.

Levine has shown that the S-equivalence classes of non-singular Seifert matrices correspond biunivoguely to the isotopy classes of simple $(2q-1)$-knots [Le, Theorems 1, 2, 3]. Note that this implies that irreducible Seifert matrices correspond to irreducible knots. Let K_1, L_1, K_2, L_2 be the simple $(2q-1)$-knots corresponding to $A_1, -A_1, A_2, -A_2$ respectively. These knots are irreducible, because A_1 and A_2 are irreducible. By Lemma 1,

$$\begin{pmatrix} A_1 & 0 \\ 0 & -A_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_2 & 0 \\ 0 & -A_2 \end{pmatrix}$$

are integrally congruent, as $z_1 = z_2$. Therefore they are S-equivalent. So by [Le, Theorem 3] the connected sum of K_1 and L_1 is isotopic to the connected sum of K_2 and L_2. On the other hand, [Le, Theorem 1] shows that K_1 is not isotopic either to K_2 or to L_2, as the Seifert matrices are not S-equivalent.

Proof of Lemma 1. Let A be a Seifert matrix, and let

$$M_1 = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & (-1)^q(1-z^t) \\ z & 0 \end{pmatrix}.$$

Then M_1 and M_2 are integrally congruent. Indeed, let

$$X=\begin{pmatrix}1-z & (-1)^q S^{-1} \\ -z & (-1)^q S^{-1}\end{pmatrix}=\begin{pmatrix}I & I \\ 0 & I\end{pmatrix}\begin{pmatrix}I & 0 \\ 0 & (S^{-1})^t\end{pmatrix}\begin{pmatrix}I & 0 \\ (-1)^{q+1}A & I\end{pmatrix}$$

where I is the identity matrix.

One checks by direct computation that $M_2=X^t M_1 X$ (it is useful to note that $1-z^t=AS^{-1}$, and that $(1-z^t)A=Az$). This proves Lemma 1, as $z_1=z_2=z$,

$$\begin{pmatrix}A_1 & 0 \\ 0 & -A_1\end{pmatrix} \text{ and } \begin{pmatrix}A_2 & 0 \\ 0 & -A_2\end{pmatrix}$$

are both congruent to

$$\begin{pmatrix}0 & (-1)^q(1-z^t) \\ z & 0\end{pmatrix}.$$

Proof of Lemma 2. Let ϕ be the cyclotomic polynomial corresponding to the 15^{th} roots of unity. Let t be the Jordan matrix associated with ϕ, and let $z=(1-t)^{-1}$. Note that $\det(z)=1$: indeed, $\det(1-t)=\phi(1)=1$.

Let ζ be a primitive 15^{th} root of unity. Sending ζ to ζ^{-1} induces a non-trivial involution on $\mathbf{Z}[\zeta]$. We shall denote this involution by an overbar.

Let $\Delta=\{x\in\mathbf{Q}(\zeta)\mid \mathrm{Tr}_{\mathbf{Q}(\zeta)/\mathbf{Q}}(x\mathbf{Z}[\zeta])\subset\mathbf{Z}\}$ be the inverse different of the extension $\mathbf{Q}(\zeta)/\mathbf{Q}$. We have: $\bar{\Delta}=\Delta$.

DEFINITION. Let V be a torsion free $\mathbf{Z}[\zeta]$-module of finite rank. We shall say that a hermitian or skewhermitian form

$$h: V\times V\to\Delta$$

is *unimodular*, if

$$\begin{aligned}\mathrm{ad}(h): V &\to \mathrm{Hom}_{\mathbf{Z}[\zeta]}(V_1\Delta)\\ x &\to h(\ ,x)\end{aligned}$$

is an isomorphism.

Claim 1. The integral congruence classes of Seifert matrices A such that

$$(A+(-1)^q A^t)^{-1}A=z \tag{1}$$

($z=(1-t)^{-1}$ as above, fixed; q a fixed integer) are in bijection with the isometry classes of $(-1)^q$-hermitian unimodular forms

$$h:\mathbf{Z}[\zeta]\times\mathbf{Z}[\zeta]\to\Delta.$$

Proof of Claim 1. Let A be a Seifert matrix with property (1), and let $S=A+(-1)^qA^t$. Rank $(A)=$ degree $(\phi)=8$. Let V be a free $\mathbf{Z}$-module of rank 8. We can consider S as a $(-1)^q$-symmetric form $S:V\times V\to\mathbf{Z}$, and $t=1-z^{-1}:V\to V$ will be an isometry for S.

