Pompeiu's problem on symmetric spaces.

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Pompeiu's problem on symmetric spaces

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This paper, the promised sequel to [2], continues our study [30], [1], [2] of the Pompeiu problem and related matters. In it, we extend results established for spaces of constant curvature in [2] to the agreeably general context of two-point homogeneous spaces and, in particular, to arbitrary (globally) symmetric spaces of rank one. Briefly, this is accomplished by reducing the question to a problem of spectral analysis, which can then be settled using the classical theory for $\mathbf{R}^1$.

Problems of Pompeiu type are, in fact, closely related to questions of spectral analysis and spectral synthesis for mean-periodic functions. The failure of spectral synthesis for $\mathbf{R}^n$ $(n>1)$ [15] raises the question of describing those situations in which spectral synthesis does hold and, more generally, of finding the "correct" generalization of the one-variable theory. Various versions of the Pompeiu property comprise one class of such positive results. They also suggest that an appropriate generalization may be obtained by replacing $\mathbf{R}$ by symmetric spaces or semisimple Lie groups or (real) rank one.

The plan of the paper is as follows. Sections 1 and 2 provide brief introductions to the Pompeiu problem and question of spectral synthesis, respectively. In Sections 3 and 4, we show that for $X=G/K$, G a separable unimodular Lie group and K a compact subgroup, the Pompeiu property may be reformulated as a question concerning the coincidence of two spaces of distributions on G. Section 5 completes the reduction to a problem of spectral analysis in the case in which X is a (noncompact) symmetric space. Section 6 contains a detailed discussion of concrete examples in the situation where X has rank one, with an emphasis on making the rather generally formulated results of previous sections as explicit as possible; in addition to the theorems of Pompeiu type, analogues of Delsarte's two radius theorem for the spaces in question are obtained. Section 7 continues the discussion, focusing on analogues of Pizzetti's formula [31, p. 342]; and Section 8 deals with the case of compact spaces, not treated above. In Section 9, we show how the results of Sections 3 and 4 can be applied to treat the Pompeiu

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problem on $\mathbf{R}^n$. The paper concludes (Section 10) with a brief comment on the general setting of our results and a suggestion for future research.

In writing this paper, we have tried to take account of the fact that relatively few analysts interested in problems of Pompeiu type have any extensive background in the theory of Lie groups. Accordingly, we have attempted to render references in the text as explicit as possible and, in Sections 3 and 4 at least, to suppress only those calculations which are genuinely routine. Our notation is standard, based on [27] and [17], to which the reader should refer for undefined terms.

1.

For purposes of orientation, it will be convenient to formulate the Pompeiu problem in a form somewhat more general than usually considered. Accordingly, let X be a locally compact Hausdorff space and μ a positive Baire measure on X. A collection $C=\{A\}$ of compact subsets $A\subset X$ is said to have the Pompeiu property with respect to (X, μ) if the condition

$$\int_A f\,d\mu=0 \qquad A\in C$$

implies that f vanishes identically whenever $f\in C(X)$.

The situation of greatest interest occurs when X is a Riemannian manifold admitting a transitive group G of isometries. In this case, it is natural to take μ to be volume measure on X and the collection C to be invariant under the action of G. Typically, one chooses a finite collection $\mathcal{P}$ of subsets of X and puts $C=\{gA : A\in\mathcal{P},\ g\in G\}$. In this case, we say that the family $\mathcal{P}$ has the Pompeiu property with respect to G. By the Pompeiu problem we understand the somewhat vague quest for explicit conditions insuring that a family $\mathcal{P}$ possess the Pompeiu property.

The examples presented below give an indication of the spirit which animates the study of the Pompeiu problem.

1. Let $X=\mathbf{R}^n$ and let G be the group of all translations. Fix $r_1, r_2>0$ and let $\mathcal{P}=\{D_1, D_2\}$, where D_j is a closed ball of radius r_j. Then $\mathcal{P}$ has the Pompeiu property if and only if r_1/r_2 is not a quotient of zeroes of the Bessel function $J_{n/2}(z)$ [30, p. 247].

2. Let $X=\mathbf{R}^2$ and let G be the group of all rigid motions of the plane. Let D be a noncircular ellipse. Then $\mathcal{P}=\{D\}$ has the Pompeiu property [4, p. 143].

3. Take X and G as in the previous example, and let D be a compact convex

set with nonempty interior which fails to have a unique line of support at some boundary point. Then $\mathcal{P}=\{D\}$ has the Pompeiu property. In particular, all convex polygons have the Pompeiu property [4, p. 150]. Here, as in the previous example, one can replace G by the smaller group of translations at the expense of enlarging $\mathcal{P}$ to the (infinite) set of all rotations of D.

4. Let $X=\mathbf{R}^2$ and let G be the group of all translations. Fix $a_1, a_2, a_3>0$ and let $\mathcal{P}=\{Q_1, Q_2, Q_3\}$, where Q_j is a square of side a_j having sides parallel to the coordinate axes. Then $\mathcal{P}$ has the Pompeiu property if and only if the ratios a_1/a_2, a_2/a_3, and a_3/a_1 are all irrational [1, p. 253].

5. Let $X=S(n,-\alpha^2)$, the (unique, up to isometric equivalence) complete, simply-connected n-dimensional Riemannian manifold of constant negative curvature $-\alpha^2$. Let G be the group of isometries of X and let $\mathcal{P}=\{B_1, B_2\}$, where B_j is a geodesic ball in X of radius r_j. Then $\mathcal{P}$ has the Pompeiu property if and only if the equations

$$P_z^{-n/2}(\cosh \alpha r_j)=0 \qquad j=1,2$$

have no common solution $z\in\mathbf{C}$. Here $P_z^{-n/2}(x)$ is the associated Legendre function of the first kind [2, p. 125].

6. Let $X=S^n(1/\alpha)$, the n-sphere of radius $1/\alpha$ with the metric structure it inherits as a subset of $\mathbf{R}^{n+1}$. Let G be the group of all rotations of X and let B be an n-dimensional spherical cap of (geodesic) radius r. Then $\mathcal{P}=\{B\}$ has the Pompeiu property if and only if r is not a zero of any of the functions

$$C_m^{(n+1)/2}(\cos \alpha r) \qquad m=1,2,3,\ldots.$$

Here $C_m^{(n+1)/2}(x)$ is a Gegenbauer polynomial [28], [25] (where different notations are used), and [2, p. 128].

In subsequent sections we shall show how most of the above results can be recaptured in a *uniform* fashion.

2.

Let $\mathscr{E}=\mathscr{E}(\mathbf{R}^n)$ denote the locally convex space of all infinitely differentiable functions on $\mathbf{R}^n$ with the topology of uniform convergence on compacta. A translation invariant subspace $\mathcal{M}\subset\mathscr{E}$ is said to admit *spectral analysis* if $\mathcal{M}$ contains an exponential, i.e. if there exists $z\in\mathbf{C}^n$ such that $f(x)=e^{i(x\cdot z)}(x\cdot z=x_1z_1+\cdots+x_nz_n, x\in\mathbf{R}^n)$ belongs to $\mathcal{M}$. If the exponential polynomials belonging to $\mathcal{M}$ are dense in $\mathcal{M}$, we say that $\mathcal{M}$ admits *spectral synthesis.*

(Recall that an exponential polynomial is a finite sum of terms of the form $p(x)e^{i(x\cdot z)}$, where $x\in\mathbf{R}^n$, $z\in\mathbf{C}^n$, and p is a polynomial.) In case every translation invariant subspace admits spectral analysis (synthesis), we say that spectral analysis (synthesis) holds in $\mathscr{E}$.

In a celebrated paper [26], Laurent Schwartz proved that spectral synthesis (and, a fortiori, spectral analysis) holds in $\mathscr{E}(\mathbf{R})$. Thus, for any collection $\mathscr{P}$ of distributions of compact support on $\mathbf{R}$ the system of convolution equations in $\mathscr{E}(\mathbf{R})$

$$f*\mu=0 \qquad \mu\in\mathscr{P} \tag{2.1}$$

has only the trivial solution $f=0$ if and only if there are no solutions of the form $f(x)=e^{i(x\cdot z)}$. Equivalently, the equations (2.1) have no common solution $f\neq 0$ if and only if the Fourier transforms $\mu(z)=\langle\mu, e^{i(x\cdot z)}\rangle\ (z\in\mathbf{C})$ have no common zeroes.

To make the connection with the Pompeiu problem, let us suppose that G consists of all translations on $\mathbf{R}^n$. For $f\in\mathscr{E}(\mathbf{R}^n)$, the condition

$$\int_{g(A)} f(x)\,dx=0 \qquad A\in\mathscr{P},\ g\in G$$

is precisely the assertion that

$$(f*\check{\chi}_A)(x)=0 \qquad A\in\mathscr{P}, \tag{2.2}$$

where χ_A denotes the characteristic function of the set A and $\check{h}(x)=h(-x)$.

Clearly, if (2.2) is to force f to vanish identically, the Fourier transforms $(\check{\chi}_A)^\wedge(z)\ (z\in\mathbf{C}^n)$ must have no common zeroes; otherwise an appropriate exponential satisfies (2.2). (It is at this point that the special arithmetic conditions on the radii of balls, sides of squares, etc. become relevant; they insure that the associated Fourier transforms have no common zeroes.) Proving the sufficiency of this condition depends in general on certain symmetry conditions on the family $\mathscr{P}$ which allow reduction to the situation on the line. Schwartz's theorem can then be invoked to show that $f=0$. The case of general (not necessarily smooth) f follows from a standard approximation argument. This general line of reasoning (with variations) underlies most previous work in this area, and we shall follow it here. However, since the spaces we shall be considering are not, in general, Euclidean, considerable preparation is required before we can effect the reduction to Schwartz's theorem. We turn to this task now.

3.

