A quick proof of the 4-dimensional stable surgery theorem.

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A quick proof of the 4-dimensional stable surgery theorem

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In 1971 Cappel and Shaneson published a proof that if $f:(M^4, \partial) \to (X, \partial)$ is a smooth surgery problem with trivial obstruction $(\sigma(f) = o \in L_4^s(\pi_1 X))$ then a stable solution for f exists. That is, for some k the map $f \# id:(M \# k(S^2 \times S^2), \partial) \to (X \# k(S^2 \times S^2), \partial)$ is normally bordant relative to the boundary to a simple homotopy equivalence.

At about the time of the Cappel–Shaneson result the second author discovered a homotopy theoretic proof of a closely related factorization result for surgery maps. The purpose of this note is to give a short geometric proof of this factorization result, and to observe that it implies the stable surgery theorem.

We shall call a surgery map *prepared* if it induces an isomorphism on π_0 and π_1, and the intersection form on the kernel $K_2(M)$ is a direct sum of standard planes. There is no difficulty in constructing a normal bordism of a map with trivial obstruction to a prepared one: First, surgeries on 0 and 1-spheres are used to achieve the homotopy conditions. The surgery obstruction is then defined to be the stable equivalence class of the intersection form on $K_2(M)$ [Wall]. Vanishing of the obstruction means that after addition of trivial planes, this kernel is isomorphic to a sum of planes. Since surgery on a trivial 1-sphere in M has the effect of adding a plane to $K_2(M)$, repetition of this operation yields a prepared map.

PROPOSITION 1. *Any prepared f factors up to homotopy as a surgery map through a simple homotopy equivalence* g:

$$\begin{array}{ccc} (M^4, \delta) & \xrightarrow{\quad f \quad} & (X, \delta) \\ & {\scriptstyle g}\searrow_{\simeq} \qquad \nearrow{\scriptstyle \text{projection}} & \\ & (X \# k(S^2 \times S^2), \delta) & \end{array}$$

Part of the data of a surgery map is a vector bundle ξ over X and a bundle

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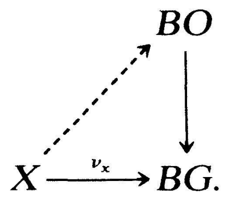
map over f, $b: \nu_M \to \xi$. By factoring "as a surgery map" we mean that this bundle map also factors through a map to the pull-back; $c: \nu_M \to p^*\xi$, where p is the projection.

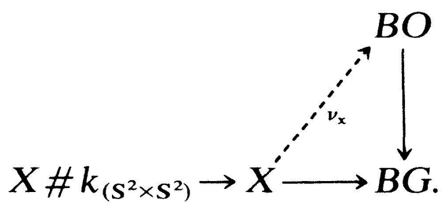
PROPOSITION 2. *Proposition* 1 *implies the stable surgery theorem.*

Proof of Proposition 2. Suppose f is a surgery map with trivial obstruction. As explained above we may assume f is prepared. We show that $f \# id_{k(S^2 \times S^2)}$ is normally bordant to the map g of Proposition 2. Since g is a simple homotopy equivalence this constitutes a solution of the stabilized surgery problem.

Normal bordism classes (rel ∂) correspond to lifts (rel ∂) of the classifying map for the normal fibration of X to BO;



(The uniqueness theorem for the normal fibration gives a fiber homotopy equivalence $\nu_x \simeq \xi$, which defines a lift.) Both g and $f \# id_{k(S^2 \times S^2)}$ have lifts obtained from the lift for f by composition with the projection:



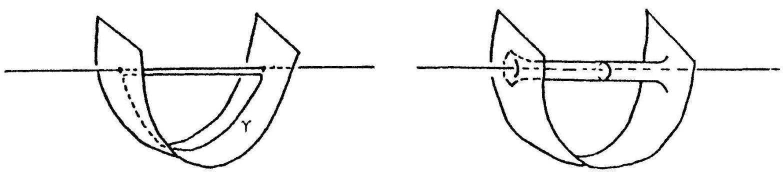
This is the lift corresponding to $f \# id_{k(S^2 \times S^2)}$ by direct inspection, and it corresponds to g by the existence of the factorization of the bundle map into $c: \nu_M \to p^*\xi$ and the pullback map. Since these maps correspond to the same lift, they are normally cobordant.

The philosophical significance is that the troublesome surgeries on 2-spheres are unnecessary for the stable surgery theorem. Once the surgery map is prepared, its domain *is* the domain of a stable solution; only a little tinkering is required to find the map g.

