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Introduction

In this paper it is shown that, contrary to previous beliefs, link cobordism does not reduce to knot cobordism. This is a consequence of a study of link cobordism that applies the global methods of homology equivalences introduced in [CS2].

An m-link in the $(n+2)$-sphere S^{n+2} is a smooth oriented sub-manifold $\Sigma^n \subset S^{n+2}$, where $\Sigma^n = \Sigma_1^n \cup \cdots \cup \Sigma_m^n$ is the ordered disjoint union of m manifolds that are piecewise linearly homeomorphic[1] to the n-sphere. Two m-links $\Sigma_0 = \Sigma_{0,1} \cup \cdots \cup \Sigma_{0,m} \subset S^{n+2}$ and $\Sigma_1 \subset S^{n+2}$ are said to be cobordant if there is a smooth oriented submanifold $V \subset S^{n+2} \times [0, 1]$, piecewise linearly homeomorphic[1] to $\Sigma_0 \times [0, 1]$, which meets the boundary transversely in ∂V, so that $V \cap (S^{n+2} \times i) = \Sigma_i$ for $i = 0, 1$. Let $C(n, m)$ denote the set of cobordism classes[2] of m-links in S^{n+2}. Thus $C(n, 1) = C_n$ is the usual knot cobordism group.

Links can arise from singularities of complex algebraic hyper-surfaces. More generally, recall that the global understanding of knot cobordism via homology equivalences, given in [CS2], was a key ingredient for the study of piecewise linear embeddings and immersions and their singularities [CS3] [CS4] [CS5] [CS6] [CS7] [CS8] [CS9]. A similar global point of view on links could enhance the study of singularities and multiple points of P.L. singularities and could also serve as a point of departure for the study of embeddings, immersions and singularities of a class of objects wider than the class of manifolds. In addition, a description by invariants of the Seifert surface type[3] as in [L] for knot cobordism, does not yet exist for links.

A link $\Sigma^n \subset S^{n+2}$ is said to be *split* if the components Σ_i, $1 \le i \le m$, are contained in mutually disjoint disks in S^{n+2}. It apparently has been believed for some time (see [G1] [G2] and [G3], and the review of [G3], MR54 #3709) that

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[1] This is the same as homeomorphic except possibly for $n = 4,3$.

[2] By smoothing theory, $C(n, m)$ can also be described as concordance classes [H] of P.L. locally flat embeddings of a union of disjoint copies of S^n in S^{n+2}.

[3] "Local invariants", from our point of view.

every link (or at least every boundary link, but see [G2]) is cobordant to a split link, for $n>1$. This is equivalent to the assertion that the map

$$\phi : C_n \times \cdots \times C_n = (C_n)^m \to C(n, m),$$

given by placing knotted spheres into disjoint disks, is a bijective map (it is obviously injective). Thus questions about higher dimensional link cobordism would reduce to the case of knots.

This paper begins the study of links from the global point of view of [CS2]. In particular we show:

THEOREM 1. *For $m \geq 2$ and $n > 4$ odd, there exist infinitely many distinct cobordism classes of m-links in S^{n+2}, none of which contains a split link.*

In other words, the map ϕ is actually very far from surjective. The cobordism classes constructed will actually contain *boundary links* that also have the property that each component is unknotted; i.e., *isotopic* to the trivial knot. With more care, one can arrange examples with each $(m-1)$-sublink trivial, given m.

Theorem 1 has an interesting interpretation in terms of non-locally flat piecewise linear cobordism. It is well-known that, if the smoothness hypotheses are dropped from the definition, every P.L. (not necessarily *locally* flat or smoothable) knot is concordant to the trivial knot (see [H]). From Theorem 1 it is not hard to show that arbitrary P.L. link cobordism is highly non-trivial; in fact, the set of cobordism classes of P.L. (not necessarily locally smoothable) m-links in S^{n+2} will not be finite, for $m \geq 2$ and $n > 2$.

An m-link $\Sigma^n \subset S^{n+2}$ is called a *boundary link* if there are smooth disjoint orientable submanifolds $U_1, \ldots, U_m$ with $\partial U_i = \Sigma_i^n$. Equivalently, let F_m be the free group on generators $x_1, \ldots, x_m$. Then $\Sigma^n \subset S^{n+2}$ is a boundary link if and only if there is a homorphism of $\pi_1(S^{n+2} - \Sigma^n)$ onto F_m that sends a meridian[(4)] about Σ_i^n to x_i (see 1.1 below and [G1]). Similarly one defines boundary cobordism of boundary links; see §1 for the exact definition. Let $B(n, m)$ denote the boundary cobordism classes of boundary m-links in S^{n+2}. Let $\psi : B(n, m) \to C(n, m)$ be the natural map; note that for $m = 1$, ψ is an isomorphism.

This paper determines $B(n, m)$ in terms of the algebraic K-theoretic objects introduced in [CS2], and uses them for some explicit calculations. Let

$$\mathcal{F}_m : Z[F_m] \to Z$$

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[4] As long as one keeps the requirement that the homomorphism be onto, it is equivalent to require merely that a meridian map to a conjugate of x_i. Recall that a meridian is a fibre of a tubular neighborhood of Σ_i.

be the augmentation map of the integral group ring of F_m. Let $\Gamma_j(\mathcal{F}_m)$ denote the algebraically defined abelian group given in [CS2], and let $\tilde{\Gamma}_j(\mathcal{F}_m)$ denote the cokernel of the natural map from the L-group $L_j(F_m)$ to $\Gamma_j(\mathcal{F}_m)$. Let $\mathcal{A}_m$ denote the automorphisms of F_m that carry each x_i to a conjugate of itself, modulo inner automorphisms. By naturality $\mathcal{A}_m$ acts on $\tilde{\Gamma}_j(\mathcal{F}_m)$. Let

$$P_j = \begin{cases} Z \\ 0 \\ Z_2 \\ 0 \end{cases} \text{if } j = \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \bmod 4.$$

THEOREM 2. *For $n \geq 2$, $B(n, m)$ is isomorphic*[5] *to*

$$(\tilde{\Gamma}_{n+3}(\mathcal{F}_m)/\mathcal{A}_m) \times mP_{n+1}$$

In particular, it follows (see Theorem 6.1 below) from calculations in [CS2] that $B(n, m) = 0$ for n even, a result of Kervaire [K] for $m = 1$ and of Gutierrez [G1] for $m > 1$. The above theorem for $m = 1$ is a result of [CS2].

The precise nature of the isomorphism of Theorem 2 is discussed below.

However, recall that the Γ-groups represent obstructions for normal maps to be cobordant to homology equivalences with prescribed coefficients (trivial integer coefficients in the present case). The isomorphism of Theorem 2 then arises from the view-point that the question of whether two links are cobordant is essentially the same as the question of whether their closed complements are cobordant as manifolds, relative boundary, *via* a cobordism that has the homology of a product with [0, 1].

Now, an algebraic calculation given below shows that the natural map from m copies of $\tilde{\Gamma}_{n+3}(\mathcal{F}_1)$ to $\tilde{\Gamma}_{n+3}(\mathcal{F}_m)$ has a non-finitely generated cokernel. (On the i^{th} copy, this map is induced by mapping F_1 to the subgroup generated by x_i. Actually the equivalent fact that the natural map

$$\tilde{\Gamma}_{n+3}(\mathcal{F}_m) \rightarrow (\tilde{\Gamma}_{n+3}(\mathcal{F}_1))^m$$

has a big kernel is what appears below). It follows that $B(n, m)$ cannot split as a sum of copies of $B(n, 1) = C_n$.

If the map $\psi: B(n, m) \rightarrow C(n, m)$ that forgets the "boundaryness" happens to be an isomorphism, then Theorem 2 provides a complete algebraic description of

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[5] For boundary links one can show that connected sum along curves that miss the interiors of bounding surfaces induces a group structure. But the proof of this seems to require some modification to make the fundamental group of the complement free; see below or [G1].

link cobordism, and Theorem 1 then follows from what has just been said. However, it does not seem to be definitely known whether or not ψ is bijective.[6] Let $\mathcal{F}_{m,ab}: Z[F_m/[F_m, F_m]] \to Z$ be the augmentation of the free abelian group. To prove Theorem 1 we show (in §7) that it is possible to detect many elements in $C(n, m)$, and non-splitting in particular, by passing to $\tilde{\Gamma}_{n+3}(\mathcal{F}_{m,ab})$. Because the free abelian group on more than one generator is not a high dimensional link group, this requires a number of delicate arguments. The invariants involved in the calculation mentioned above actually detect elements in $\tilde{\Gamma}_{n+3}(\mathcal{F}_{m,ab})$, modulo the image of $(\tilde{\Gamma}_{n+3}(\mathcal{F}_1))^m$, so that Theorem 1 follows.

The paper concludes with some remarks on the map ψ and its possible relations to algebraic questions.