Let G be a separable unimodular Lie group with Haar measure dg, K a compact subgroup of G with normalized Haar measure dk ($\int_K dk = 1$), and $X = G/K$ the homogeneous space of right cosets gK with natural projection $\pi : G \to X$. Denote by dx the measure on X defined by

$$\int_X f(x)\,dx = \int_G (f \circ \pi)(g)\,dg \qquad f \in \mathcal{D}(X). \tag{3.1}$$

Here, as for any second-countable smooth manifold M, $\mathcal{D}(M)$ is the space of all C^∞ functions on M of compact support with the usual topology; its dual $\mathcal{D}'(M)$ is the space of (Schwartz) distributions on M. similarly, $\mathcal{E}(M)$ is the space of all C^∞ functions on M with dual $\mathcal{E}'(M)$, the space of distributions of compact support.

Functions of compact support on G can be convolved according to the rule

$$(f * \varphi)(g) = \int_G f(gh^{-1})\varphi(h)\,dh, \tag{3.2}$$

and this extends as usual to distributions. Observe that, since G is unimodular, one has also

$$(f * \varphi)(g) = \int_G f(h)\varphi(h^{-1}g)\,dh. \tag{3.3}$$

Introducing the distribution

$$\delta_K : f \mapsto \int_K f(k)\,dk \tag{3.4}$$

on G, we can associate to each function $f \in \mathcal{E}(G)$ a function $f_\pi \in \mathcal{E}(X)$ defined by

$$f_\pi \circ \pi = f * \delta_K; \tag{3.5}$$

cf. [19, p. 453]. Hence, each $T \in \mathcal{D}'(X)$ lifts to a distribution $\tilde{T}$ on G given by

$$\tilde{T}(f) = T(f_\pi) \qquad f \in \mathcal{D}(G). \tag{3.6}$$

If a function φ on X is regarded as a distribution, then

$$\tilde{\varphi}(g) = \varphi(\pi(g)), \qquad (\tilde{\varphi})_\pi = \varphi; \tag{3.7}$$

thus functions (and so, distributions) on X can be identified with functions (distributions) on G which are right-invariant under K, i.e., which satisfy $\varphi(gk) = \varphi(g)$ for all $k \in K$.

The lifting also induces a notion of convolution in $\mathscr{D}'(X)$ by

$$\widetilde{T_1 * T_2} = \tilde{T}_1 * \tilde{T}_2 \tag{3.8}$$

if one of the distributions T_j has compact support. This convolution is associative and satisfies $T * \delta = \delta * T = T$, where δ is the distribution on X given by $\langle \delta, f \rangle = f(\pi(e))$, e the identity in G. While this convolution can be used to treat aspects of the problems to be studied here, we prefer to carry out our preliminary analysis on the group G itself.

An element $g \in G$ acts diffeomorphically on X via

$$\tau(g)(x) = g\bar{x}K \qquad x = \bar{x}K; \tag{3.9}$$

we write $\tau(g)(x) = \tau(g)x = g \cdot x$ when no confusion is possible. Following standard conventions, we write $f^\tau = f \circ \tau^{-1}$, where $f \in \mathscr{E}(X)$ and τ is an arbitrary diffeomorphism of X. For $T \in \mathscr{D}'(X)$ we define

$$T^\tau(f) = T(f^{\tau^{-1}}) = T(f \circ \tau), \tag{3.10}$$

which agrees with the definition for functions when τ leaves dx invariant, as is the case for $\tau = \tau(g)$. If A is a compact subset of X, integration over A defines a distribution of compact support

$$T_A(f) = \int_A f(x)\, dx \qquad f \in \mathscr{E}(X). \tag{3.11}$$

It is clear that T_A acts on continuous (or even locally integrable) functions on X, and an easy calculation shows that, with the obvious notation,

$$T_A^{\tau(g)} = T_{g \cdot A} \qquad g \in G. \tag{3.12}$$

Suppose now that a family $\mathscr{P}$ of compact subsets of X is given. The Pompeiu problem for $\mathscr{P}$ is the problem of deciding whether the family $\mathscr{P}$ has the Pompeiu property, i.e. whether all solutions $f \in C(X)$ to

$$T_{g \cdot A}(f) = 0 \qquad A \in \mathscr{P},\ g \in G \tag{3.13}$$

must vanish identically. This can be reformulated as a question concerning a system of convolution equations on the group G.

To see this, observe first that for an arbitrary compactum $B \subset X$ we have

$$\tilde{T}_B(\varphi) = \int_{\tilde{B}} \varphi(g)\, dg \qquad \varphi \in \mathscr{E}(G), \tag{3.14}$$

where $\tilde{B} = \pi^{-1}(B)$. Indeed, denoting characteristic functions by χ, we have

$$\begin{aligned}
\tilde{T}_B(\varphi) &\overset{(3.6)}{=} T_B(\varphi_\pi) \overset{(3.11)}{=} \int_B \varphi_\pi(x)\, dx = \int_X \varphi_\pi(x)\chi_B(x)\, dx \\
&\overset{(3.1)}{=} \int_G (\varphi_\pi \circ \pi)(g)(\chi_B \circ \pi)(g)\, dg = \int_G (\varphi * \delta_K)(g)\chi_{\tilde{B}}(g)\, dg \\
&= \int_{\tilde{B}} \left(\int_K \varphi(gk^{-1})\, dk \right) dg = \int_K \left(\int_{\tilde{B}} \varphi(gk^{-1})\, dg \right) dk \\
&= \int_K \left(\int_{\tilde{B}k^{-1}} \varphi(g)\, dg \right) dk = \int_K \left(\int_{\tilde{B}} \varphi(g)\, dg \right) dk = \int_{\tilde{B}} \varphi(g)\, dg
\end{aligned}$$

as required.

Suppose now that $f \in \mathscr{E}(X)$ and write, as usual, $\check{\varphi}(h) = \varphi(h^{-1})$. Then we have

$$\begin{aligned}
\tilde{f} * \check{\chi}_{\tilde{A}}(g) &= \int_G \tilde{f}(gh^{-1})\check{\chi}_{\tilde{A}}(h)\, dh = \int_G \tilde{f}(gh)\chi_{\tilde{A}}(h)\, dh \\
&= \int_{\tilde{A}} \tilde{f}(gh)\, dh = \int_{g\tilde{A}} \tilde{f}(h)\, dh \\
&= \int_{\widetilde{g \cdot A}} \tilde{f}(h)\, dh = \tilde{T}_{g \cdot A}(\tilde{f}),
\end{aligned}$$

the last equality holding by (3.14). But $\tilde{T}_{g \cdot A}(\tilde{f}) = T_{g \cdot A}((\tilde{f})_\pi) = T_{g \cdot A}(f)$ by (3.6) and (3.7). It follows that

$$T_{g \cdot A}(f) = \tilde{f} * \check{\chi}_{\tilde{A}}(g). \tag{3.15}$$

It is worth observing that this formula cannot in general be expressed in the form $f * T_C$ for some set $C \subseteq X$ unless a certain symmetry is assumed for A. We also note that (3.15) defines a function of $g \in G$ and not of $g (\text{mod}\, K)$ in X.

Equation (3.15) shows that, at least for smooth functions, (3.13) may be interpreted as a system of convolution equations

$$\tilde{f} * \check{\chi}_{\tilde{A}} = 0 \qquad A \in \mathscr{P} \tag{3.16}$$

on G. In the next section we shall reduce the study of this system to a problem of spectral analysis.

4.

Keeping the notation of the previous section, let $\mathcal{V}$ be the set of all right-invariant functions in $\mathscr{E}(G)$ for which

$$\varphi * \check{\chi}_{\tilde{A}} = 0 \tag{4.1}$$

for all $A \in \mathcal{P}$. In the terminology of [10], $\mathcal{V}$ is a left-variety, i.e. a closed subspace of $\mathscr{E}(G)$ such that $\mathscr{E}'(G) * \mathcal{V} \subseteq \mathcal{V}$. (Actually, equality holds, since $\delta_e \in \mathscr{E}'(G)$.) Solving the Pompeiu problem consists in finding conditions under which $\mathcal{V} = \{0\}$.

Let $\hat{K}$ be the set of all equivalence classes of (continuous, finite-dimensional) irreducible unitary representations of the compact group K [29]. For $\sigma \in \hat{K}$, we denote by $d(\sigma)$ its degree and by α_σ its character. Thus each representation in σ maps $k \in K$ to a $d(\sigma) \times d(\sigma)$ unitary matrix having trace $\alpha_\sigma(k)$. Set

$$\xi_\sigma(k) = d(\sigma)\alpha_\sigma(k^{-1}) = d(\sigma)\overline{\alpha_\sigma(k)} \qquad k \in K. \tag{4.2}$$

Integration against $\xi_\sigma(k)\, dk$ defines a distribution on G which is supported on K; making the usual identification, we call this distribution ξ_σ. In particular, if 1 denotes the trivial representation, ξ_1 coincides with the distribution δ_K defined in (3.4). For future reference we record the following lemma.