Proof of Proposition 1. Assume f is prepared. $K_2(M)$ has a preferred basis represented by framed immersed spheres $a_1, \ldots, a_k, b_1, \ldots, b_k$ with algebraic intersections $\lambda(a_j, a_j) = \lambda(b_j, b_j) = 0$, $\mu(a_i) = \mu(b_i) = 0$ and $\lambda(a_i, b_i) = \delta_{ij} \in Z[\pi_1 X]$.

The framing of each sphere's normal bundle is determined by null homotopies for these spheres in X together with the bundle map $b: \nu_M \to \xi$ covering f.

In dimension four this data may not be sufficient to produce disjointly embedded wedges of spheres. However, we can find framed disjointly embedded wedges of oriented surfaces $A_1 \vee B_1, \ldots, A_k \vee B_k$ representing the preferred basis, which are nullhomotopic in X. Suppose we have an algebraically cancelling pair of intersection points. Choose an arc between these points on one surface, and modify the other surface by an ambient o-surgery: replace discs by the normal sphere bundle restricted to the arc. Algebraically cancelling means first



the intersection points have opposite sign (so the result of the o-surgery is oriented) and second the loop formed by arcs on the two surfaces (γ in the picture) is nullhomotopic. The nullhomotopy may be used to construct a homotopy of the surged surface into the original one. Therefore nullhomotopy in X is also preserved by this operation.

Again these surfaces are framed by the nullhomotopy in X and the bundle map. The framing determines maps on the closed regular neighborhoods $h_i: (\mathscr{n}(A_i \vee B_i), \partial) \to (S^2 \times S^2 - \operatorname{int} D^4, \partial)$, $1 \leq k$.

Assume, as in [Wall, Chapter 2] that X has a top 4-cell. Let $D_1, \ldots, D_k$ be disjoint 4-discs in the top cell. Then there is a map f' homotopic (rel ∂) to f such that $(f')^{-1}(D_i, \partial) = (\mathscr{n}(A_i \vee B_i), \partial)$: First find a map f'' (using the nullhomotopies) such that $f''(\mathscr{n}(A_i \vee B_i), \partial) = (D_i, \partial)$ and (by transversality) the rest of the inverse image of D_i consists of discs mapping difformorphically to D_i. Since f'' is degree 1, the extra discs may be cancelled by a further homotopy. The result is f'.

The factorization g is constructed by cutting and pasting: $g = f' \mid (M - \amalg \mathscr{n}(A_i \vee B_i)) \cup \amalg h_i$. This does not change the isomorphism on π_1, and the following homology calculation (with $Z[\pi_1 X]$ coefficients) shows that g is a simple homotopy equivalence.

Let $\mathscr{n}$ denote $\amalg_{i=1}^{k} \mathscr{n}(A_i \vee B_i)$, $M^- = M$-int $\mathscr{n}$. From the Mayer–Vietoris sequences of kernel modules of

$$K_2^f(\partial \mathscr{n}) \to K_2^f(\mathscr{n}) \oplus K_2^f(M^-) \to K_2^f(M) \to 0$$

we see that

$$K_2^f(\partial \mathscr{N}) \xrightarrow{\text{inc}_*} K_2^f(M^-)$$

is onto, the middle arrow having been constructed to be a simple isomorphism when restricted to the first summand. Now consider the same sequence replacing f by g. $K_2^f(\partial \mathscr{N}) = K_2^g(\partial \mathscr{N})$ and $K_2^f(M^-) = K_2^g(M^-)$ so the map

$$K_2^g(\partial \mathscr{N}) \xrightarrow{\text{inc}_*} K_2^g(M^-)$$

remains an epimorphism. By construction $K_2^g(\mathscr{N}) \cong 0$. Consequently $K_2^g(M) \cong 0$. A standard argument using Poincaré duality shows that g induces an isomorphism on $H_*(\ ; Z[\pi_1 X])$ for all $*$ and by Whitehead's theorem must be a homotopy equivalence. The simplicity of $K_2^f(\mathscr{N}) \to K_2^f(M)$ implies that g is in fact a simple homotopy equivalence.

Finally the nullhomotopy of the $A_i \vee B_i$ in X, and the bundle map $b : \nu_M \to \xi$ define a framing of the restriction of ν_M to the neighborhood $\mathscr{N}_i$. This can be interpreted as a factorization of b through the pullback $p^*\xi$, since this pullback is trivial on the summands $\# S^2 \times S^2$.

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