§1. Boundary links

Let $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_m \subset S^{n+2}$ be an m-link. Then Σ has a tubular neighbourhood

$$\Sigma = \Sigma \times 0 \subset \Sigma \times D^2 \subset S^{n+2},$$

D^2 the 2-disk, and for $n \geq 2$ the embedding of $\Sigma \times D^2$ is unique up to ambient isotopy relative Σ. For $x_i \in \Sigma_i$, $x_i \times \partial D^2$ is called a *meridian* of Σ_i or an i^{th} meridian of Σ. The orientation of Σ, a fixed orientation of S^{n+2}, and a convention (assumed settled once and for all) give an orientation to each meridian. Thus, if each meridian is connected to a basepoint, one obtains elements of $\pi_1(S^{n+2} - \Sigma)$, well-defined up to conjugation. These are also called *meridians*, or *meridianal elements*. For a fixed choice of meridianal elements we thus obtain a homomorphism

$$\tau: F_m \to \pi_1(S^{n+2} - \Sigma) = \pi_\Sigma.$$

PROPOSITION 1.1. (Compare [G1]). *$\Sigma \subset S^{n+2}$ is a boundary link if and only if τ splits for some choice of meridians.*

Proof. (Outline) If τ splits, realize the splitting as a map of $S^{n+2} - \Sigma$ to the 1-point union $S^1 \vee \ldots \vee S^1$ of m circles and apply transversality (away from the common point) to obtain disjoint surfaces bounding the components.

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[6] See [G2]. The failure of ψ to be surjective would provide different types of examples of non-splittable cobordism classes.

Conversely, say $\Sigma_i = \partial V_i$, $1 \le i \le m$, with V_i disjoint. Suppose that $V \cap T$, T as above, is a boundary collar of V, and let $\bar{V} = V - \operatorname{Int} V \cap T$. Apply the Thom-Pontrjagin construction (see e.g. [Sto]), in a relative form to $(\bar{V}, \partial\bar{V})$ in $(S^{n+2} - \operatorname{Int} \Gamma, \partial T)$. The result is a map of $S^{n+2} - \operatorname{Int} T$ to $S^1 \vee \ldots \vee S^1$. The induced map on fundamental group is easily seen to carry suitable meridians to generators $x_1 \ldots x_m$, where $F_m = \pi_1(S^1 \vee \ldots \vee S^1)$, x_i represented by the i^{th} circle.

By Stalling's theorem [St], a map τ given as above by a choice of meridians induces a monomorphism of F_m into $(\pi_\Sigma)/(\pi_\Sigma)_\omega$, $(\pi_\Sigma)_\omega$ the intersection of the terms in the lower central series. Thus if τ splits, it induces an isomorphism of F_m with $(\pi_\Sigma)/(\pi_\Sigma)_\omega$. Let a map $\theta : \pi_\Sigma \to F_m$ be called a *splitting map* (for the link Σ) if it is surjective and carries meridians to conjugates of the generators.[7] Then θ induces an isomorphism $(\pi_\Sigma)/(\pi_\Sigma)_\omega \to F_m$. The next result follows:

PROPOSITION 1.2. *Any two splitting maps for an m-link Σ in S^{n+2} differ by an automorphism of F_m that sends x_i to a conjugate of x_i, $1 \le i \le m$.*

Results analogous to 1.1 and 1.2 also hold for boundary cobordisms, which will be defined momentarily. The details of 1.2 are left to the reader.

An F_m-link will be defined as an m-link $\Sigma^n \subset S^{n+2}$, together with a splitting map θ. Two F_m-links $(\Sigma_i \subset S^{n+2}, \le_i)$, $i = 0, 1$, are said to be *cobordant* if there is a cobordism $V \subset S^{n+2} \times I$ of $\Sigma_0 \subset S^{n+2}$ with $\Sigma_1 \subset S^{n+2}$ and a surjective map of $\pi_1(S^{n+2} \times I - V) \to F_m$ that agrees with θ_0 and θ_1 under composition with the natural maps from π_{Σ_0} and π_{Σ_1} to $\pi_1(S^{n+2} \times I - V)$, up to an inner automorphism[8] of F_m. Such a cobordism will be called an F_m-cobordism. Let $C_n(F_m)$ denote the F_m-cobordism classes of F_m-links in S^{n+2}. Also, define boundary links to be boundary cobordant if they have splitting maps for which the resulting links are F_m-cobordant.

Let $\mathcal{A}_m$ denote the group of automorphisms of F_m that map each x_i to a conjugate, modulo inner automorphisms. Clearly $\mathcal{A}_m$ acts on $C_n(F_m)$ by composition with the splitting map.

PROPOSITION. 3. $B(n, m) = C_n(F_m)/\mathcal{A}_m$.

This follows easily from 1.1 and 1.2

Note. Suppose that π is a link group with given normal generators $\xi_1, \ldots, \xi_m$, i.e., by [K], π is finitely generated, $H_1(\pi) = Z^m$, $H_2(\pi) = 0$. Then $C_n(\pi)$ can be defined, similarly, as cobordism classes of links with maps of the group of the complement onto π that carry an i^{th} meridian to a conjugate of ξ_i.

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[7] See footnote 4.

[8] This is necessary because of ambiguity in the choice of basepoint.

§2. Characteristic and complementary maps for links

Let $\bar{X}_*(m, n) = S_1^1 \vee \cdots \vee S_m^1 \vee S_1^{n+1} \vee \cdots \vee S_{m-1}^{n+1}$ be the indicated one-point union of (oriented) circles and $(n+1)$-spheres. Let $Y_*(m, m)$ be the disjoint union of m-copies $(S^n \times S^1)_i$, $1 \le i \le m$ of $S^n \times S^1$. Let $g : Y_*(m, n) \to \bar{X}_*(m, n)$ be the map defined as follows: for $i \ne 1, m$, define $g \mid (S^1 \times S^n)_i$ by first collapsing $\mathrm{pt} \times S^n$ to a point, to obtain $S^1 \vee S^{n+1}$. Then map the circle summand to S_i^1 homeomorphically with degree one, and the $(n+1)$-sphere so as to represent the difference of the homotopy class[9] represented by S_i^{n+1} and S_{i-1}^{n+1}; i.e., $[S_i^{n+1}] - [S_{i-1}^{n+1}]$. For $i = 1$, do the same thing, but map the $(n+1)$-sphere so as to represent $[S_1^{n+1}]$, and, for $i = m$, map it to represent $-[S_{m-1}^{n+1}]$. This defines g up to homotopy. Let $X_*(m, n)$ be the mapping cylinder of g.

PROPOSITION 2.1. *$(X_*(m, n),\ Y_*(m, n))$ is a simple Poincare pair (as defined in* [W], *for example), of dimension $(n+2)$.*

In fact (X_*, Y_*) has the (simple) homotopy type of the m-fold interior connected sum of $S^1 \times D^{n+1}$ with itself. In the present paper, however, only the following easy fact is used essentially: (X_*, Y_*) satisfies Poincare duality with *respect to integer coefficients*; i.e., there is a class in $H_{n+2}(X_*, Y_*; Z)$, cap product with which induces isomorphisms of $H^i(X_*; Z)$ with $H_{n+2-i}(X_*, Y_*; Z)$.

We will identify $\pi_1 X_*$ with F_m, so that S_i^1 represents x_i.

If π is any link group, a similar construction using $K(\pi, 1)$ instead of a wedge of circles yields $(X_*(\pi), Y_*)$. If π has no higher integral homology, then one still obtains an integral Poincare complex.

PROPOSITION 2.2. *Let $\Sigma^n \subset S^{n+2}$ be an m-link with tubular neighborhood $T = \Sigma \times D^2 \subset S^{n+2}$ and let $\alpha_\partial : \partial T \to Y_* = Y_*(m, n)$ have the form $K \times id_{S^1}$ on each component, K a degree one map of (oriented) homotopy n-spheres. Let θ be a splitting map for $\Sigma^n \subset S^{n+2}$. Then α_∂ extends to a map*

$$\alpha : (S^{n+2} - \mathrm{Int}\, T) \to X_*(m, n) = X_*$$

that induces θ on fundamental groups and that induces isomorphisms on homology groups with integer coefficients.

The map α will be called a complementary map for the F_m-link $(\Sigma^n \subset S^{n+2}, \theta)$. Note that it has degree one, because its restriction to the boundary does.

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[9] The common point in the 1-pt. union is the basepoint.

Consider the union of X_* with m-copies of $S^n \times D^2$ attached along Y_*, Z_* say. Then Z_* has the homotopy type of S^{n+2} (this is easy to see) and contains a link Σ_*, the (ordered) union of m copies of $S^n \times 0 \subset S^n \times D^2$. Clearly α can be completed to a degree-one map $\bar{\alpha}: S^{n+2} \to Z_*$, transverse regular to Σ_* with $(\bar{\alpha})^{-1}\Sigma_* = \Sigma$. The map $\bar{\alpha}$ is called a *characteristic map* of the link $\Sigma \subset S^{n+2}$; it expresses Σ as the inverse image of a trivial link, in view of the discussion following Proposition 2.1.

Proof of 2.2. For $1 < i \leq n$, $\pi_i(X_*)$ is trivial, and $\pi_{n+1}(X_*)$ is the free module over the integral group ring $Z[F_m]$, generated by the classes $[S_1^{n+1}], \ldots [S_{m-1}^{n+1}]$. The map α_∂ can easily be extended to the relative 2-skeleton of $(X, \partial T)$, where $X = S^{n+2} - \operatorname{Int} T$, so as to induce θ on the fundamental group.