LEMMA 1. *Let $\sigma, \rho \in \hat{K}$. Then*

$$\xi_\sigma * \xi_\rho = \begin{cases} \xi_\sigma & \sigma = \rho \\ 0 & \sigma \neq \rho \end{cases}$$

Proof. Since the distributions ξ_σ and ξ_ρ are supported on K, it suffices to calculate their convolution (as elements of $\mathscr{E}'(G)$) on K. Thus

$$\begin{aligned} \xi_\sigma * \xi_\rho(k_0) &= \int_K \xi_\sigma(k_0 k^{-1}) \xi_\rho(k)\, dk = \int_K \xi_\sigma(k_0 k)\overline{\xi_\rho(k)}\, dk \\ &= d(\sigma)\, d(\rho) \int_K \overline{\alpha_\sigma(k_0 k)}\alpha_\rho(k)\, dk \\ &= d(\sigma)\, d(\rho) \int_K \overline{\operatorname{tr}(\sigma(k_0)\sigma(k))} \operatorname{tr} \rho(k)\, dk. \end{aligned} \tag{4.3}$$

Now recall the Schur orthogonality relations [29]. Denoting (unitary) representatives for σ and ρ again by σ, ρ (so that $\sigma(k)=\|\sigma_{ij}(k)\|$, $i, j=1, 2, \ldots, d(\sigma)$ and similarly for $\rho(k)$), we have

$$\int_K \sigma_{ij}(k)\overline{\rho_{l,m}(k)}\, dk = 0 \qquad i, j, l, m; \sigma \neq \rho \tag{4.4}$$

and

$$\int_K \sigma_{ij}(k)\overline{\sigma_{lm}(k)}\, dk = \begin{cases} \dfrac{1}{d} & (i, j)=(l, m) \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

Now

$$\operatorname{tr} \sigma(k_0)\sigma(k) = \sum_j \sum_l \sigma_{jl}(k_0)\sigma_{lj}(k)$$

so that if $\sigma \neq \rho$.

$$\xi_\sigma * \xi_\rho(k_0) = d(\sigma)\, d(\rho) \sum_{j, l} \overline{\sigma_{jl}(k_0)} \int_K \overline{\sigma_{lj}(k)} \operatorname{tr} \rho(k)\, dk = 0$$

by (4.4). Similarly, by (4.3), (4.5), and (4.2)

$$\begin{aligned} \xi_\sigma * \xi_\sigma(k_0) &= d(\sigma)^2 \sum_{j, l} \overline{\sigma_{jl}(k_0)} \int_K \overline{\sigma_{lj}(k)} \operatorname{tr} \sigma(k)\, dk \\ &= d(\sigma) \sum_j \overline{\sigma_{jj}(k_0)} = d(\sigma) \overline{\operatorname{tr} \sigma(k_0)} = d(\sigma)\overline{\alpha_\sigma(k_0)} = \xi_\sigma(k_0) \end{aligned}$$

as required.

Since the ξ_σ are compactly supported, the convolutions $f * \xi_\sigma$ exist for any f in $\mathscr{E}(G)$ or, indeed, in $\mathscr{D}'(G)$. In fact, we can decompose an arbitrary distribution f into a series

$$f = \sum_{\sigma \in \hat{K}} f * \xi_\sigma \tag{4.6}$$

which converges in the topology of whichever of the spaces $\mathscr{E}(G)$, $\mathscr{D}(G)$, $\mathscr{E}'(G)$, or $\mathscr{D}'(G)$ f belongs to [16, p. 13].

Now let $\mathscr{E}_0(G)$ be the closed subspace of $\mathscr{E}(G)$ consisting of all functions which are bi-invariant with respect to K; $f \in \mathscr{E}(G)$ belongs to $\mathscr{E}_0(G)$ if and only if

$f(k_1 g k_2) = f(g)$ for all $k_1, k_2 \in K$. The dual space of $\mathscr{E}_0(G)$ is naturally identified with the space $\mathscr{E}'_0(G)$ of bi-invariant distributions in $\mathscr{E}'(G)$.

Let $\mathscr{U}$ be the closure in $\mathscr{E}'_0(G)$ of the linear space spanned by all distributions of the form

$$S = \check{\chi}_{\tilde{A}} * \xi_\sigma * T * \xi_1 \tag{4.7}$$

where $A \in \mathscr{P}$, $\sigma \in \hat{K}$, and $T \in \mathscr{E}'(G)$. (Each such distribution belongs to $\mathscr{E}'_0(G)$ since $\check{\chi}_{\tilde{A}}$ is left-invariant and ξ_1 is right-invariant.) It is clear from (4.1) that $\mathscr{V} * \mathscr{U} = \{0\}$. Since $\delta_e \in \mathscr{E}'_0(G)$, it follows that if $\mathscr{U} = \mathscr{E}'_0(G)$ then $\mathscr{V} = \{0\}$. The converse holds as well.

PROPOSITION. $\mathscr{V} = \{0\}$ *if and only if* $\mathscr{U} = \mathscr{E}'_0(G)$.

Proof. Define an operator $r(g)$ $(g \in S)$ on smooth functions S by

$$r(g)S(h) = S(hg) \qquad h \in G \tag{4.8}$$

and extend its action to distributions in the obvious fashion. Then $r(g_2) \circ r(g_1) = r(g_1 g_2)$ and $r(g^{-1})S = S * \delta_g$, where δ_g is the Dirac distribution at g. If $S = \check{\chi}_{\tilde{A}} * \xi_\sigma * T * \xi_1$ we have

$$r(g)S * \xi_1 = \check{\chi}_{\tilde{A}} * \xi_\sigma * (T * \xi_1 * \delta_{g^{-1}}) * \xi_1,$$

which is again of the same general form. It follows that

$$r(g)S * \xi_1 \in \mathscr{U} \qquad S \in \mathscr{U} \tag{4.9}$$

for all $g \in G$.

Now if $\mathscr{U} \neq \mathscr{E}'_0(G)$ there exists a nonzero function $\varphi \in \mathscr{E}_0(G)$ such that

$$\varphi * S = 0 \qquad S \in \mathscr{U}. \tag{4.10}$$

Indeed, since $\mathscr{U} \neq \mathscr{E}'_0(G)$ we can find, by duality, a nonzero function $\psi \in \mathscr{E}_0(G)$ which satisfies

$$S(\psi) = 0 \qquad \text{all} \qquad S \in \mathscr{U}. \tag{4.11}$$

Let $\varphi = \check{\psi}$. Then we have, using integral notation,

$$(\varphi * S)(g) = \int_G \varphi(h)S(h^{-1}g)\,dh = \int_G \psi(h)S(hg)\,dh$$
$$= \int_K \left(\int_G \psi(hk)S(hkg)\,dh\right) dk = \int_G \left(\int_K \psi(hk)S(hkg)\,dk\right) dg$$
$$= \int_G \psi(h)\left(\int_K S(hkg)\,dk\right) dg$$

since ψ is right invariant. Since

$$\int_K S(hkg)\,dk = \int_K r(g)S(hk)\,dk = \int_K r(g)S(hk^{-1})dk$$
$$= (r(g)s * \xi_1)(h),$$

we obtain

$$(\varphi * S)(g) = \int_G \psi(h)(r(g)S * \xi_1)(h)\,dh,$$

which vanishes by (4.9) and (4.11).

To complete the proof we shall show that $\varphi \in \mathcal{V}$, i.e. that φ is right-invariant and satisfies (4.1). Right invariance is immediate since $\varphi = \check{\psi}$ and ψ is left-invariant. To establish (4.1), it is sufficient by (4.6) to prove that

$$\varphi * \check{\chi}_A * \xi_\sigma = 0 \tag{4.12}$$

for all $A \in \mathcal{P}$, $\sigma \in \hat{K}$. For this, we need a companion formula to (4.6). Let

$$\pi_{\alpha,\beta}(f) = \xi_\alpha * f * \xi_\beta \qquad \alpha, \beta \in \hat{K}. \tag{4.13}$$

Then

$$f = \sum_{\alpha,\beta \in \hat{K}} \pi_{\alpha,\beta}(f) \tag{4.14}$$

where the convergence is again in the topology of the space of functions or distributions to which f belongs [16, p. 14].

For $\alpha \in \hat{K}$ denote by $\bar{\alpha}$ the contragredient representation, given (in terms of representatives) by $\bar{\alpha}(k) = (\alpha^t)^{-1}(k) = \overline{\alpha(k)}$. Then $\bar{\alpha} \in \hat{K}$ and $\xi_{\bar{\alpha}}(k) = \overline{\xi_\alpha(k)} = \xi_\alpha(k^{-1})$. Write $\mathscr{E}_{\alpha,\beta} = \pi_{\alpha,\beta}(\mathscr{E})$, $\mathscr{E}'_{\alpha,\beta} = \pi_{\alpha,\beta}(\mathscr{E}')$. It is easy to see that $\mathscr{E}_0 = \mathscr{E}_{1,1}$, $\mathscr{E}'_0 = \mathscr{E}'_{1,1}$.

LEMMA 2. (a) $\pi_{\alpha,\beta}\circ\pi_{\alpha,\beta}=\pi_{\alpha,\beta}$; $\pi_{\alpha,\beta}\circ\pi_{\gamma,\delta}=0$, $(\alpha,\beta)\neq(\gamma,\delta)$.
(b) $\mathscr{E}_{\alpha,\beta}\cap\mathscr{E}_{\gamma,\delta}=0$ *if* $(\alpha,\beta)\neq(\gamma,\delta)$
(c) $\pi_{\alpha,\beta}(T)^{\vee}=\pi_{\bar\beta,\bar\alpha}(\check T)$, $T\in\mathscr{E}'$
(d) $(\mathscr{E}_{\alpha,\beta})'=\mathscr{E}_{\bar\alpha,\bar\beta}$.

Proof. Part (a) is immediate from Lemma 1, and (b) follows from (a). To prove (c), it is enough to show that it holds pointwise for functions. We have

$$\begin{aligned}
\pi_{\alpha,\beta}(T)^{\vee}(g) &= \pi_{\alpha,\beta}(T)(g^{-1}) = (\xi_\alpha * T * \xi_\beta)(g^{-1})\\
&= \int_K \xi_\alpha(k_1)(T*\xi_\beta)(k_1^{-1}g^{-1})\,dk_1\\
&= \int_{K\times K} \xi_\alpha(k_1)\xi_\beta(k)T(k_1^{-1}g^{-1}k^{-1})\,dk\,dk_1\\
&= \int_{K\times K} \xi_\alpha(k_1)\xi_\beta(k)\check T(kgk_1)\,dk\,dk_1\\
&= \int_{K\times K} \xi_\alpha(k_1^{-1})\xi_\beta(k^{-1})\check T(k^{-1}gk_1^{-1})\,dk\,dk_1\\
&= \int_{K\times K} \xi_{\bar\alpha}(k_1)\xi_{\bar\beta}(k)\check T(k^{-1}gk_1^{-1})\,dk\,dk_1\\
&= \pi_{\bar\beta,\bar\alpha}(\check T)(g),
\end{aligned}$$

as required. Finally, to verify (d) observe that for $f\in\mathscr{E}$, $T\in\mathscr{E}'$ we have

$$\begin{aligned}
T(\pi_{\alpha,\beta}(f)) &= \int_G T(g)[\xi_\alpha * f * \xi_\beta(g)]\,dg\\
&= \int_G T(g)\left(\int_{K\times K} \xi_\alpha(k_1)\xi_\beta(k)f(k_1^{-1}gk^{-1})\,dk\,dk_1\right)dg\\
&= \int_G T(g)\left(\int_{K\times K} \xi_\alpha(k_1^{-1})\xi_\beta(k^{-1})f(k_1gk)\,dk\,dk_1\right)dg\\
&= \int_G T(g)\left(\int_{K\times K} \xi_{\bar\alpha}(k_1)\xi_{\bar\beta}\ (k)f(k_1gk)\,dk\,dk\right)dg\\
&= \int_G f(g)\left(\int_{K\times K} \xi_{\bar\alpha}(k_1)\xi_{\bar\beta}(k)T(k_1^{-1}gk^{-1})\,dk\,dk_1\right)dg\\
&= \int_G f(g)[\xi_{\bar\alpha} * T * \xi_{\bar\beta}(g)]\,dg\\
&= \pi_{\bar\alpha,\bar\beta}(T)(f).
\end{aligned}$$