Setting $\zeta\cdot v=t(v)$ for v in V makes V into a $\mathbf{Z}[\zeta]$-module. As t corresponds to the Jordan matrix of ϕ, V is isomorphic to $\mathbf{Z}[\zeta]$.

As in [B–M, §1], we associate to S a $(-1)^q$-hermitian form

$$h:\mathbf{Z}[\zeta]\times\mathbf{Z}[\zeta]\to\Delta$$

such that

$$\mathrm{Tr}_{\Phi(\zeta)/\Phi}\, h(\alpha x, y)=S(\alpha x, y)\ \forall\alpha\in\Phi(\zeta)\ \forall x, y\in V. \tag{2}$$

It is easy to check that h is unimodular and that congruent Seifert matrices determine isometric $(-1)^q$-hermitian forms.

Conversely, given a unimodular $(-1)^q$-hermitian form $h:\mathbf{Z}[\zeta]\times\mathbf{Z}[\zeta]\to\Delta$, the formula (2) determines a $(-1)^q$-symmetric matrix S such that $\det(S)=\pm1$ and t is an isometry for S. Set $A=Sz$. Then $A+(-1)^qA^t=S$, therefore A is a Seifert matrix satisfying (1). Isometric $(-1)^q$-hermitian forms determine congruent Seifert matrices.

Claim 2. The isometry classes of unimodular $(-1)^q$-hermitian forms

$$h:\mathbf{Z}[\zeta]\times\mathbf{Z}[\zeta]\to\Delta$$

are in bijection with $U_0/N(U)$, where U is the group of units of $\mathbf{Z}[\zeta]$, U_0 is the group of units of $\mathbf{Z}[\zeta+\bar{\zeta}]$, and $N:U\to U_0$, $N(u)=u\bar{u}$, is the norm map.

Proof of Claim 2. Let g be the minimal polynomial of $\zeta+\bar{\zeta}$, and let

$$\alpha_0=\frac{1}{g'(\zeta+\bar{\zeta})}\frac{1}{\zeta-\bar{\zeta}}.$$

Let Δ_1 be the inverse different of the extension $\mathbf{Q}(\zeta)/\mathbf{Q}(\zeta+\bar{\zeta})$, and Δ_2 the inverse different of $\mathbf{Q}(\zeta+\bar{\zeta})/\mathbf{Q}$. Then $\Delta=\Delta_1\cdot\Delta_2$ [L, III. §1, Proposition 5] and

$$\Delta_1=\frac{1}{\zeta-\bar{\zeta}}\mathbf{Z}[\zeta]$$

$$\Delta_2=\frac{1}{g'(\zeta+\bar{\zeta})}\mathbf{Z}[\zeta+\bar{\zeta}]$$

[L, III. §1 Corollary of Proposition 2]. Therefore $\Delta=a_0\mathbf{Z}[\zeta]$. Notice that $\bar{a}_0=-a_0$.

Let $h:\mathbf{Z}[\zeta]\times\mathbf{Z}[\zeta]\to\Delta$ be a unimodular $(-1)^q$-hermitian form. We have: $h(x, y)=ax\bar{y}$ for some a in Δ such that $\bar{a}=(-1)^q a$.

As we can identify $\mathrm{Hom}_{\mathbf{Z}[\zeta]}(\mathbf{Z}[\zeta], \Delta)$ with Δ, the unimodularity of h implies that $a\mathbf{Z}[\zeta]=\Delta$. Therefore $a\mathbf{Z}[\zeta]=a_0\mathbf{Z}[\zeta]$. This implies that aa_0^{-1} is a unit. We have $\overline{aa_0^{-1}}=(-1)^{q+1}aa_0^{-1}$.

Set

$$u=\begin{cases} aa_0^{-1} & \text{if } q \text{ is odd} \\ aa_0^{-1}(\zeta-\bar{\zeta}) & \text{if } q \text{ is even} \end{cases} \tag{3}$$

$\zeta-\bar{\zeta}$ is a unit: $(\zeta-\bar{\zeta})^2(\zeta+\bar{\zeta})(-\zeta-\bar{\zeta}+1)=1$.

Therefore u is in U_0 in both cases. Conversely, to $u\in U_0$ we associate the $(-1)^q$-hermitian form $h(x, y)=ax\bar{y}$ where a is given by (3). One checks easily that two $(-1)^q$-hermitian forms are isometric if and only if the corresponding units are in the same class in $U_0/N(U)$.