There remains a single obstruction to completing this extension to all of X, $o(\alpha_\partial, \theta) \in H^{n+2}(X, \partial T; \pi_{n+1}(X_*))$. By Poincare duality, this homology group with local coefficients is isomorphic to $H_0(X; \pi_{n+1}(X_*))$. Since θ is onto, it is an exercise in the definitions of homology with local coefficients that the coefficient homomorphism $a: \pi_{n+1}(X_*) \to \pi_{n+1}(X_*) \otimes_{Z[F_m]} Z = Z^{m-1}$ induces an isomorphism of $H_0(X; \pi_{n+1}(X_*))$ and the homology group $H_0(X; Z^{m-1})$ with trivial coefficients. So the map

$$a_*: H^{n+2}(X, \partial T; \pi_{n+1}(X_*)) \to H^{n+2}(X, \partial T; Z^{m-1})$$

induced by a is also an isomorphism. Thus it suffices to show that $a_*(o(a_\partial, \theta)) = 0$.

The target of a_* can be thought of as the free abelian group generated by $[S_1^{n+1}], \ldots, [S_{m-1}^{n+1}]$. Let $\pi_j: X_* \to S_j^{n+1}$ denote the obvious map. Write

$$a_*(o(\alpha_\partial; \theta)) = \sum_{j=1}^{m-1} \beta_j [S_j^{n+1}].$$

Then $\beta_j[S_j^{n+1}] = (\pi_j)_* a_* o(f, \theta)$. But by naturality of obstructions, the right hand term is just the obstruction in $H^{n+2}(X, \partial T; \pi_{n+1}(S_j^{n+1})) = H^{n+2}(X, \partial T; Z)$ to extending $\pi_j \alpha_\partial$ to all of X. From degree considerations it follows easily that this obstruction vanishes. Thus $a_*(o(\alpha_\partial, \theta)) = 0$. Thus α exists. Since it has degree one (as α_∂ does), it induces a surjection on integral homology, by 2.1. But X and X_* have isomorphic (and finitely generated) homology, by Alexander duality for X and direct calculations for X_*. The final statement of 2.2 follows.

A relative form of 2.2 is also needed, to apply to a cobordism. However, in the relative case there are some new twists.

PROPOSITION 2.3. *Let $(\Sigma_i \subset S^{n+2}, \theta_i)$, $i = 0, 1$, $n \geq 2$ be F_m-links in S^{n+2}, and suppose that they represent the same elements in $C_n(F_m)$. Let $T_i = \Sigma_i \times D^2 \subset$*

S^{n+2} be tubular neighborhoods, and let $\alpha_i : S^{n+2} - \mathrm{Int}\, T_i \to X_*(m, n) = X_*$ be complementary maps, as in 2.2. Then there is a cobordism $V \subset S^{n+2} \times [0, 1]$ of $\Sigma_0 \subset S^{n+2}$ and $\Sigma_1 \subset S^{n+2}$, with tubular neighborhood $T = V \times D^2$ meeting $S^{n+2} \times i$ in T_i for $i = 0, 1$, and maps

$$A : S^{n+2} \times [0, 1] - \mathrm{Int}\, T \to X_*$$

$$\beta : (X_*, Y_*) \to (X_*, Y_*)$$

with the following properties:

(i) $A \mid S^{n+2} \times 1 - \mathrm{Int}\, T_1 = \alpha_1$;
(ii) $A \mid S^{n+2} \times 0 - \mathrm{Int}\, T_0 = \beta \circ \alpha_0$;
(iii) $A(V \times S^1) \subset Y_*$ *and* $A \mid V \times S^1$ *has the form* $K \times \mathrm{id}_{S^1}$;
(iv) $\beta \mid Y_*$ *is the identity,* β *induces the identity*[10] *on* $\pi_1 X_*$, *and* β *is a (simple) homotopy equivalence; and*
(v) A *induces isomorphisms on integral homology groups.*

The map A will be called a complementary map for the cobordism whose existence Prop. 2.3 asserts. The map β (which could be the identity, of course) will actually be the identity outside the $(n+2)$-cell in a cell decomposition of X_* *relative* Y_*. Further, β will actually have the property that β^2 is homotopic to the identity relative Y_*. Therefore composing A with β would change α_1 to $\beta\alpha_1$ in (i) and $\beta\alpha_0$ to α_0 in (ii).

Proof of 2.3. Given any cobordism $V \subset S^{n+2} \times [0, 1]$ from $\Sigma_0 \subset S^{n+2}$ to $\Sigma_1 \subset S^{n+2}$, it is a standard fact that there is a tubular neighborhood $T = V \times D^2$ that meets $S^{n+2} \times i$ in a tubular neighborhood T_i of $\Sigma_i \times D^2$, $i = 0, 1$. By hypothesis, we have a cobordism which admits a map

$$\theta : \pi_1(S^{n+2} \times [0, 1] - \mathrm{Int}\, T) \to F_m$$

whose composition with the inclusion induced maps agree with θ_i, $i = 0, 1$, up to inner automorphisms.

Consider the map $g = \alpha_0 \cup \alpha_1 \cup K \times \mathrm{id}_{S^1}$, defined on $X_0 \cup V \times S^1 \cup X_1 = \partial Q$, where $X_i = S^{n+2} \times i - \mathrm{Int}\, T_i$, $i = 0, 1$, and $Q = S^{n+2} \times [0, 1] - \mathrm{Int}\, T$. It is not hard to see that a K exists that makes g a well-defined mapping. The existence of θ implies that g can be extended over the relative 2-skeleton of $(Q, \partial Q)$. (This is shown by an essentially standard argument; note, however, that it involves

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[10] With respect to a given basepoint in $\mathrm{Int}\, X_* = X_* - Y_*$.

homotopies of α_0 and α_1 to themselves, relative boundary, that move basepoints around elements in $\pi_1(X_0)$ and $\pi_1(X_1)$ that map by the surjections θ_0 and θ_1 to elements of F_m that induce by conjugation the inner automorphisms mentioned in the preceding paragraph.)

Now let us assume temporarily that θ is actually an isomorphism; note that it is automatically surjective, as θ_0 is. The first possibly non-zero obstruction to extending g to all of Q lies in the homology group with local coefficients $H^{n+2}(Q, \partial Q;\ \pi_{n+1}X_*)$. By Poincare duality, this group is isomorphic to $H_1(Q, \pi_{n+1} X_*)$, which is just $(m-1)$ copies of $H_1(Q; Z[F_m])$, as $\pi_{n+1}X_*$ is free over $Z[F_m]$ of rank $m-1$. But if θ is assumed to be an isomorphism, then $H_1(Q; Z[F_m])$ is just H_1 of the universal covering space of Q and so trivial.

The only remaining obstruction is an element $o(g, \theta)$ in $H^{n+3}(Q, \partial Q;\ \pi_{n+2}X_*)$. The connecting homomorphism δ maps $H^{n+2}(\partial Q;\ \pi_{n+2}X_*)$ onto this group, as an $(n+3)$-manifold with boundary has the homotopy type of an $(n+2)$-complex. (Using Poincare duality, one can show that δ is actually an isomorphism.) Suppose that $\beta:(X_*, Y_*)\to(X_*, Y_*)$ is any map satisfying (iv) in the statement of 2.3. As $H^{n+1}(X_*, Y_*;\ \pi_{n+1} X_*)=0$ (either by direct calculation or Poincare duality and 2.1) the only obstruction $o(\beta)$ for β to be homotopic to the identity relative Y_* lies in $H^{n+2}(X_*, Y_*;\ \pi_{n+2} X_*)$.

Let j^* be the composite

$$H^{n+2}(X_0, \partial X_0;\ \pi_{n+2} X_*)\to H^{n+2}(\partial Q, X_1\cup V\times S^1;\ \pi_{n+2} X_*)$$
$$\to H^{n+2}(\partial Q;\ \pi_{n+2} X_*)$$

of the excision isomorphism and the natural map. As for δ, j^* is easily seen to be surjective.

Let $g_\beta=\beta\alpha_0\cup\alpha_1\cup K\times \mathrm{id}_{S^1}$. Then from the difference and composition formulae for obstructions (see e.g. [Stn]),

$$o(g_\beta;\theta)-o(g;\theta)=\delta j^*\alpha_0^*o(\beta).$$

Now, $\alpha_0^*:H^{n+2}(X_*, Y_*;\ \pi_{n+2}X_*)\to H^{n+2}(X_0, \partial X_0;\ \pi_{n+2}X_*)$ is also an isomorphism. This follows easily from 2.1, Poincare duality, and the fact that α_0 has degree one. (Again use of 2.1 and duality can be replaced by a direct calculation.)

We assert that β can be chosen with $o(\beta)$ arbitrary. In fact, choose a relative cell decomposition of (X_*, Y_*) with a single oriented $(n+2)$-cell. The $(n+2)$-cochains for this cell complex, with coefficients in $\pi_{n+2}X_*$, can then be identified with $\mathrm{Hom}_{Z[F_m]}(Z[F_m];\ \pi_{n+2}(X_*))=\pi_{n+2}X_*$ in the obvious way. Given $\gamma\in \pi_{n+2}X_*$, there is a map $\beta:(X_*, Y_*)\to(X_*, Y_*)$, that is the identity outside the $(n+2)$-cell, such that the difference cocycle for homotopy of β to the identity,

$c(\beta, \mathrm{id})$, is precisely γ. For example, just compose $\mathrm{id} \vee \gamma$ with the map $(X, Y) \to (X \vee S^{n+2}, Y)$ obtained by pinching the boundary of a smaller $(n+2)$-cell to a point. Of course, $o(\beta)$ is just the class represented by $c(\beta, \mathrm{id})$. Obviously $\beta \mid Y_* = \mathrm{id}$ and β induces the identity on the fundamental group. It is not hard to check directly that β is a simple homotopy equivalence. But instead, note that $\pi_{n+2}X_* = (Z_2[F_m])^{m-1}$ is all 2-torsion, so that β^2 is homotopic to the identity, rel Y_*. Thus β is a homotopy equivalence, and $\mathrm{Wh}(F_m) = 0$ (see [Ba]). Thus every co-cycle, and hence every cohomology class, has the form $o(\beta)$.