Any $T \in (\mathscr{E}_{\alpha,\beta})'$ has an extension T' to all of $\mathscr{E}$; and for $f \in \mathscr{E}$ we have

$$T(\pi_{\alpha,\beta}(f)) = T'(\pi_{\alpha,\beta}(f)) = \pi_{\bar{\alpha},\bar{\beta}}(T')(f).$$

The correspondence $T \rightarrow \pi_{\bar{\alpha},\bar{\beta}}(T')$ is linear, one-to-one, and onto and thus provides the required identification of $(\mathscr{E}_{\alpha,\beta})'$ and $\mathscr{E}_{\bar{\alpha},\bar{\beta}}$.

Returning to the proof of (4.12), let $\Phi = \varphi * \check{\chi}_{\bar{A}} * \xi_\sigma$ for some choice of $A \in \mathscr{P}$, $\sigma \in \hat{K}$. Since φ is left-invariant, $\xi_1 * \varphi = \varphi$ so that

$$\begin{aligned}\pi_{1,\sigma}(\Phi) &= \xi_1 * \varphi * \check{\chi}_{\bar{A}} * \xi_\sigma * \xi_\sigma = (\xi_1 * \varphi) * \check{\chi}_{\bar{A}} * (\xi_\sigma * \xi_\sigma)\\ &= \varphi * \check{\chi}_{\bar{A}} * \xi_\sigma = \Phi,\end{aligned}$$

whence $\Phi \in \mathscr{E}_{1,\sigma}$ and $\pi_{\alpha,\beta}(\Phi) = 0$, $(\alpha, \beta) \neq (1, \sigma)$. Taking $T \in \mathscr{E}'$, we have by part (c) of Lemma 2

$$\begin{aligned}\pi_{1,\bar{\sigma}}(T)(\Phi) &= (\Phi * \pi_{1,\bar{\sigma}}(T)^{\check{}})(e)\\ &= \Phi * \pi_{\sigma,1}(\check{T})(e).\end{aligned}$$

But

$$\begin{aligned}\Phi * \pi_{\sigma,1}(\check{T}) &= (\varphi * \check{\chi}_{\bar{A}} * \xi_\sigma) * (\xi_\sigma * \check{T} * \xi_1)\\ &= \varphi * (\check{\chi}_{\bar{A}} * \xi_\sigma * \check{T} * \xi_1) = 0\end{aligned}$$

by (4.7) and (4.10). Thus

$$\pi_{1,\bar{\sigma}}(T)(\Phi) = 0$$

for all $T \in \mathscr{E}'$. Since $\Phi \in \mathscr{E}_{1,\sigma}$ is killed by each distribution in $\mathscr{E}'_{1,\bar{\sigma}} = (\mathscr{E}_{1,\sigma})'$, we must have $\Phi = 0$. That completes the proof.

Remarks 1. If a right-invariant distribution $f \in \mathscr{D}'(G)$ satisfies $f * \check{\chi}_{\bar{A}} = 0$ then for any $\varphi \in \mathscr{D}(G)$ we have $\varphi * f \in \mathscr{V}$. Thus the condition $\mathscr{U} = \mathscr{E}'_0(G)$ is also necessary and sufficient that each solution $f \in \mathscr{D}'(X)$ to (3.13) vanish. This shows that the hypothesis of smoothness in (3.13) may be relaxed to continuity or even local integrability.

2. More generally, if the family $\{T_A : A \in \mathscr{P}\}$ is replaced by an arbitrary family of distributions in $\mathscr{E}'(X)$ the analogue of Proposition 1 holds (with obvious modifications in the definition of $\mathscr{U}$).

The reduction of the Pompeiu problem accomplished above raises the question of finding concrete conditions which determine whether or not $\mathscr{U} = \mathscr{E}'_0(G)$.

When $X=\mathbf{R}^n$, an appropriate necessary condition is provided by the nonexistence of common zeroes for a certain family of holomorphic functions (which arise as Fourier transforms of distributions associated with the family $\mathcal{P}$). Whether such a condition is sufficient for the Pompeiu property to hold for a family $\mathcal{P}$ of subsets of $\mathbf{R}^n$ remains an open question. Of course, the choice $G=\mathbf{R}^n$, $K=\{0\}$ yields $\mathcal{E}'_0(G)=\mathcal{E}'(\mathbf{R}^n)$; and, as mentioned above, there is a counterexample to spectral analysis in $\mathbf{R}^n$ $(n\geq 2)$. However, that counterexample does not involve distributions of the form χ_A.

In general, progress beyond Proposition 1 requires extra assumptions on X; otherwise, $\mathcal{E}'_0(G)$ may even fail to be commutative. Accordingly, we shall assume henceforth that (G, K) is a symmetric pair such that X is a Riemannian globally symmetric space.

5.

Suppose now that G is a connected non-compact semi-simple Lie group with finite center, and let K be a maximal compact subgroup of G. Then $X=G/K$ is a globally symmetric space of non-compact type, and each such symmetric space can be realized in this fashion.

For $f\in\mathcal{D}_0(G)$, i.e. for $f\in\mathcal{D}(G)$ and K-biinvariant, we have a *spherical Fourier transform*

$$(\mathcal{F}f)(\lambda)=\int_G f(g)\varphi_\lambda(g^{-1})\,dg \qquad \lambda\in\mathfrak{A}^*. \tag{5.1}$$

Here $\mathfrak{A}$ is a real vector space of dimension l, the rank of X (and the real rank of G) and the φ_λ are the *spherical functions* of G [17, p. 398]. Denoting Lebesgue measure divided by $(2\pi)^{l/2}$ by $d\lambda$, we have the inversion formula [20, p. 35]

$$f(g)=\frac{1}{w}\int_{\mathfrak{A}^*}(\mathcal{F}f)(\lambda)\varphi_\lambda(g)\,|c(\lambda)|^{-2}\,d\lambda, \tag{5.2}$$

where $c(\lambda)^{-1}$ is a certain analytic function on $\mathfrak{A}^*$ and w is the order of the Weyl group W, a certain finite group (generated by reflections) of automorphisms of the complexified space $\mathfrak{A}^*_{\mathbf{C}}$.

The functions φ_λ are defined for $\lambda\in\mathfrak{A}^*_{\mathbf{C}}$, and $(\mathcal{F}f)(\lambda)$ extends to an entire function on $\mathfrak{A}^*_{\mathbf{C}}$. Regarded as functions of λ, the φ_λ are invariant under the action of W; thus $\mathcal{F}f$ is also W-invariant. In fact, one has an analogue of the Paley–Wiener theorem: the spherical Fourier transform establishes a bijection of $\mathcal{D}_0(G)$

onto the space of W-invariant entire functions on $\mathfrak{A}_{\mathbf{C}}^*$ of exponential type which are rapidly decreasing on $\mathfrak{A}^*$ [20, p. 37]. This bijection extends to a vector-space isomorphism between $\mathscr{E}_0'$ and the space $\mathscr{F}(\mathscr{E}_0')$ of W-invariant entire functions on $\mathfrak{A}_{\mathbf{C}}^*$ of exponential type which are slowly increasing on $\mathfrak{A}^*$ [8].

Now the spherical Fourier transform of f is the composition of the classical Fourier transform on the Euclidean space $\mathfrak{A}^*$ with the Abel transform F_f of f (see [17, p. 429]). Since $F_{f*g} = F_f * F_g$ [17, p. 454], one has

$$\begin{aligned}\mathscr{F}(f*g)(\lambda) &= (F_{f*g})^\wedge(\lambda) = (F_f * F_g)^\wedge(\lambda)\\ &= (F_f)^\wedge(\lambda)(F_g)^\wedge(\lambda) = \mathscr{F}f(\lambda)\mathscr{F}g(\lambda),\end{aligned}$$

so the correspondence between $\mathscr{E}_0'$ and $\mathscr{F}(\mathscr{E}_0')$ is an algebra isomorphism as well. Finally, we observe that the isomorphism is topological. This is surely well-known, but we have been unable to find a simple proof in the literature; for completeness, we sketch the proof.

The space $\hat{\mathscr{E}}'(\mathbf{R}^n)$ carries the topology that makes it isomorphic to $\mathscr{E}'(\mathbf{R}^n)$. For any constant $A > 0$ the set $\mathscr{B}_A$ of all entire functions in $\mathbf{C}^n$ satisfying

$$|f(z)| \le A(1+|z|)^A \exp A\,|\mathrm{Im}\, z| \tag{5.3}$$

is a bounded subset of $\hat{\mathscr{E}}'(\mathbf{R}^n)$, furthermore, this topology is characterized by the fact that every bounded set is a subset of some $\mathscr{B}_A$ [9, Lemma 5.18]. The space $\mathscr{F}(\mathscr{E}_0')$ is a closed complemented subspace of $\hat{\mathscr{E}}'(\mathbf{R}^n)$ and we consider it with the relative topology. The open mapping theorem shows that all we have to prove is that the map $\mathscr{F}: \mathscr{E}_0' \to \mathscr{F}(\mathscr{E}_0')$ is continuous. since $\mathscr{E}_0'$ is bornological [27] the problem reduces to showing that for any given bounded set $\mathscr{B}$ in $\mathscr{E}_0'$ we have $\mathscr{F}(\mathscr{B}) \subseteq \mathscr{B}_A$ for some $A > 0$. If $D_1, \ldots, D_l$ are the generators of the algebra of biinvariant differential operators in G ($l =$ rank of X), then there is a compact subset C and constants A_1, N such that $T \in \mathscr{B}$ implies

$$|T(f)| \le A_1 \max\{|D_i^j f(g)| : 1 \le i \le l,\ 1 \le J \le N,\ g \in C\}$$

for any $f \in \mathscr{E}_0(G)$. Since [17, p. 431]

$$D_i^j \varphi_\lambda(g) = [p_i(\lambda)]^j \varphi_\lambda(g)$$

for some W-invariant polynomials p_i, we have that (5.3) holds for all the functions $\mathscr{F}T(\lambda) = T(\varphi_\lambda(g^{-1}))$, $T \in \mathscr{B}$, for a sufficiently large constant A. Hence the isomorphism $\mathscr{E}_0' \cong \mathscr{F}\mathscr{E}_0'$ is topological.