Let us determine the cardinality of $U_0/N(U)$. We have

$$[U_0:N(U)]=\frac{[U_0:U_0^2]}{[N(U):U_0^2]}.$$

Using the theorem of Dirichlet on the rank of the group of units, we see that $[U_0:U_0^2]=16$.

Let μ be the group of roots of unity in $\mathbf{Q}(\zeta)$. Then

$$[N(U):U_0^2]=[U:\mu U_0]=Q$$

and $Q=2$ [L1, Chap. 3, Theorem 4.1]. So $[U_0:N(U)]=8$. (We shall actually exhibit 8 distinct classes of $U_0/N(U)$ in the next section.)

Applying Claim 1 and Claim 2, we see that there are 8 non-congruent Seifert matrices A such that

$$(A+(-1)^q A^t)^{-1} A = z. \tag{1}$$

Therefore it is possible to choose A_1 and A_2 satisfying (1), and such that A_1 is not congruent either to A_2 or to $-A_2$. But congruence and S-equivalence are the same in this case, because the Seifert matrices are unimodular (see [T, Proposition 4.3]). A_1 and A_2 are irreducible, as their Alexander polynomial is irreducible.

Explicit examples

Let $\zeta = e^{2i\pi/15}$, and let $u_1 = 1$, $u_2 = \zeta + \zeta^{-1}$. We have $u_2(-u_2^3 + u_2^2 + 4u_2 - 4) = 1$, therefore u_2 is a unit. But u_2 is not in $N(U)$: indeed, u_2 is conjugate to $\zeta^7 + \zeta^{-7}$ which is negative. Clearly $-u_2$ is also negative, therefore not in $N(U)$. Using similar methods for the units $u_3 = \zeta^2 + \zeta^{-2}$, $u_4 = u_2 u_3 = \zeta + \zeta^{-1} + \zeta^3 + \zeta^{-3}$, we see that u_1, $-u_1$, u_2, $-u_2$, u_3, $-u_3$, u_4, $-u_4$ are all in different classes of $U_0/N(U)$. In the proof of Lemma 2 we have seen that the cardinality of $U_0/N(U)$ is 8, therefore we have a complete set of representants of $U_0/N(U)$.

Using the method given in the proof of Lemma 2, let us associate the Seifert matrices A_i to the units u_i, $i = 1 \cdots 4$.

Then the $\begin{pmatrix} A_i & 0 \\ 0 & -A_i \end{pmatrix}$ are all different factorisations of the same Seifert matrix B (see Lemma 1). Moreover, B has no other factorisations than these four. Direct computation gives the following matrices for A_1 and A_2:

q odd:

$$A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & -1 & -2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 1 & 1 & 1 & 1 \\ -2 & -2 & -1 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$A_2=\begin{pmatrix} 2 & 2 & 2 & 2 & 1 & 0 & -2 & -3 \\ 2 & 2 & 2 & 2 & 2 & 1 & 0 & -2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 1 & 0 \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\ 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\ -2 & 0 & 1 & 2 & 2 & 2 & 2 & 2 \\ -3 & -2 & 0 & 1 & 2 & 2 & 2 & 2 \\ -4 & -3 & -2 & 0 & 1 & 2 & 2 & 2 \end{pmatrix}$$

q even

$$A_1=\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$A_2=\begin{pmatrix} 0 & 0 & 0 & -1 & -2 & -3 & -3 & -2 \\ 0 & 0 & 0 & 0 & -1 & -2 & -3 & -3 \\ 1 & 0 & 0 & 0 & 0 & -1 & -2 & -3 \\ 2 & 1 & 0 & 0 & 0 & 0 & -1 & -2 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & -1 \\ 3 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 3 & 2 & 1 & 0 & 0 \end{pmatrix}$$

2. Factorisation is not necessarily unique for $(2q)$-knots, $q \geqq 4$

Let $q \geqq 4$ be an integer. Let $\Lambda = \mathbf{Z}[t, t^{-1}]$, and let T be a finitely generated $\mathbf{Z}$-torsion Λ-module such that $(1-t): T \to T$ is an isomorphism.