Therefore, choose β with $o(\beta) = -o(g, \theta)$. Then $o(g_\beta, \theta) = 0$. Therefore g_β extends to all of Q; let A be an extension. Clearly A satisfies (i)–(iv) in 2.3. By Alexander duality, it follows that $X_0 \subset Q$ induces an isomorphism of integral homology groups. By 2.2, α_0 also induces such isomorphisms. Therefore by (i) so does A; i.e., A satisfies (V).

It remains to show that there is a cobordism in which the splitting map θ is an isomorphism. So suppose that $V \subset S^{n+2} \times [0, 1]$, T, etc., are all as above, but that θ is not necessarily one-to-one; of course θ is onto because θ_0(or θ_1) is. Let $\xi_1, \ldots, \xi_r \in \pi_1(S^{n+2} \times [0, 1] - \mathrm{Int}\, T)$ be normal generators for the kernel of θ (see [Ku]). Of course, $Q = S^{n+2} \times [0, 1] - \mathrm{Int}\, T$ is parallelizable, as $S^{n+2} \times [0, 1]$ is. Let Q' be obtained from Q by framed surgery on circles representing $\xi_1, \ldots, \xi_r$. In other words, choose disjoint embeddings $S_i^1 \times D^{n+2} \subset \mathrm{Int}\, Q$ representing ξ_i, and let Q' be obtained from the disjoint union of $Q - \mathrm{Int}\, \widehat{\bigcup}_{i=1}^r S_i^1 \times D^{n+2}$ and $\bigcup_{i=1}^r D_i^2 \times S^{n+1}$ by identifying the corresponding boundary components $S_i^1 \times S^{n+2}$, $1 \le i \le r$. Here D_i^2 is a copy of a 2-disk, with boundary S_i^1. This can be done in such a way that Q' is also (stably) parallelizable $[M]$.

By an application of Van-Kampen's theorem, $\pi_1 Q'$ is the quotient of $\pi_1 Q$ by the normal subgroup generated by $\xi_1, \ldots, \xi_r$. Hence θ induces an isomorphism $\theta' : \pi_1 Q' \to F_m$. Obviously, the compositions of θ' with the inclusion induced maps $\pi_1 X_i \to \pi_1 Q'$ are the same as the compositions of θ with the maps $\pi_1 X_i \to \pi_1 Q$, $i = 0, 1$.

Since the circles $S_i^1 \times 0$ represent zero in homology $H_1(Q)$ with integer coefficients, it follows from the same type of argument as used in [K] that $H_2(Q)$ is a free abelian group generated by $\eta_1, \ldots, \eta_r$ say, and that $H_i(Q') = H_i(Q)$ (=0 by Alexander duality) for $3 \le i \le n$. The Hopf sequence $\pi_2(Q') \to H_2(Q') \to H_2(\pi_1 Q') = H_2(F_m) = 0$ shows that each η_i is spherical, and so they can be represented by disjointly embedded spheres, as $n \ge 2$. Again we can take appropriate tubular neighborhoods and perform (interior) framed surgery to obtain a (stably) parallelizable manifold Q''. By general position or Van-Kampen, $\pi_1 Q' = \pi_1 Q''$, so we have $\theta'' : \pi_1 Q'' \to F_m$, an isomorphism, with the compositions with the inclusion induced maps from $\pi_1 X_i$ still unchanged. Again by the same type of argument as in [K], one calculates that $H_i Q'' = 0$ for $2 \le i \le n$.

Of course $\partial Q = \partial Q''$. Let $W = Q'' \bigcup_{V\times S_1} T$. Since $S^{n+2}\times[0, 1] = Q \bigcup_{V\times S_1} T$, it follows from Van-Kampen's theorem that $\pi_1 Q$ is normally generated by m meridianal classes, which of course can be represented by circles in the boundary (connected to a base-point). Therefore $\pi_1 Q'' = \pi_1 Q/\langle \xi_1, \ldots, \xi_r\rangle$ is normally generated by classes represented by the same circles. Hence by Van-Kampen, $\pi_1 W$ is trivial. By the Meyer-Vietoris sequence, it is not hard to see that $H_i(W) = 0$ for $1 \leq i \leq n$. Since $n \geq 2$ (so that $n \geq [n/2]$), it then follows from Poincaré duality that W is an h-cobordism.

Thus, by the h-cobordism theorem, W is diffeomorphic to $S^{n+2}\times[0, 1]$. (For the case $n = 2$, see [S2].) Since any diffeomorphism of S^{n+2} is *isotopic* to the identity in the complement of any point or cell (by uniqueness of smooth disks in manifolds), the cobordism, obtained as a composition of inclusions,

$$V \subset T \subset W = S^{n+2}\times[0, 1]$$

is a cobordism of links isotopic[11] to $\Sigma_0 \subset S^{n+2}$ and $\Sigma_1 \subset S^{n+2}$. Since θ'' was an isomorphism, this implies that the desired cobordism exists.

§3. An invariant for F_m-links.

Let $(\Sigma^n \subset S^{n+2}, \theta)$ be an F_m-link. Let $X = S^{n+2} - \operatorname{Int} T$, $T = \Sigma^n \times D^2 \subset S^{n+2}$ a tubular neighborhood. Let

$$\alpha : (X, \partial X) \to (X_*, Y_*) = (X_*(m, n), Y_*(m, n))$$

be a complementary map. Let $\alpha_0 : (X_0, \partial X_0) \to (X_*, Y_*)$ be a fixed complementary map for the trivial link (i.e., a link of m components that bound disjoint disks.) Note that α_0 is a (simple)[12] homotopy equivalence. A *complementary normal cobordism for* $(\Sigma^n \subset S^{n+2}, \theta)$ is defined to be any normal cobordism [B1] (H, B) from α_0 to α; i.e.,

$$H : (W; \partial_- W, \partial_+ W, \partial_0 W) \to (X_*\times[0, 1], X_*\times 0, X_*\times 1, Y_*\times[0, 1]),$$

with $\partial_- W = X_0$, $\partial_+ W = X$, $H \mid \partial_- W = \alpha_0$, $H \in \partial_+ W = \alpha$, and B is a stable linear bundle map from the (stable) normal bundle of W to a bundle over $X_*\times[0, 1]$. (We assumed fixed an orientation of X_*, we require all normal maps to have

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[11] And so ambient isotopic by isotopy extension.

[12] $Wh(F_m) = 0$.

degree +1, and we adopt the convention $\partial(M\times[0,1])=[M\times 1]-[M\times 0]$ for oriented manifolds.)

LEMMA 3.1. *A complementary normal cobordism always exists.*

Proof. Let $\tilde{\alpha}$ and $\tilde{\alpha}_0$ be characteristic maps corresponding to α and α_0. Since they have the same degree, they are homotopic; let $H_1: S^{n+2}\times[0,1]\to Z_*\times[0,1]$ be a homotopy; i.e. $H_1(x,0)=(\alpha_0(x),0)$, $H_1(x,1)=(\alpha(x),1)$. Let B_1 be a bundle map, covering H_1, of trivial bundles. Clearly $H \mid S^n\times\{0,1\}$ is already transverse to $\Sigma_*\times[0,1]$. By a small homotopy of H_1, relative the boundary, we may assume that H_1 is actually transverse to $\Sigma_*\times[0,1]$. Further, it may be supposed that $H_1^{-1}(\Sigma_*\times D^2\times[0,1])$ is a tubular neighborhood for $H_1^{-1}(\Sigma_*\times[0,1])$, extending $T\cup T_0$, and that the restriction of H_1 to this neighborhood is an $SO(2)$ bundle map. (Here T_0 is a tubular neighborhood of the trivial link. Recall also that $Z_*=\Sigma_*\times D^2\cup_{Y_*}X_*$ and $Y_*=\Sigma_*\times S^1$.).

Now let $W=S^{n+2}\times[0,1]-\mathrm{Int}\,(\mathrm{H}^{-1}(\Sigma_*\times \mathrm{D}^2\times[0,1])$. then it is not hard to check that $(H_1\mid W, B_1\mid W)=(H,B)$ is the desired normal cobordism. This proves 3.1.