It is now clear that a necessary condition for $\mathcal{U} = \mathcal{E}_0'$ is that the variety $Z(\mathcal{U})$ of common zeroes of the entire functions $\mathcal{F}S$ from (4.7) must be empty. When $l = 1$, this condition is also sufficient. In that case, W contains a single nontrivial transformation, which may be taken to be multiplication by -1; thus, $\mathcal{F}(\mathcal{E}_0')$ consists precisely of the even functions of exponential type on $\mathbf{C}$ which are slowly increasing on $\mathbf{R}$. applying Schwartz's theorem [26] together with a simple averaging process completes the proof of the sufficiency.

6.

Calculating the spherical Fourier transforms of the distributions in (4.7) is most agreeable in precisely those cases of the greatest geometric interest (in which $\mathcal{P}$ consists of spheres, balls, or other spherically symmetric distributions). As in earlier work, it turns out that knowledge of the integrals of a function over all balls of two distinct radii is, in general, sufficient to determine the function uniquely.

Suppose then that $X = G/K$ is a non-compact rank one symmetric space and let $\mathcal{P}$ consist of a pair of geodesic balls B_1, B_2 centered at $\pi(e) = 0 \in X$ having radii r_j $(j = 1, 2)$. To verify that the transforms of the distributions in (4.7) have no common zeroes, it clearly suffices to show that the transforms $\mathcal{F}\chi_{\tilde{B}_j}$ have no common zeroes $\lambda \in \mathbf{C}$. The spherical functions on G are the lifts to G of functions $\varphi(x)$ on X which depend only on the distance t between x and 0. Making the natural identifications and writing φ_λ indifferently for functions on G, X, or $\mathbf{R}^+$, we have

$$\mathcal{F}\chi_{\tilde{B}}(\lambda) = \int_G \chi_{\tilde{B}}(g)\varphi_\lambda(g^{-1})\,dg = \int_{\tilde{B}} \varphi_\lambda(g^{-1})\,dg = \int_{\tilde{B}} \varphi_\lambda(g)\,dg$$

$$= \int_B \varphi_\lambda(x)\,dx = \int_0^r \varphi_\lambda(t)A(t)\,dt, \tag{6.1}$$

where $A(t)$ is the area of the sphere of radius t in X.

Further calculation depends on the explicit form of $\varphi_\lambda(t)$ and $A(t)$. These are given [18], [12] by

$$\varphi_\lambda(t) = \varphi_\lambda^{(\alpha,\beta)}(t) = F(\tfrac{1}{2}(\rho + i\lambda), \tfrac{1}{2}(\rho - i\lambda); \alpha + 1; -\sinh^2 \kappa t)$$
$$= R_{\frac{1}{2}(i\lambda-\rho)}(\cosh 2\kappa t), \tag{6.2}$$

where $\rho = \alpha + \beta + 1$ and

$$R_{\mu}^{(\alpha,\beta)}(x) = F\left(-\mu, \mu + \alpha + \beta + 1; \alpha + 1; \frac{1-x}{2}\right)$$

and by

$$A(t) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \left(\frac{\sinh \kappa t}{\kappa}\right)^{2\alpha+1} (\cosh \kappa t)^{2\beta+1}. \tag{6.3}$$

Here, $F(a, b; c; z)$ denotes the usual hypergeometric function; n is the real dimension of X; $\alpha = n/2 - 1$ and β are real parameters depending on X; and κ is a real parameter (the appearance of which as an argument of φ we suppress) whose dependence on the metric of X is given by $m = -4\kappa^2$, where m is the maximum sectional curvature of X. Ordinarily, we may take $\kappa = 1$.

The functions φ_λ satisfy

$$\Delta_t \varphi_\lambda + (\rho^2 + \lambda^2)\kappa^2 \varphi_\lambda = 0 \tag{6.4}$$

where

$$\Delta_t = \frac{\partial^2}{\partial t^2} + \frac{A'(t)}{A(t)} \frac{\partial}{\partial t} \tag{6.5}$$

is the "radial part" of the Laplace–Beltrami operator Δ on X.

The simply connected rank one symmetric spaces of non-compact type are the real, complex, and quaternionic hyperbolic spaces $H^n(\mathbf{R})$, $H^n(\mathbf{C})$, and $H^n(\mathbf{H})$ and the Cayley hyperbolic plane $H^{16}(\mathrm{Cay})$ [17]. Realizations of these spaces as G/K and the corresponding values of the parameters α, β, and n are exhibited in the following table; cf. [17, p. 354], [21, p. 239]

X	G/K	α	β	n
$H^n(\mathbf{R})$	$SO_0(n, 1)/SO(n)$	$\frac{n}{2}-1$	$\frac{n}{2}-1$	$2, 3, 4, \ldots.$
$H^n(\mathbf{C})$	$SU(n/2, 1)/S(U_{n/2} \times U_1)$	$\frac{n}{2}-1$	0	$4, 6, 8, \ldots.$
$H^n(\mathbf{H})$	$Sp(n/4, 1)/Sp(n/4) \times Sp(1)$	$\frac{n}{2}-1$	1	$8, 12, 16, \ldots.$
$H^{16}(\mathrm{Cay})$	$F_{4(-20)}/SO(9)$	7	3	16

A direct calculation (cf. [11, 2.8 (27); 2.1.4 (23)]) yields

$$\int_0^r \varphi_\lambda^{(\alpha,\beta)}(t)A(t)\,dt = \frac{\pi^{n/2}}{\Gamma(n/2+1)}\left(\frac{\sinh \kappa r}{\kappa}\right)^n \varphi_\lambda^{(\alpha+1,-\beta-1)}(r)$$

$$= \frac{\pi^{n/2}}{\Gamma(n/2+1)}\left(\frac{\sinh \kappa r}{\kappa}\right)^n (\cosh \kappa r)^{2\beta+2}\varphi_\lambda^{(\alpha+1,\beta+1)}(r). \tag{6.6}$$

Accordingly, we have the following

THEOREM 1. *Let X be a noncompact symmetric space of rank one. Suppose $u \in L^1_{\mathrm{loc}}(X)$ and*

$$\int_B u(x)\,dx = 0 \tag{6.7}$$

for each geodesic ball in X having radius r_1 or r_2. Then $u=0$ so long as the equations

$$\varphi_\lambda^{(\alpha+1,\beta+1)}(r_j) = 0 \qquad j=1,2 \tag{6.8}$$

have no common solution $\lambda \in \mathbf{C}$.

For $X = H^n(\mathbf{R})$, this result was obtained in [2, p. 122]; note that

$$\varphi_\lambda^{(n/2,\,n/2)}(t) = \frac{2^{n/2}\Gamma(n/2+1)P^{-n/2}_{1/2(i\lambda-1)}(\cosh 2\kappa t)}{(\sinh 2\kappa t)^{n/2}}, \tag{6.9}$$

where $P_\nu^\mu(z)$ is an associated Legendre function; cf. [13, p. 248], [11, 3.2(7)].

When (6.8) does have a common solution $\lambda_0 \in \mathbf{C}$, it is relatively easy to exhibit a nonzero function in $C(X)$ which satisfies (6.7). Indeed, take $f(x) = \varphi_{\lambda_0}^{(\alpha,\beta)}(t)$, where as before $t = \operatorname{dist}(x, 0)$. Spherical functions on G satisfy (and, indeed, are characterized by) the identity

$$\int_K \varphi(gkh)\,dk = \varphi(g)\varphi(h) \qquad g, h \in G; \tag{6.10}$$

cf. [17, p. 399]. Taking φ in (6.10) as the lift to G of $\varphi_\lambda^{(\alpha,\beta)}(x) \equiv \varphi_\lambda^{(\alpha,\beta)}(t)$, we may interpret the left-hand side of (6.10) as the mean value of $\varphi_\lambda^{(\alpha,\beta)}$ over the sphere in X centered at $x = \pi(g)$ having radius $s = \operatorname{dist}(\pi(g), \pi(h))$; cf. [17, p. 434]. Thus, denoting the ball of (geodesic) radius r centred at $x \in X$ by $B_r(x)$, we have, by (6.10), (6.6) and (6.8),

$$\int_{B_r(x)} \varphi_{\lambda_0}^{(\alpha,\beta)}(y)\,dy = \int_0^r \varphi_{\lambda_0}^{(\alpha,\beta)}(x)\varphi_{\lambda_0}^{(\alpha,\beta)}(s)A(s)\,ds$$
$$= \frac{\pi^{n/2}}{\Gamma(n/2+1)}\left(\frac{\sinh \kappa r}{\kappa}\right)^n$$
$$\times (\cosh \kappa r)^{2\beta+2}\varphi_{\lambda_0}^{(\alpha,\beta)}(x)\varphi_{\lambda_0}^{(\alpha+1,\beta+1)}(r) = 0$$

for all $x \in X$, whenever $r = r_1$ or r_2.