DEFINITION. $L: T\times T\to \mathbf{Q}/\mathbf{Z}$ is a *Levine pairing* if L is $\mathbf{Z}$-bilinear, non-singular, $(-1)^{q+1}$-symmetric, such that

$$L(tx, ty) = L(x, y) \quad \text{for} \quad x, y \text{ in } T.$$

In [Le 1] Levine associates to every $(2q)$-knot K a Levine pairing on the $\mathbf{Z}$-torsion part T of $H_q(\tilde{X})$, $\tilde{X}$ being the maximal abelian cover of $X = S^{2q+2}\backslash K$. Isotopic knots have isometric pairings. He also shows that every Levine pairing can be realized by a simple $(2q)$-knot [Le 1, Theorem 13.1]. Conversely, Kojima has shown that if $H_q(\tilde{X})$ is finite and 2-torsion free and if $q\geqq 4$, then simple $(2q)$-knots having isometric Levine pairings are isotopic [Ko, Theorem 1]. Therefore, the following examples determine simple $(2q)$-knots which factorise in more than one way:

q odd

Let $T=\mathbf{Z}/5$, and let $t(x) = -x$ for x in T. Then $L_1, L_2: T\times T\to \mathbf{Q}/\mathbf{Z}$ given by $L_1(x, y) = \frac{1}{5}xy$, $L_2(x, y) = \frac{2}{5}xy$ are Levine pairings. Clearly L_1 is not isometric either to L_2 or to $-L_2$. But

$$\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1/5 & 0 \\ 0 & 4/5 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1/5 \\ 1/5 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2/5 & 0 \\ 0 & 3/5 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix},$$

and the isomorphisms obviously commute with $t\oplus t$.

q even

Let $T=\mathbf{Z}/5\oplus\mathbf{Z}/5$, $t: T\to T$ given by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$$

and let $L_1, L_2: T\times T\to \mathbf{Q}/\mathbf{Z}$ be the Levine pairings given by the matrices

$$\begin{pmatrix} 0 & 1/5 \\ 4/5 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 2/5 \\ 3/5 & 0 \end{pmatrix}$$

respectively.

L_1 is not Λ-isometric either to L_2 or to $-L_2$. Indeed, suppose that L_1 is Λ-isometric to $\varepsilon\cdot L_2$, for $\varepsilon = +1$ or -1, and let X be the matrix corresponding to this isometry. Then $\det(X) = 2\varepsilon$.

Let

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The relation $tX = Xt$ implies $\det(X) = (a+b)^2$. But $(a+b)^2 = 2\varepsilon$ is impossible. $L_1 \oplus -L_1$ and $L_2 \oplus -L_2$ are both Λ-isometric to

$$\begin{pmatrix} 0 & 0 & 0 & 2/5 \\ 0 & 0 & 3/5 & 0 \\ 0 & 2/5 & 0 & 0 \\ 3/5 & 0 & 0 & 0 \end{pmatrix}$$

the isometries are given by

$$\begin{pmatrix} I & I \\ -I & I \end{pmatrix} = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 2I & I \end{pmatrix}\begin{pmatrix} 2I & 0 \\ 0 & I \end{pmatrix}$$

and

$$\begin{pmatrix} 3I & I \\ 2I & I \end{pmatrix} = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 2I & I \end{pmatrix}$$

respectively, with $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

REFERENCES

B–M E. BAYER and F. MICHEL, *Finitude du nombre des classes d'isomorphisme des structures isométriques entières*, Comment. Math. Helv. *54* (1979), 378–396.

K C. KEARTON, *Factorisation is not unique for 3-knots*, Indiana Univ. Math. Jour. *28* (1979), 451–452.

K1 ——, *The factorisation of knots* (preprint).

Ko S. KOJIMA, *Classification of simple knots by Levine pairings*, Comment. Math. Helv. *54* (1979) 356–367.

L S. LANG, *Algebraic number theory*, Addison-Wesley Publishing Company

L1 ——, *Cyclotomic fields*, Springer-Verlag (1978).

Le J. LEVINE, *An algebraic classification of some knots in codimension two*, Comment. Math. Helv. *45* (1970), 185–198.

Le1 ——, *Knot modules*, Trans. of the A.M.S. *229* (1977), 1–50.

S H. SCHUBERT, *Die eindeutige Zerlegbarkeit eines Knotens in Primknoten*, S.-B. Heidelberger Akad. Wiss. Math.-Natur. Kl. 1, *3* (1949), 57–104.

So A. B. SOSINSKII, *Decomposition of knots*, Math. USSR Sbornik *10* (1970), 139–150.
T H. F. TROTTER, *Homology of group systems with applications to knot theory*, Ann. of Math. *76* (1962), 464–498.

Mathematisches Institut der Universität Münster
Federal Republic of Germany

Current address:
Université de Genève, Section de Mathématiques
2–4 rue du Lièvre
1211 *Genève* 24
Switzerland

Received May 15, 1980.