Given an F_m-link $(\Sigma\subset S^{n+2},\theta)$, let (G,C) be a complementary normal cobordism,

$$G:(W,\partial_-W,\partial_+W,\partial_0W)\to(X_*\times[0,1], X_*\times 0, X_*\times 1, Y_*\times[0,1]),$$

as above. Then $G\mid\partial_-W$ is a homotopy equivalence, and $G\mid\partial_\pm W$, by 2.2, induces isomorphisms of homology groups. $G\mid\partial(\partial_+W)$ is also a homotopy equivalence. Let $\Phi(=\Phi(m))$ be the diagram

$$\begin{array}{ccc} Z(\pi(Y_*)) & \xrightarrow{id} & Z(\pi(Y_*)) \\ \big\downarrow{\scriptstyle i_*=i_*(m)} & & \big\downarrow{\scriptstyle a} \\ Z\pi_1X_* & \xrightarrow{a} & Z \end{array}$$

be the indicated diagram of integral group rings of fundamental groupoids, with a the augmentation map and i_* induced by inclusion of the boundary components. (Thus $\Phi(m)$ can be identified with the result of taking integral rings in the diagram

$$\begin{array}{ccc} Z\cup\ldots\cup Z & \xrightarrow{id} & Z\cup\ldots\cup Z \\ \big\downarrow{\scriptstyle i_*} & & \big\downarrow \\ F_m & \longrightarrow & \{e\}\ ). \end{array}$$

It follows by [CS2] that the obstruction

$$\sigma(G,C)\in\Gamma_{n+3}\ (\Phi)$$

is defined. (Since $Wh(F_m)=0$, we omit the s or h superscript.) This is the obstruction[13] to finding (G', C') normally cobordant to (G, C) relative $\partial_- W \cup \partial_+ W$, with G' inducing isomorphisms of integral homology groups and with G' restricting to a homotopy equivalence on the part of the boundary corresponding to $Y_* \times [0, 1]$.

Let $L_{n+3}(i_*)$ denote the relative L-group [W] of the inclusion induced map. By [CS2] this is the same as Γ_{n+3} of the diagram

$$\begin{array}{ccc} Z[Y_*] & \xrightarrow{id} & Z[Y_*] \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\ Z[F_m] & \xrightarrow{id} & Z[F_m]. \end{array}$$

Hence there is a functiorial map $L_{n+3}(i_*) \to \Gamma_{n+3}(\Phi)$.

PROPOSITION 3.2. *Modulo the image of $L_{n+3}(i_*)$, the obstruction $\sigma(G, C)$ is independent of the choice of complementary normal cobordism and in fact depends only upon the F_m-cobordism class of $(\Sigma \subset S^{n+2}, \theta)$.*

Proof. First observe that if β is as in Proposition 2.3, then $\sigma(\beta \circ G, \hat{\beta} \circ C) = \sigma(G, C)$ for $\hat{\beta}$ any bundle map covering β; $\hat{\beta}$ is easily seen to exist, with domain the target bundle of C. This equation holds because of functorial properties of this obstruction [CS2] and because β induces the identity on the fundamental group.

Now suppose that $(\Sigma_1 \subset S^{n+2}, \theta_1)$ is F_m-cobordant to $(\Sigma \subset S^{n+2}, \theta)$, and let (G_1, C_1) be any complementary normal cobordism for $(\Sigma_1 \subset S^{n+2}, \theta_1)$. Then to prove 3.2 we must show that

$$\sigma(G_1, C_1) \equiv \sigma(G, C) \text{ modulo Image } L_{n+3}(i_*).$$

Write $G_1 : (W_1, \partial_- W_1, \partial_0 W_1, \partial_+ W_1) \to (X_* \times [0, 1], X_* \times 0, X_* \times 1, Y_* \times [0, 1])$, as for G; e.g., $\partial_- W_1 = X_0$, $\partial_+ W_1 = X_1 = S^{n+2} - \text{Int } T_1$ a tubular neighborhood of $\Sigma_1 \subset S^{n+2}$.

Let $V \subset S^{n+2} \times [0, 1]$ be the type of cobordism of $\Sigma \subset S^{n+2}$ with $\Sigma_1 \subset S^{n+2}$ whose existence[14] is asserted in Prop. 2.3. Let Q be the complement of a tubular neighborhood of V, so that $\partial Q = X \cup V \times S^1 \cup X_1$, and let

$$A : Q \to X_*$$

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[13] In Chapter I of [CS2] we assumed for simplicity that the "movable" part of the boundary was connected, but the theory is similar in general. Compare [W, §9].

[14] But note the slight change in notation, due to the fact that in the current discussion X_0 is used for the trivial link complement.

and β be as in 2.3, with respect to the complementary maps $G \mid X$ and $G_1 \mid X_1$; i.e., $A \mid X = G \mid X$ and $A \mid X_1 = G_1 \mid X_1$. In view of the opening observation, it may be supposed that β is the identity.

Let $U = W \bigcup_X Q \bigcup_{X_1} W_1$

be obtained from the disjoint union by the indicated identification of boundary components, Thus ∂U contains $\partial_0 U = \partial_0 W \bigcup_{\partial X} V \times S^1 \bigcup_{\partial X_1} \partial_0 W_1$ and $\partial U - \operatorname{Int} \partial_0 U$ is just two disjoint copies of X_0.

Let $\bar{G} : Q \to X_* \times [0, 3]$ be defined as follows:
First, $\bar{G} \mid W = G$. Then $G \mid Q = (A, \phi)$, where $\phi : (Q, X, X_1) \to ([1, 2], 1, 2)$ is a Morse function that restricts to a Morse function (without critical points) on the boundary. Finally, $G \mid W_1 = \gamma \circ G_1$, where $\gamma(x, t) = (x, 3\text{-}t)$ for $x \in X_*$, $t \in [0, 1]$. We also wish to find a bundle map $\bar{C}$ extending C and C_1. Now, $G \mid X$ is a homology equivalence and hence induces an isomorphism of stable bundle theories; i.e., of real K-theory and in particular of K_0 and K_{-1}. From this it is an exercise to check that $\bar{C}$ exists, at least if one is willing to replace C by $\delta \circ C$, δ a stable bundle map bundle map over the identity of $X_* \times [0, 1]$. From the opening observation again (with $\beta =$ identity), such a change doesn't affect $\sigma(G, C)$; thus it may be assumed that $\bar{C}$ exists.[15]

By additivity [CS2] $\sigma(\bar{G}, \bar{C}) = \sigma(G, C) + \sigma((A, \phi), \bar{C} \mid Q) - \sigma(G_1, C_1)$. The sign is due to the reversal of orientation of W_1 required to orient U.

But by 2.3(v), (A, ϕ) induces isomorphisms of integral homology, and from 2.3(iii) it follows that (A, ϕ) restricts to a homotopy equivalence of $\partial Q - \operatorname{Int}(X_0 \cup X_1)$ with $Y_* \times [0, 1]$. Therefore, $\sigma((A, \phi), \bar{C} \mid Q) = 0$.

Further, the restriction of $\bar{G}$ to $\partial U - \operatorname{Int} \partial_0 U$ is just two copies of the complementary map of the trivial link, mapping to $X_* \times 0$ and $X_* \times 3$. Thus this restriction is actually a (simple) homotopy equivalence. Therefore $(\bar{G}, \bar{C})$ actually has an obstruction, in $L_{n+3}(i_*)$, which vanishes if and only if $(\bar{G}, \bar{C})$ is normally cobordant to a homotopy equivalence, relative $\partial U - \operatorname{Int} \partial_0 U$. By naturality, $\sigma(\bar{G}, \bar{C})$ is the image of this obstruction under the natural map $L_{n+3}(i_*) \to \Gamma_{n+3}(\Phi)$. This completes the proof of 3.2.

§4. Calculation of F_m-link cobordism

In view of 3.2, the assignment of the obstruction $\sigma(G, C)$ of a complementary normal cobordism to an F_m-link induces a well-defined map

$$\rho = \rho(m, n) : C_n(F_m) \to \Gamma_{n+3}(\Phi(m))/L_{n+3}(i_*)$$

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[15] We may write $(\bar{G}, \bar{C}) = (G, C) \bigcup_X ((A, \phi), \bar{C} \mid Q) \bigcup_{X_1} -(G_1, C_1)$, taking account of orientation.

(The use of quotient notation on the right will be justified in §6, when it will be shown that $L_{n+3}(i_*)$ maps monomorphically to $\Gamma_{n+3}(\Phi)$.)

THEOREM 4.1. *For $n \geq 2$, but $n \neq 3$, the map $\rho(m, n)$ is an isomorphism, for all m. For $n = 3$ it is a monomorphism onto a subgroup of index 2^m.*

The proof for the case $n=2$ involves some special arguments about low dimensional normal maps and so will be omitted.[16] For $n = 3$, the image of ρ is the kernel of the composition natural map to $L_5(Z \cup \cdots \cup Z)(= Z^m$ by [S1]) and reduction mod 2. If one passes to topological links, ρ extends to an isomorphism of $C_3^{TOP}(F_m)$ with $\Gamma_6(\Phi)/L_6(i_*)$, and the quotient $C_3^{TOP}(F_m)/C_3(F_m)$ is mapped isomophically to Z_2^m by applying the map of [CS1] to each component. This situation is in fact analoguous to what happens for the case $m = 1$, discussed in [CS1] and [CS2]. Therefore we will also not discuss in detail the case $n = 3$. The proof that $\rho(3, m)$ is monic is almost the same as for higher n.