The case of spherical means is implicit in the discussion given above and requires no further calculation. Let $u \in C(X)$ and denote by $U(x, r)$ the mean value of u taken over the geodesic sphere in X of radius r centered at x. Then we have

THEOREM 2. *Let X be a noncompact symmetric space of rank one. Suppose $u \in C(X)$ and that there exist $r_1, r_2 > 0$ such that*

$$U(x, r_j) = 0 \qquad j = 1, 2 \tag{6.11}$$

for all $x \in X$. Then $u \equiv 0$ so long as the equations

$$\varphi_{\lambda}^{(\alpha,\beta)}(r_j) = 0 \qquad j = 1, 2 \tag{6.12}$$

have no common solution $\lambda \in \mathbf{C}$.

Should the system (6.12) have a solution λ_0, then $f(x) = \varphi_{\lambda_0}^{(\alpha,\beta)}(t)$ is a nonzero smooth function on X whose mean value over each sphere of radius r_1 and r_2 is zero; cf. (6.10) *et seq.*

Theorems 1 and 2 have analogues in which the hypotheses that integrals of u over balls or spheres vanish is replaced by the assumption that u satisfies a mean-value condition and the conclusion is changed correspondingly to assert that u is harmonic, i.e., $\Delta u = 0$. These results extend the celebrated two radius theorem of Delsarte [6], [7] (cf. [30], [2]) to noncompact symmetric spaces of rank one. (That harmonic functions on such spaces possess the mean-value property is, of course, well-known [23], and in any case, follows from (7.5) below.)

We state the result for spheres.

THEOREM 3. *Let $u \in C(X)$ and suppose that*

$$U(x, r) = u(x) \qquad r = r_1, r_2 \tag{6.13}$$

for all $x \in X$. *Then* $\Delta u = 0$ *so long as the equations*

$$\varphi_\lambda^{(\alpha,\beta)}(r_1) = 1 \qquad \varphi_\lambda^{(\alpha,\beta)}(r_2) = 1 \tag{6.14}$$

have no common solution $\lambda \in \mathbf{C} \setminus \{\pm i(n/2 + \beta)\}$.

For $X = H^n(\mathbf{R})$, this result was obtained in [2, p. 121]; cf. (6.9).

Proof. Consider the radial distributions on X given by $S_j = \Omega_j - \delta_0$, where Ω_j is normalized surface area on the geodesic sphere of radius r_j centered at $0 \in X$. The hypothesis of Theorem 3 is that $S_j^\tau(u) = 0$ $(j = 1, 2)$ for all $\tau \in G$. This translates to the equations $\tilde{u} * \tilde{S}_j = 0$ $(j = 1, 2)$ in G; cf. (3.10)–(3.16). Identifying the collection of radial distributions in $\mathscr{E}'(X)$ with $\mathscr{E}'_0(G)$ and taking spherical Fourier transforms, we have

$$\begin{aligned}(\mathscr{F}\tilde{S}_j)(\lambda) &= (\mathscr{F}S_j)(\lambda) = \varphi_\lambda(r_j) - 1 \\ &= -(\rho^2 + \lambda^2)\kappa^2 F_j(\lambda) = \mathscr{F}(\Delta T_j)(\lambda).\end{aligned}$$

Here $\rho = \alpha + \beta + 1 = (n/2 + \beta)$; Δ is the Laplace–Beltrami operator on X, given by (6.5) for radial distributions; and $T_j \in \mathscr{E}'_0(G)$ is determined by $\mathscr{F}T_j = F_j$.

Now

$$F_j\left(\pm i\left(\frac{n}{2} + \beta\right)\right) = \frac{1}{2n}\left(\frac{\sinh \kappa r_j}{\kappa}\right)^2 \neq 0,$$

and by hypothesis the F_j can have no other common zero. It follows that the closure of the ideal in $\mathscr{E}'_0(G)$ generated by T_1 and T_2 is all of $\mathscr{E}'_0(G)$; hence $\Delta\delta_0$ belongs to the ideal generated by $S_j = \Delta T_j$ $(j = 1, 2)$. We conclude that $\Delta u = 0$, as required.

7.

Knowledge of the spherical functions enables us to derive explicit representations, analogous to the classical formula of Pizzetti [22], for the spherical means of functions defined on rank one symmetric spaces; cf. [31], [2] and [5]. For X such a space of noncompact type and $u \in C(X)$ we denote by $U(x, r)$ the mean value of u over the sphere of (geodesic) radius r centered at $x \in X$. When $u \in C^2(X)$, U

satisfies the differential equation

$$\begin{aligned} &\Delta_r U(x, r) = \Delta U(x, r) \\ &U(x, 0) = u(x), \qquad \frac{\partial U}{\partial r}(x, 0) = 0. \end{aligned} \tag{7.1}$$

Here Δ denotes the Laplace–Beltrami operator on X and $\Delta_r = \partial^2/\partial r^2 + [A'(r)/A(r)]\partial/\partial r$ (cf. (6.5)) is the radial Laplacian, acting on functions of r. Equation (7.1) can be written more concretely as

$$\frac{\partial^2 U}{\partial r^2} + [(n-1)\kappa \coth \kappa r + (2\beta+1)\kappa \tanh \kappa r]\frac{\partial U}{\partial r} = \Delta U. \tag{7.2}$$

where β depends on X and κ depends on the normalization of the metric; cf. Section 6. In particular, if u satisfies $\Delta u + \mu^2 u = 0$ then $U(x, r) = U(r)u(x)$, where $U(r)$ satisfies the ODE

$$\begin{aligned} &U''(r) + [(n-1)\kappa \coth \kappa r + (2\beta+1)\kappa \tanh \kappa r]U'(r) + \mu^2 U(r) = 0, \\ &\quad U(0) = 1 \end{aligned}$$

with solution

$$F\left(\frac{1}{2}\left(\frac{n}{2}+\beta+\sqrt{\left(\frac{n}{2}+\beta\right)^2-(\mu/\kappa)^2}\right), \frac{1}{2}\left(\frac{n}{2}+\beta-\sqrt{\left(\frac{n}{2}+\beta\right)^2-(\mu/\kappa)^2}\right); \frac{n}{2}; -\sinh^2 \kappa r\right). \tag{7.3}$$

An equivalent representation is given by

$$\frac{F\left(\frac{1}{2}\left(\frac{n}{2}-\beta+\sqrt{\left(\frac{n}{2}+\beta\right)^2-(\mu/\kappa)^2}\right), \frac{1}{2}\left(\frac{n}{2}-\beta-\sqrt{\left(\frac{n}{2}+\beta\right)^2-(\mu/\kappa)^2}\right); \frac{n}{2}; -\sinh^2 \kappa r\right)}{(\cosh \kappa r)^{2\beta}}; \tag{7.4}$$

cf. [11, 2.1.4 (23)].

Replacing μ^2 by $-\Delta$ in (7.3) and (7.4) and applying the operational expression obtained to u, we obtain the expansions

$$U(x,r)=\Gamma\left(\frac{n}{2}\right)\left\{\sum_{m=0}^{\infty}\left(\frac{\sinh\kappa r}{2\kappa}\right)^{2m}\right.$$
$$\left.\times\frac{\Delta[\Delta-(2n+4\beta+4)\kappa^2]\cdots[\Delta-(m-1)(2n+4\beta+4m-4)\kappa^2]}{m!\,\Gamma\left(\frac{n}{2}+m\right)}u(x)\right\} \tag{7.5}$$

and

$$U(x,r)=(\cosh\kappa r)^{-2\beta}\Gamma\left(\frac{n}{2}\right)\left\{\sum_{m=0}^{\infty}\left(\frac{\sinh\kappa r}{2\kappa}\right)^{2m}[\Delta+2n\beta\kappa^2]\right.$$
$$\left.\times\frac{[\Delta+(2n\beta-2n+4\beta-4)\kappa^2]\cdots[\Delta+(2n\beta-2(m-1)(n-2\beta+2m-2))\kappa^2]}{m!\,\Gamma\left(\frac{n}{2}+m\right)}u(x)\right\}, \tag{7.6}$$

where the empty differential operator ($m=0$) is understood as the identity.

These formulas are valid for u real-analytic and r sufficiently small; for functions with less smoothness truncated expansions (with remainder) hold. Taking $\beta=n/2-1$ and setting $\kappa=i\sqrt{k}/2$, we obtain the expansions for $H(\mathbf{R}^n)$ given already in [2, p. 119]. Note also that letting κ tend to 0 in either (7.5) or (7.6) leads to the classical Pizzetti formula

$$U(x,r)=\Gamma\left(\frac{n}{2}\right)\sum_{m=0}^{\infty}\left(\frac{r}{2}\right)^{2m}\frac{\Delta^m u(x)}{m!\,\Gamma\left(\frac{n}{2}+m\right)}, \tag{7.7}$$

valid for functions defined in $\mathbf{R}^n$.

Expansions analogous to (7.5) and (7.6) can also be obtained for $V(x,r)$, the (volume) integral of u over the geodesic ball of radius r centered at x. Thus, direct integration of (7.6) gives

$$V(x,r)=\pi^{n/2}\left(\frac{\sinh\kappa r}{\kappa}\right)^{n}\left\{\sum_{m=0}^{\infty}\left(\frac{\sinh\kappa r}{2\kappa}\right)^{2m}[\Delta+2n\beta\kappa^2]\right.$$
$$\left.\times\frac{[\Delta+(2n\beta-2n+4\beta-4)\kappa^2]\cdots[\Delta+(2n\beta-2(m-1)(n-2\beta+2m-2))\kappa^2]}{m!\,\Gamma\left(\frac{n}{2}+m+1\right)}u(x)\right\}, \tag{7.8}$$

which corresponds to the first expression on the right hand side of (6.6); cf. [2, p. 124]. Corresponding to the second expression in (6.6) and to (7.5), we have the expansion

$$V(x, r) = \pi^{n/2}\left(\frac{\sinh \kappa r}{\kappa}\right)^n (\cosh \kappa r)^{2(\beta+1)}\left\{\sum_{m=0}^{\infty}\left(\frac{\sinh \kappa r}{2\kappa}\right)^{2m}\right.$$

$$\left.\times\frac{[\Delta-(2n+4\beta+4)\kappa^2]\cdots[\Delta-m(2n+4\beta+4m)\kappa^2]}{m!\,\Gamma\left(\frac{n}{2}+m+1\right)}u(x)\right\}.$$

Related expansions have been obtained by Gray and Willmore [14].

8.

To complete the foregoing discussion, let us say a few words about rank one symmetric spaces of *compact* type. These spaces, which are in one-to-one correspondence with their non-compact duals discussed in Section 6, consist of

(1) The spheres, $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ $(n = 2, 3, 4, \ldots)$;
(2) the complex projective spaces, $P^n(\mathbf{C}) = \mathrm{SU}(n/2)/\mathrm{S}(\mathrm{U}_{n/2}\times \mathrm{U}_1)$ $(n = 4, 6, 8, \ldots)$;
(3) the quaternionic projective spaces,

$$P^n(\mathbf{H}) = \mathrm{Sp}\left(\frac{n}{4}+1\right)\Big/\mathrm{Sp}\left(\frac{n}{4}\right)\times \mathrm{Sp}(1) \qquad (n = 8, 12, 16, \ldots)$$

and

(4) the Cayley projective plane, $P^{16}(\mathrm{Cay}) = \mathrm{F}_{4(-52)}/\mathrm{SO}(9)$, all of which are simply connected.

To complete the list we must add

(5) the real projective spaces, $P^n(\mathbf{R}) = \mathrm{SO}(n+1)/\mathrm{O}(n)$ $(n = 2, 3, 4, \ldots)$.

So far as the local expansions obtained in Section 7 are concerned, there is little to add: these formulas retain their validity in the compact case under the simple change of variable $\kappa \mapsto i\kappa$.

The situation as regards the results of Section 6 depends upon global properties of the spaces involved and is correspondingly more delicate. Thus, in the compact case, all geodesics are closed and have the same finite length [17, p. 356]. This fact may be conceived of as imposing on radial functions an additional requirement of periodicity, which reduces by one the number of conditions needed for a positive result. It turns out that a condition involving only *a single radius* is sufficient to ensure an affirmative solution to the Pompeiu problem. The role of the Fourier transform is taken over by expansion in series of spherical and associated spherical functions; in particular, the exceptional set is the collection of zeroes of a certain family of Jacobi polynomials. The case $X = S^n$, treated by Ungar [28], Schneider [24], [25], and the authors [2] is typical and already contains the essential features of the general case. Accordingly, we shall content ourselves with a brief sketch and statement of the results. The reader intent on working out the details should find [21] an instructive reference.

For X a compact symmetric space of rank one, *viz.* any of the spaces listed in (1)–(5) above, the corresponding spherical functions are given by

$$\varphi_m(t) = R_m^{(\alpha,\beta)}(\cos 2\kappa t) \qquad m = 0, 1, 2, \ldots;$$

cf. (6.2). Here $R_m^{(\alpha,\beta)}(x) = F(-m, m+\alpha+\beta+1; \alpha+1; (1-x)/2)$ is, up to normalization, a Jacobi polynomial; and t denotes the geodesic distance from $0 = \pi(e) \in X$. As before, $\alpha = n/2 - 1$ and β are parameters depending on X; we have $\beta = n/2 - 1, 0, 1, 3$, or $-\frac{1}{2}$ as $X = S^n$, $P^n(\mathbf{C})$, $P^n(\mathbf{H})$, $P^n(\mathrm{Cay})$ or $P^n(\mathbf{R})$, respectively. Finally, κ is a real parameter which may now be interpreted as $\pi/2L$, where L is the diameter (maximum distance between two points) of X.

Corresponding to Theorems 1 and 2 of Section 6 we have

THEOREM 4. *Let X be a compact rank one symmetric space. Suppose $u \in L^1(X)$ and*

$$\int_B u(x)\, dx = 0 \tag{8.1}$$

for each geodesic ball in X of (fixed) radius r. Then $u = 0$ so long as r is not a zero of any of the functions

$$R_m^{(\alpha+1,\beta+1)}(\cos 2\kappa t) \qquad m = 1, 2, 3, \ldots. \tag{8.2}$$

More generally, if (8.1) *holds for all geodesic balls of radii $r_1, r_2, \ldots, r_l$ and the* *equations*

$$R_m^{(\alpha+1,\beta+1)}(\cos 2\kappa r_j)=0 \qquad j=1,2,\ldots,l \tag{8.3}$$

have no common solution for $m=1,2,\ldots,$ *then* $u=0$.

THEOREM 5. *Let* X *be a compact rank one symmetric space. Suppose* $u\in C(X)$ *and that*

$$U(x,r)=0 \tag{8.4}$$

for all $x\in X$ *and some fixed* r. *Then* $u=0$ *unless* r *is a zero of one of the functions*

$$R_m^{(\alpha,\beta)}(\cos 2\kappa t) \qquad m=1,2,3,\ldots. \tag{8.5}$$

Similarly, if (8.4) *holds for* $r=r_1,r_2,\ldots,r_l$ *and the equations*

$$R_m^{(\alpha,\beta)}(\cos 2\kappa r_j)=0 \qquad j=1,2,\ldots,l \tag{8.6}$$

have no common solution for $m=1,2,\ldots,$ *then* $u\equiv 0$.

Examples analogous to those given in Section 6 show that the exceptional set cannot be dispensed with.

9.

The discussion in Sections 3 and 4 can also be applied to recapture the central result (Theorem 4.1) of [4], dealing with functions defined on $\mathbf{R}^n$. We sketch the details, as they relate to the Pompeiu problem, below.

Write $\mathbf{R}^n=G/K$, where $G=M(n)$, the group of euclidean motions, and $K=SO(n)$. Let a family $\mathcal{P}$ of compact sets $A\subset\mathbf{R}^n$ be given. To settle the Pompeiu problem for $\mathcal{P}$ we must determine whether or not the closed ideal $\mathcal{U}$ generated by distributions of the form $\check{\chi}_A*\xi_\sigma*T*\xi_1$ exhausts $\mathscr{E}'_0(G)$. Formula (4.14) shows that it suffices to consider the convolutions $\check{\chi}_{\tilde{A}}*T$, where T ranges over all right-invariant distributions in $\mathscr{E}'(G)$. Since $\chi_{\tilde{A}}$ is left-invariant, $\check{\chi}_{\tilde{A}}*T$ is bi-invariant and so may be identified with a radial function on $\mathbf{R}^n$.

Now for radial functions, the euclidean Fourier transform takes the guise of a Bessel transform. Indeed, suppose $F(x)=F(r)$, $|x|=r$; then writing $x=r\omega$, $\xi=$

$R\omega'$ ($R^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2$), we have (cf. [3, p. 69])

$$\begin{aligned}\hat{F}(\xi) &= \int e^{-ix\cdot\xi} F(x)\,dx \\ &= \int_0^\infty \int_{|\omega|=1} e^{-irR\omega\cdot\omega'} F(r\omega)\,d\omega r^{n-1}\,dr \\ &= (2\pi)^{n/2} \int_0^\infty F(r) \jmath_{(n-2)/2}(Rr) r^{n-1}\,dr,\end{aligned}$$

where $\jmath_l(z) = J_l(z)/z^l$. Clearly, the Fourier transform maps radial functions to radial functions. In fact, it is an isomorphism between the space $\mathscr{E}_0'(G)$ and the even functions in $\hat{\mathscr{E}}'(\mathbf{R})$ (see [27] or [4, p. 134]).

Consider the collection of Fourier transforms $(\check{\chi}_A * T)^\wedge$ obtained as A ranges over $\mathscr{P}$ and T varies over all right-invariant distributions in $\mathscr{E}'(G)$, these are functions of $\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2 \in \mathbf{C}$. In case these transforms have a common zero $\alpha \in \mathbf{C}$, it is clear that $\mathscr{U}$ cannot coincide with $\mathscr{E}_0'(G)$, since $1 \in \mathscr{U}$; thus the Pompeiu property fails for $\mathscr{P}$. If, on the other hand, the transforms have no common zero we may, in view of the isomorphism between $\mathscr{E}_0'(G)$ and the even functions in $\hat{\mathscr{E}}'(\mathbf{R})$, apply Schwartz' one-variable theorem to conclude that $\mathscr{P}$ has the Pompeiu property.

To obtain a more tractable condition than the vanishing of the transforms $(\check{\chi}_A * T)^\wedge$ requires some additional calculation. Recall that $M(n)$ can be represented as the group of $(n+1)\times(n+1)$ matrices of the form $g = \begin{Vmatrix} k & x \\ 0 & 1 \end{Vmatrix}$, where $k \in \mathrm{SO}(n)$ and $x \in \mathbf{R}^n$ (as a column vector). Then

$$g^{-1} = \begin{Vmatrix} k^{-1} & -k^{-1}x \\ 0 & 1 \end{Vmatrix} \quad \text{and} \quad dg = dk\,dx,$$

where dk is normalized Haar measure on $\mathrm{SO}(n)$ and dx is Lebesgue measure on $\mathbf{R}^n$. Suppose f is K left-invariant and T is K right-invariant. Then $F(g) = (f * T)(g)$ is bi-invariant and so depends only on $|x|$. Abusing notation, we write $F(x)$ $(x \in \mathbf{R}^n)$.