LEMMA 4.2. *For $n \geq 3$, let $\xi \in L_{n+3}(i_*)(=L_{n+3}(i_*(m)))$. Then the trivial m-link has a complementary normal cobordism, (G_ξ, C_ξ) say, with $\sigma(G_\xi, C_\xi) = \xi$.*

Assuming the lemma, let $(\Sigma \subset S^{n+2}, \theta)$ and $(\Sigma_1 \subset S^{n+2}, \theta_1)$ represent elements of $C_n(F_m)$, $n \geqq 3$, with the same image under ρ. Let (G, C) and (G_1, C_1) be respective complementary normal cobordisms. By hypothesis,

$$\sigma(G, C) - \sigma(G_1, C_1) = \xi \in L_{n+3}(i_*).$$

Let (G_ξ, C_ξ) be as in the lemma. Then the normal cobordism

$$-(G_\xi, C_\xi) \cup_{X_0} (G, C)$$

(see footnote 15 for this notation) is also a complementary normal cobordism for $(\Sigma \subset S^{n+2}, \theta)$. Note that, as in the proof of 3.2, it may be necessary, in order to take this union, to alter C_ξ by composition with a bundle map over the identity of X_*; this doesn't change $\sigma(G_\xi, C_\xi)$. By additivity,

$$\sigma(-(G_\xi, C_\xi) \cup_{X_0} (G, C)) = -\xi + \sigma(G, C).$$

Thus we may assume without loss of generality that $\sigma(G, C) = \sigma(G_1, C_1)$.

By additivity

$$\sigma(-(G, C) \cup_{X_0} (G_1, C_1)) = 0;$$

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16 In fact, $C_2(F_m) = 0$.

hence this normal map is normally cobordant to a normal map (H, B), *relative* $X \cup X_1$ (X_1 = complement of tubular neighborhood T_1 of $\Sigma_1 \subset S^{n+2}$),

$$H:(W; X, X_1, \partial_0 W) \to (X_* \times [-1, 1], X_* \times -1, X_* \times 1, Y_* \times [-1, 1]),$$

say, with H inducing a homotopy equivalence of $\partial_0 W$ with $Y_* \times [-1, 1]$ and an isomorphism of all integral homology groups. Further, by the addendum to 3.3 of [CS2], it may be assumed that H induces an isomorphism of $\pi_1 W$ with $\pi_1 X_* = F_m$.

By the s-cobordism theorem, there exists a diffeomorphism $\phi: \partial_0 W \to \partial T \times [-1, 1] = \Sigma \times S^1 \times [-1, 1]$, with $\phi(x) = (x, -1)$ for $x \in \partial_0 T \subset \partial_0 W$. (for $n = 3$ one would use, for example, [S3].) Let $U = W \bigcup_\phi T \times [-1, 1]$; clearly we have

$$\Sigma \times [-1, 1] = \Sigma \times 0 \times [-1, 1] \subset \Sigma \times D^2[-1, 1] = T \times [-1, 1] \subset U.$$

Further, one component of ∂U, $\partial_- Y$ say, is just $S^{n+2} = T \bigcup_{\partial T} X$ and meets $\Sigma \times [-1, 1]$ in $\Sigma \subset S^{n+2}$. Further, by an agrument involving Van-Kampen's Theorem, U is easily seen to be simply connected, because H induces an isomorphism of fundamental groups. By Meyer-Vietoris, it then follows that U is actually a simply connected h-cobordism. Hence there is a diffeomorphism

$$\psi:(U, \partial_- U, \partial_+ U) \to (S^{n+2} \times [0, 1], S^{n+2} \times 0, S^{n+2} \times 1),$$

with $\psi(x) = (x, 0)$ for $x \in \partial_- U = S^{n+2}$ and with $\partial_+ U = \partial U - \partial_- U$. Therefore $\psi(\Sigma \times [-1, 1])$ is a cobordism of $\Sigma \subset S^{n+2}$ to the link that is the image of Σ under the composite

$$\Sigma \subset \Sigma \times D^2 \bigcup_{\phi_1} X_1 = \partial_+ U \xrightarrow{\psi \mid \partial_+ U} S^{n+2}, \phi_1 = \phi \mid \partial T \times 1.$$

From the fact that ϕ_1 is the restriction of ϕ, it follows easily (e.g., because $M \times I$ retracts to $M \times 0 \cup \partial M \times I$ for any M) that ϕ_1 extends to an orientation preserving map from $\Sigma_1 \times D^2$ to $\Sigma \times D^2$ (recall $\partial X_1 = \Sigma_1 \times S^1 = \partial T_1$). Therefore ([B1] [LS] see also [Ka]), ϕ_1^{-1} extends to a homomorphism $\lambda: \Sigma \times D^2 \to \Sigma_1 \times D^2$ with $\lambda(\Sigma_1 \times 0) = \Sigma \times 0$, that is actually smooth on the complement of a point, say near the boundary.[17] Thus we have the diagram

$$\begin{tikzcd}
\Sigma \subset \Sigma \times D^2 \bigcup_{\phi_1} X_1 \arrow[r, "\psi \mid \partial_+ U"] \arrow[d, "\lambda/\Sigma \quad\quad \lambda \cup \mathrm{id}"] & S^{n+2} \arrow[dl, "\omega"] \\
\Sigma_1 \subset \Sigma_1 \times D^2 \cup X_1 = S^{n+2} &
\end{tikzcd}$$

[17] Actually [B1] [LS] only consider the case of $S^n \times S^1$, but the same arguments apply. Alternatively, by [B1] [LS] [Ka], ϕ_1 and $h \times \mathrm{id}_{S^1}$ are piecewise linearly pseudo-isotopic, h: $\Sigma_1 \to \Sigma$ a P.L. homeomorphism. The assertion then follows from a standard smoothing theory argument (see [LR] [HM] for basic smoothing theory.)

where ω is the indicated composition. In other words, the link $\Sigma_1 \subset S^{n+2}$ is obtained from $(\psi \mid \partial_+ U)(\Sigma) \subset S^{n+2}$ by composition with an orientation preserving homeomorphism that is a diffeomorphism outside a finite set of points. In particular the restriction of ω to a smooth disk containing $(\psi \mid \partial_+ U)(\Sigma)$ is isotopic to the identity, by uniqueness of disks. Therefore these two links are isotopic and so cobordant. Hence $\Sigma \subset S^{n+2}$ and $\Sigma_1 \subset S^{n+2}$ are cobordant; this proves that ρ is one-to-one, assuming Lemma 4.2.

To prove that ρ is surjective (for $n \geq 4$), the realization theorem 3.4 of [CS2] will be applied. Let $\xi \in \Gamma_{n+3}(\Phi(m))$. Let $\alpha_0 : (X_0, Y_0) \to (X_*, Y_*)$ be a complementary map for the trivial link as in §3. By the realization theorem, there exists a normal cobordism (H, B),

$$H : (W; \partial_- W, \partial_+ W, \partial_0 W) \to (X_* \times [0, 1], X_* \times 0, X_* 1, Y_* \times [0, 1]),$$

with the following properties: $(X = \partial_+ W)$

(i) $\partial_- W = X_0$ and $H \mid \partial_- W = \alpha_0$;

(ii) $H \mid X : X \to X_* \times 1$ is 2-connected and induces isomorphisms of homology groups;

(iii) $H \mid \partial(\partial_+ W) : \partial(\partial_+ W) \to Y_* \times 1$ is a homotopy equivalence; and

(iv) $\sigma(H, B) = \xi$. (This invariant is defined in view of (i)–(iii).)

Now $\partial X = M_1 \cup \cdots \cup M_m$, each M_i homotopy equivalent to $S^n \times S^1$, via the restriction of H. Therefore $H \mid M_i$ is homotopic to a P. L. homeomorphism, ϕ_i say. Consider $(S \times D^2) \bigcup_\phi X$, where S is the union of m copies of S^n and $\phi = \phi_1 \cup \cdots \cup \phi_m$. By (ii), Van-Kampen's theorem and the Meyer-Vietoris sequence $S \times D^2 \bigcup_\phi X$ has the homotopy type of S^{n+2}; hence by the generalized Poincare conjecture it is *P.L.* homeomorphic is S^{n+2}. Hence we obtain a P.L. (F_m)-link $S \subset S^{n+2}$ with complement X and characteristic map $\alpha = H \mid X$. By smoothing theory [LR, HM], this link has a smoothing $\Sigma \subset S^{n+2}$, with a smoothing of X as the complement of a tubular neighborhood. In fact, using the normal map to obtain a tangential retraction of W to X, it follows that this smoothing of X extends over W. Since X_0 has a unique smoothing, a (smooth) complementary normal map for $\Sigma \subset S^{n+2}$ is thus obtained, with obstruction ξ. This proves that ρ is surjective.

§5. The unlinking theorem and Lemma 4.2

THEOREM 5.1. *Let $\Sigma^n \subset S^{n+2}$, $n \geq 3$ be an m-link and suppose that $\pi_1(S^{n+2} - \Sigma) = F_m$ generated by meridians and $\pi_i(S^{n+2} - \Sigma) = 0$ for $1 < i \leq [n/2] + 1$. Then $\Sigma \subset S^{n+2}$ is trivial (i.e., the components bound disjoint disks.)*

For $m=1$, this is the unknotting result of [L2] [S3]. The general statement appeared in [G1]. We briefly outline a proof of 5.1. Clearly $\Sigma \subset S^{n+2}$ is a boundary link. Let $\alpha:(X, \partial X) \to (X_*, Y_*)$ be a complementary map, inducing an isomorphism of the fundamental groups, X the complement of a tubular neighborhood of T. One deduces from the hypotheses and Poincare duality that α is a (simple) homotopy equivalence. (X_*, Y_*) has the homotopy type of a connected sum, in the interior, of copies of $S^1 \times D^{n+1}$. By the splitting theorem of [C], it follows that X is a connected sum of homotopy $S^1 \times D^{n+1}$. To these one can apply the fibering theorem [BL] to deduce the result, provided $n \geq 4$. For $n=3$, one argues essentially as in [S3], using [C] as well as [S1].