Taking $g = \begin{Vmatrix} I & x \\ 0 & 1 \end{Vmatrix}$, we have

$$F(x) = \int_G f(gh^{-1}) T(h)\,dh$$

where

$$h = \begin{Vmatrix} k & y \\ 0 & 1 \end{Vmatrix} \quad \text{and} \quad dh = dk\,dy.$$

Since T is right invariant we may write

$$\tau(y)=T\left(\begin{Vmatrix} I & y \\ 0 & 1 \end{Vmatrix}\right)=T(h).$$

Moreover,

$$gh^{-1}=\begin{Vmatrix} I & x \\ 0 & 1 \end{Vmatrix}\begin{Vmatrix} k^{-1} & -k^{-1}y \\ 0 & 1 \end{Vmatrix}=\begin{Vmatrix} k^{-1} & x-k^{-1}y \\ 0 & 1 \end{Vmatrix},$$

so that

$$f(gh^{-1})=f\left(\begin{Vmatrix} k^{-1} & x-k^{-1}y \\ 0 & 1 \end{Vmatrix}\right)=f\left(\begin{Vmatrix} I & kx-y \\ 0 & 1 \end{Vmatrix}\right)$$

since f is left-invariant. Writing $\varphi(x)=f(\begin{Vmatrix} I & x \\ 0 & 1 \end{Vmatrix})$, we obtain

$$F(x)=\int_G \varphi(kx-y)\tau(y)\,dk\,dy,$$

so that the Fourier transform is given by

$$\hat{F}(\xi)=\int_{\mathbf{R}^n} e^{-ix\cdot\xi}\int_{\mathbf{R}^n}\int_{\mathrm{SO}(n)} \varphi(kx-y)\tau(y)\,dk\,dy\,dx.$$

Set $x'=kx$; then $x\cdot\xi=k^{-1}x'\cdot\xi=x'\cdot(k^{-1})^t\xi=x'\cdot k\xi$ and $dx=dx'$ since $k\in \mathrm{SO}(n)$. Interchanging order of integration and writing x for x', we obtain

$$\begin{aligned}\hat{F}(\xi)&=\int_{\mathrm{SO}(n)}\int_{\mathbf{R}^n} e^{-ix\cdot k\xi}\int_{\mathbf{R}^n}\varphi(x-y)\tau(y)\,dy\,dx\,dk\\&=\int_{\mathrm{SO}(n)}\hat{\varphi}(k\xi)\hat{\tau}(k\xi)\,dk.\end{aligned}$$

This is the desired formula.

Setting $f=\check{\chi}_{\tilde{A}}$, we have

$$\varphi(x)=\check{\chi}_{\tilde{A}}\left(\begin{Vmatrix} I & x \\ 0 & 1 \end{Vmatrix}\right)=\chi_{\tilde{A}}\left(\begin{Vmatrix} I & -x \\ 0 & 1 \end{Vmatrix}\right)=\chi_A(-x)$$

so that $\hat{\varphi}(\xi)=\hat{\chi}_A(-\xi)$, while τ is an arbitrary distribution of compact support in $\mathbf{R}^n$. Suppose that $\hat{\chi}_A$ vanishes on the complex sphere $M_\alpha=\{\xi_1^2+\xi_2^2+\cdots+\xi_n^2=\alpha\}$,

$\alpha \neq 0$; then (9.1) shows that, for $\xi \in M_\alpha$, $\hat{F}(\xi) = (\check{\chi}_A * T)\hat{}(\xi) = 0$, whatever the choice of T. Conversely, if $\hat{F} = 0$ on M_α for all choices of T, we have $\hat{\chi}_A = 0$ on M_α. Indeed, write $\alpha = |\alpha| e^{i\theta}$ and consider the $(n-1)$-sphere $S = \{e^{i\theta/2}x : x \in \mathbf{R}^n, |x|^2 = |\alpha|\} \subset M_\alpha$. The restrictions of holomorphic polynomials $p \in \mathbf{C}[z_1, z_2, \ldots, z_n]$ are dense in $C(S)$ because $z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}|_S = e^{i(m_1+m_2+\cdots+m_n)\theta/2} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$. Since $C[z_1, z_2, \ldots, z_n] \subset \hat{\mathscr{E}}'(\mathbf{R}^n)$ and $S = \mathrm{SO}(n) \cdot (\sqrt{|\alpha|} e^{i\theta/2}, 0, \ldots, 0)$, it follows from (9.1) that $\hat{\chi}_A = 0$ on S. An additional reasoning, based on the fact that a function analytic on a connected open set $U \subset C^m$ which vanishes on $U \cap \mathbf{R}^m$ must vanish on all of U, now shows that $\hat{\chi}_A = 0$ on the full variety M_α; we omit the details. This argument fails for $\alpha = 0$; in that case, however, $\hat{\chi}_A(0) = |A| \neq 0$, so that (9.1) yields $\hat{F}(0) = \hat{\chi}_A(0)\hat{\tau}(0) \neq 0$ as long as $\hat{\tau}(0) \neq 0$.

The discussion given above actually shows that if $\mathscr{P} = \{P\}$ is a family of distributions for which $M_\alpha \not\subset \bigcap_{\mathscr{P}} \{P^{-1}(0)\} = Z$ for all $\alpha \neq 0$ then any solution of the system $P^g(f) = 0$, $g \in M(n)$, $P \in \mathscr{P}$ is polyharmonic; i.e., there exists $m = m(\mathscr{P}) \geq 0$ such that $\Delta^m f = 0$. If $0 \notin Z$, then $m = 0$, and the family $\mathscr{P}$ possesses the Pompeiu property.

10.

The spaces considered in this paper, the symmetric spaces of rank one (compact and noncompact), together with the Euclidean spaces $\mathbf{R}^n$ $(n = 1, 2, 3, \ldots)$ and the circle S^1, comprise the *two-point homogeneous spaces* [17], [18], [23]. These are the Riemannian manifolds with the property that for any two pairs points (x_1, x_2) and (y_1, y_2) satisfying $d(x_1, x_2) = d(y_1, y_2)$, there exists an isometry mapping x_1 to y_1 and x_2 to y_2. The same collection of spaces also exhausts the class of manifolds known to be *harmonic spaces* [23]. (A Riemannian manifold X is harmonic if every function defined and harmonic on an open subset $U \subset X$ possesses the mean-value property at each point of U; cf. [23, pp. 45–52].) It would be interesting to investigate the extent to which the conditions defining either of these classes can be made to enter explicitly into the formulation and proof of the results of the present paper.

REFERENCES

[1] BERENSTEIN, C. A. and TAYLOR, B. A., *The "three-squares" theorem for continuous functions*, Arch. Rat. Mech. Anal. *63* (1977), 253–259.

[2] BERENSTEIN, C. A. and ZALCMAN, L., *Pompeiu's problem on spaces of constant curvature*, J. Analyse Math. *30* (1976), 113–130.

[3] BOCHNER, S. and CHANDRASEKHARAN, K., *Fourier Transforms*, Princeton University Press, 1949.
[4] BROWN, L., SCHREIBER, B. M. and TAYLOR, B. A. *Spectral synthesis and the Pompeiu problem*, Ann. Inst. Fourier (Grenoble) *23* (1973), 125–154.
[5] CARROLL, R. W. and SHOWALTER, R. E., *Singular and Degenerate Cauchy Problems*, Academic Press, 1976.
[6] DELSARTE, J., *Note sur une propriété nouvelle des fonctions harmoniques*, C. R. Acad. Sci. Paris *246* (1958), 1358–1360.
[7] DELSARTE, J. and LIONS, J. L., *Moyennes généralisées*, Comment. Math. Helvet. *33* (1959), 59–69.
[8] EGUCHI, M., HASHIZUME, M. and OKAMOTO, K., *The Paley–Wiener theorem for distributions on symmetric spaces*, Hiroshima Math. J. *3* (1973), 109–120.
[9] EHRENPREIS, L., *Fourier Analysis in Several Complex Variables*, Interscience, 1970.
[10] EHRENPREIS, L. and MAUTNER, F. I., *Some properties of the Fourier transform on semisimple Lie groups, II*, Trans. Amer. Math. Soc. *84* (1957), 1–55.
[11] ERDÉLYI, A. (ed.), *Higher Transcendental Functions*, vol. 1, McGraw-Hill, 1953.
[12] FLENSTED–JENSEN, M. *Paley–Wiener type theorems for a differential operator connected with symmetric spaces*, Arkiv för matematik *10* (1972), 143-162.
[13] FLENSTED-JENSEN, M. and KOORNWINDER, T., *The convolution structure for Jacobi function expansions*, Arkiv för matematik *11* (1973), 245–262.
[14] GRAY, A. and WILLMORE, T. J., *Mean-value theorems for Riemannian manifolds*, preprint.
[15] GUREVICH, D. I., *Counterexamples to a problem of L. Schwartz*, Functional Anal. Appl. *9* (1975), 116–120.
[16] HARISH-CHANDRA, *Discrete series for semisimple Lie groups*. II, Acta Math. *116* (1966), 1–111.
[17] HELGASON, S., *Differential Geometry and Symmetric Spaces*, Academic Press, 1962.
[18] ——, *The Radon transform on Euclidean spaces, compact two-point homogeneous spaces and Grassman manifolds*, Acta Math. *113* (1965), 153–180.
[19] ——, *The surjectivity of invariant differential operators on symmetric spaces*. I, Ann. of Math. *98* (1973), 451–479.
[20] ——, *A duality for symmetric spaces with applications to group representations*. I, Advances in Math. *5* (1970), 1–154.
[21] KOORNWINDER, T., *The addition formula for Jacobi polynomials and spherical harmonics*, SIAM J. Appl. Math. *25* (1973), 236–246.
[22] PIZZETTI, P., *Sulla media devi valori che una funzione dei punti dello spazio assume alla superficie di una sfera*, Rend. Lincei, ser. 5, *18* (1909), 182–185.
[23] RUSE, H. S., WALKER, A. G. and WILLMORE, T. J., *Harmonic spaces*, Edizioni Cremonese, 1961.
[24] SCHNEIDER, R., *Functions on a sphere with vanishing integrals over certain subspheres*, J. Math. Anal. Appl. *26* (1969), 381-384.
[25] ——, *Über eine Integralgleichung in der Theorie der konvexen Körper*, Math. Nachr. *44* (1970), 55–75.
[26] SCHWARTZ, L. *Théorie générale des fonctions moyenne périodiques*, Ann. of Math. *48* (1947), 857–929.
[27] ——, *Théorie des distributions*, Hermann, 1966.
[28] UNGAR, P. *Freak theorem about functions on a sphere*, J. London Math. Soc. *29* (1954), 100–103.
[29] VILENKIN, N. JA., *Fonctions spéciales et théorie de la représentation des groupes*, Dunod, 1969.
[30] ZALCMAN, L., *Analyticity and the Pompeiu problem*, Arch. Rat. Mech. Anal. *47* (1972), 237–254.
[31] ——, *Mean values and differential equations*, Israel J. Math. *14* (1973), 339–352.

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