Proof of 4.2. First assume $n \geq 4$. Let $\alpha_0:(X_0, \partial X_0) \to (X_*, Y_*)$ be a complementary map for the trivial link. Let $\xi \in L_{n+3}(i_*)$. By the realization theorem [W, §9], there is a normal cobordism of α_0 to $\alpha:(X, \partial X) \to (X_*, Y_*)$, with surgery obstruction ξ, where α is a homotopy equivalence. By the same argument as in the last part of the proof of 4.1 (that ρ is surjective), it follows that there is an m-link $\Sigma \subset S^{n+2}$ with closed complement X and with a complementary normal map having (the image of) ξ as its obstruction. But by 5.1, such a link is also trivial.

For the case $n=3$, the next result shows that there is nothing to prove.

PROPOSITION 5.2. *$L_j(i_*)=0$ for j even.*

Proof. For all j there is the exact sequence [W]

$$L_j(Z \cup \cdots \cup Z) \xrightarrow{L_j(i_*)} L_j(F_m) \to L_j(i_*) \xrightarrow{\partial} L_{j-1}(Z \cup \cdots \cup Z).$$

By definition [W], $L_j(Z \cup \cdots \cup Z) = L_j(Z) = L_j(Z) \oplus \cdots \oplus L_j(Z)$. By [C], $\tilde{L}_j(F_m) \cong \tilde{L}_j(Z) \oplus \cdots \oplus \tilde{L}_j(Z)$, where $\tilde{L}_j(\pi)$ denotes the cokernel of the natural map from $\tilde{L}_j(e)$, e the trivial group.

For j even, $\tilde{L}_j(Z)=0$ by [S1] (see also [W]), and also from [S1] $\tilde{L}_j(Z)=L_j(Z)$ for j odd, as $L_{odd}(e)=0$. Composed with the isomorphism $L_j(F_m) \cong \tilde{L}_j(Z) \oplus \cdots \oplus \tilde{L}_j(Z)$, $L_j(i_*)$ becomes the obvious map, and thus is an isomorphism for j odd. For j even, $L_j(i_*)$ can be identified with the natural map $L_j(Z \cup \cdots \cup Z) \to L_j(e)$. This proves 5.2.

§6. Discussion of and proof of Theorem 2

Let $\mathscr{F}_m : Z[F_m] \to Z$ denote the augmentation for the integral group ring.

Then (see [CS2]) there is a (natural) ladder of exact sequences:

$$\begin{array}{ccccccc}
L_{n+3}(Z\cup\cdots\cup Z) & \to & L_{n+3}(F_m) & \to & L_{n+3}(i_*(m)) & \to & L_{n+2}(Z\cup\cdots\cup Z)\\
\| & & \downarrow & & \downarrow & & \|\\
L_{n+3}(Z\cup\cdots\cup Z) & \to & \Gamma_{n+3}(\mathcal{F}_m) & \xrightarrow{\gamma} & \Gamma_{n+3}(\Phi(m)) & \xrightarrow{\Delta} & L_{n+2}(Z\cup\cdots\cup Z)
\end{array}$$

For n even, $\Gamma_{n+3}(\mathcal{F}_m)=0$, by [CS2]. Further, for n even $L_{n+2}(F_m)\to\Gamma_{n+2}(\mathcal{F}_m)$ is injective; for the natural map $L_{n+2}(F_m)\to L_{n+2}(e)$ is actually an isomorphism [C] and factors through the map to $\Gamma_{n+2}(\mathcal{F}_m)$. Thus, by a part of the 5-lemma, the natural map $L_{n+3}(i_*(m))\to\Gamma_{n+3}(\Phi(m))$ is surjective. (It is actually injective as well.) Thus we obtain

THEOREM 6.1. *For n even, $C_n(F_m)=0$. Hence (by Prop. 1.3), $B(n,m)$ is also trivial for n even.*

This proves Theorem 2 for the case n even, of course. For n odd, by 4.1 and 5.2, for $n\neq 3$ at least,

$$\rho: C_n(F_m)\to\Gamma_{n+3}(\Phi(m))$$

is an isomorphism. Let $\tilde\Gamma_{n+3}(\mathcal{F}_m)$ denote the cokernel of the natural map from $L_{n+3}(F_m)$, as in [CS2]. This equals the cokernel of the map from $L_{n+3}(Z\cup\cdots\cup Z)$, as this latter map factors surjectively through $L_{n+3}(F_m)=L_{n+3}(e)$. By [S1], $L_{n+2}(Z\cup\cdots\cup Z)$ is just mP_{n+1}, and, by [CS2], $\Gamma_{n+2}(\mathcal{F}_m)=0$. Thus one obtains

THEOREM 6.2. *For $n\neq 3$ the sequence*

$$0\to\tilde\Gamma_{n+3}(\mathcal{F}_m)\xrightarrow{\gamma} C_n(F_m)\xrightarrow{\Delta} mP_{n+1}\to 0$$

is (split) exact.

The sequence is one of abelian groups, given the group structure[18] on $C_n(F_m)$ provided by the bijection ρ. From [CS2] and definitions (compare [CS2, §13]) Δ is easily seen to be given by the Arf invariants ($n=4k-3$) or indices ($n=4k+1$) of the components, viewed as knots. In particular, the sequence of 6.2 splits; this is automatic for the index, and for the Arf invariant, see [CS3, 4.8]. Further, the

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[18] See note 5. One can use 4.1 or 6.2 and the connectivity in 3.4 of [CS2] to obtain connectivity, up to cobordism. See also the proof of 7.1 below.

sequence respects the natural action of $\mathcal{A}_m$ on $\tilde{\Gamma}_{n+3}(\mathcal{F}_m)$ and the trivial one on mP_{n+1}. Thus one has Theorem 2, except for $n=3$. In this case, the image of Δ is the subgroup of index 2, and so the result is still true for $1/2\Delta$.

In the case $n=3$, one obtains precisely the result of 6.2 by passing topological locally flat link cobordism, exactly as for knots.

Finally, let us view the m-fold product $C(n,1)\times\cdots\times C(n,1)=C(n,1)^m$ as contained in $C(n,m)$ and $B(n,m)$ by placement of m knots in disjoint disks. Then $C(n,1)=C_n(F_1)$ and the above results are natural with respect to the obvious map $\Gamma_{n+3}(\mathcal{F}_1)\times\cdots\times\Gamma_{n+3}(\mathcal{F}_1)\to\Gamma_{n+3}(\mathcal{F})$ that is induced on the i^{th} component by mapping Z into the subgroup generated by x_i. So one deduces

THEOREM 6.3. *The isomorphism of* 4.2 (*or the split exact sequence* 6.2) *induces an isomorphism*

$$C_n(F_m)/C(n,1)^m\cong\Gamma_{n+3}(\mathcal{F}_m)/(\Gamma_{n+3}(\mathcal{F}_1))^m.$$

We leave the reader to work out the details. An algebraic calculation similar to one that will be given below can be used to show that the right side is not finitely generated. Non-splitting follows for boundary cobordism of boundary links, but it will be proved for arbitrary cobordism in the remainder of the paper.

§7. Detecting elements of $C(n, m)$

Let $\mathcal{F}_{m,ab}:Z[F_{m,ab}]\to Z$ be the augmentation, where $F_{m,ab}$ is the abelianization of F_m. Let

$$\delta:C_n(F_m)\to C(n,m)$$

denote the forgetful map, and let $j_*:\Gamma_{n+3}(\mathcal{F}_m)\to\Gamma_{n+3}(\mathcal{F}_{m,ab})$ be the natural map induced by the quotient projection. Of course, j_* induces $\tilde{j}_*:\tilde{\Gamma}_{n+3}(\mathcal{F}_m)\to\tilde{\Gamma}_{n+3}(\mathcal{F}_{m,ab})$; the latter is the quotient of $\Gamma_{n+3}(\mathcal{F}_{m,ab})$ by the image of $L_{n+3}(F_{m,ab})$ under the natural map. Let $\gamma:\tilde{\Gamma}_{n+3}(\mathcal{F})\to C_n(F_m)$ be as in Theorem 6.2.

THEOREM 7.1 *Suppose $n>1$ is odd and $m\le(n-1)/2$. Let $\xi\in\tilde{\Gamma}_{n+3}(\mathcal{F}_m)$, and suppose that*[19] $\delta\gamma(\xi)=0$. *Then*[20] $\tilde{j}_*\xi=0$.

---

[19] $0\in C(n,m)$ is the class containing the trivial link.

[20] Observe also that $\mathcal{A}_m$ trivially on $\Gamma_{n+3}(\mathcal{F}_{m,ab})$.

Let $\Delta_1 : C(n, m) \to mP_{n+1}$ be the map that takes the Arf invariant $(n = 4k+1)$ or index $(n = 4k+3)$ of each component. Thus $\Delta = \Delta_1 \delta$, Δ as in §6. Then 7.1 and 6.2 imply most of the following:

COROLLARY 7.2. *If n is odd and $m \leq (n-1)/2$, then the composite $\tilde{j}_* \gamma^{-1}$ (defined on the kernel of Δ) induces an epimorphism*

$$\epsilon : \text{kernel } \Delta \to \tilde{\Gamma}_{n+3}(\mathcal{F}_{m,ab}).$$

Of course, that ϵ is actually onto follows from the fact that j_* is surjective [CS2].

These results can be summed up in a diagram

$$\begin{tikzcd}
C_n(F_m) \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\delta"] & \tilde{\Gamma}_{n+3}(\mathcal{F}_m) \oplus m\, P_{n+1} \arrow[d, "\tilde{j}_* \oplus \mathrm{id}"] \\
\operatorname{Im} \delta \arrow[r, "(\epsilon, \Delta_1)"] \arrow[d, phantom, "\cap|"] & \tilde{\Gamma}_{n+3}(\mathcal{F}_{m,ab}) \oplus m\, P_{n+1} \to 0 \arrow[d] \\
C(n, m) & 0
\end{tikzcd}$$

Note that elements of $C(n, m)$ can be detected by 7.1 or 7.2, even when $m > (n-1)/2$, by first mapping to $C(n, m')$, $m' \leq (n - 1/2) < m$ by forgetting about some components. Such a map is obviously surjective, so one obtains at least a surjection to $\tilde{\Gamma}_{n+3}(\mathcal{F}_{m',ab})$.

Proof of 7.1. Let $n = 2k-1$. Let $(\Sigma^n \subset S^{n+2}, \theta)$ be a link representing an element in $C_n(F_m)$ in the kernel of Δ; i.e., in the image of γ, that also maps to the trivial element in $C(n, m)$. Let X be the closed complement of a tubular neighborhood of Σ. Then, without loss of generality it may be assumed that $\theta : \pi_1 X \to \pi_1 X_* = F_m$ is an isomorphism and that $\pi_i X = 0$ for $1 < i \leq k-1$. In fact, in view of Theorem 4.1, and the definition of ρ, it follows directly from 3.4 of [CS2] (with $j = k$) that every link has such a highly connected representative. We leave the details to the reader (note that n in 3.4 of [CS2] is $(n+2)$ here.)

Let (H, B),

$$H : (W; \partial_- W, \partial_+ W, \partial_0 W) \to (X_* \times [0, 1], X_* \times 0 X_* \times 1, Y_* \times [0, 1])$$

be a complementary normal cobordism for $(\Sigma^n \subset S^{n+2}, \theta)$, so that $\partial_- W = X_0$, $\partial_+ W = X$, etc. Let $x \in C_n(F_m)$ be the element represented by $(\Sigma^n \subset S^{n+2}, \theta)$. Then the surgery obstruction,

$$\sigma((H, B) \mid \partial_0 W) = \Delta(x) = 0,$$

vanishes, where we identify $L_{n+2}(Z\cup\cdots\cup Z)=mP_{n+1}$. Therefore $(H,B)\,|\,\partial_0 W$ is normally cobordant relative the boundary to a homotopy equivalence. This normal cobordism can be attached to W along $\partial_0 W$; in other words, one may as well suppose at the beginning that $H\,|\,\partial_0 W:\partial_0 W\to Y_*\times[0,1]$ is a homotopy equivalence. In particular, the obstruction $\eta\in\Gamma_{n+3}(\mathcal{F}_m)$ for (H,B) to be normally cobordant to a homology equivalence, relative the boundary, is defined [CS2]. If ξ is the image of η in $\tilde{\Gamma}_{n+3}(\mathcal{F}_m)$, then it is clear from the definitions that $\gamma(\xi)=x$, and ξ is the unique such element, by 6.2. Thus, we must prove that $\tilde{j}_*\xi=0$.

Recall the definition of η. By surgery below the middle dimension, it may be assumed that H is $((k+1)$-connected. Then intersection numbers define a form

$$\phi:H_{k+1}(W;ZF_m)\times H_{k+1}(W;ZF_m)\to ZF_m.$$

Note that $H_{k+1}(X_*;ZF_m)=0$, so that $H_{k+1}(W;ZF_m)$ is the kernel of H_*. Further, let

$$\mu:H_{k+1}(W;ZF_m)\to ZF_m/\{y+(-1)^k\bar{y}\}$$

be the self-intersection form defined by (H,B); see [CS2]. Then η is represented by the form

$$\alpha_W=(H_{k+1}(W;ZF_m),\phi,\mu).$$

Now let $V\subset S^{n+2}\times[0,1]$ be a cobordism of $\Sigma^n\subset S^{n+2}$ to the trivial link. Let U be the complement of the interior of a tubular neighborhood of V that meets the boundary in a tubular neighborhood of ∂V; thus $\partial U=X\cup V\times S^1\cup X_0$. By Alexander duality, the inclusions $X\subset U$, $X_0\subset U$ induce isomorphisms on integral homology. Thus (see also 2.2), $(U,\partial U)$ has the same homology as $\#_m(S^1\times D^{n+2}, S^1\times S^{n+1})$.

Let $T^m=S^1\times\cdots\times S^1$, and let $g:U\to T^m$ induce the composite of θ with the quotient mapping $F_m\to F_{m,ab}=\pi_1 T^m$. Choose *any* bundle map c of stable normal bundles covering g; since both normal bundles are trivial, this exists. By surgery, (g,c) is normally cobordant to (g_1,c_1), relative the boundary, with g $(k+1)$-connected. Let γ_1, μ_1 be the intersection and self intersection, defined on $H_{k+1}(U_1;Z[F_{m,ab}])$. Again, μ_1 is in general defined on the kernel of $(g_1)_*$, but $H_i(T^m;Z[\pi_1 T^m])=0$ of course, for all i.

Now we assert that the form

$$\alpha_{U_1}=(H_{k+1}(U_1;Z[F_{m,ab}]),\gamma_1,\mu_1)$$

actually represents[21] an element of $\Gamma_{n+3}(\mathscr{F}_{m,ab})$. Thus, $Q1$–$Q6$ on page 286 of [CS2] must be checked (with $\eta=(-1)^{n+1}$, $\Lambda=Z$). But $Q1$–$Q5$ hold in general for highly connected normal maps of manifolds, by arguments similar to those of [W], for example. To see $Q6$, first note that, with $R=Z[F_{m,ab}]$, $H_{k+1}(U_1;R)\cong H_{k+2}(g_1;R)$ via the connecting homomorphism, by the usual long exact sequence. Also, $H_{k+2}(g_1;Z)\cong H_{k+1}(U_1;Z)$, because $H_{k+1}(T^m)=H_{k+2}(T^m)=0$ since $m\leq (n-1)/2=k-1$. Therefore by Lemma 1.4 of [CS2] we may write

$$H_{k+1}(U_1;Z)=H_{k+1}(U_1;R)\otimes_R Z,$$

where Z has the R-module structure given by the augmentation. Of course, this identification carries $(\gamma_1)_Z=\phi_1\otimes Z$ and $(\mu_1)_Z$ to the usual integral forms on $H_{k+1}(U_1;Z)$.

Thus, it is desired to show that the usual intersection form (e.g., defined by general position on chains) is unimodular on $H_{k+1}(U_1,Z)$. This form is well-known to be given by

$$(x,y)\to\langle D\ell_*(x),y\rangle,$$

$\ell: H_{k+1}(U_1)\to H_{k+1}(U_1,\partial U_1)$ the natural map, D Poincare duality, and $\langle\ ,\ \rangle$ the Kronecker product (evaluation) of cohomology on homology. Since g_1 is $(k+1)$-connected and $H_{k+1}(T^m)=H_k(T^m)=0$, $H^{k+1}(U_1)$ is free and $\langle\ ,\ \rangle$ is a duality pairing (i.e., $H^{k+1}(U_i)\cong \mathrm{Hom}_Z(H_k(U_1);Z)$), by universal coefficients. Since $\partial U_1=\partial U$ and $H\,|\,X: X\to X_*$ induces isomorphism of homology (see 2.2, 3.1), an easy argument with the Meyer-Vietoris sequence shows that $H_{k+1}(\partial U_1)=H_k(\partial U_1)=0$. Thus ℓ_* is also an isomorphism. Therefore $H_{k+1}(U_1)$ is also free, and $(\gamma_1)_Z$ is unimodular; i.e., $(Q6)$ is satisfied.

Next, it is asserted that α_{U_1} actually represents the trivial element in $\Gamma_{n+3}(\mathscr{F}_{m,ab})$. To see this, let (G,C), $G: P^{n+4}\to T^m$, be a normal cobordism of (g,c) to (g_1,c_1), relative the boundary. Thus one has $\partial P=U\cup_{\partial U}U_1$, $G\,|\,V=g$, $G\,|\,V_1=g_1$. By surgery, we may suppose that G is $(k+1)$-connected. By handle subtractions, it may also be assumed that $H_{k+1}(P,U_1;R)=0$; i.e., (P,U_1) is $(k+1)$-connected. (See [W, §1] for example.) The effect of these handle subtractions on α_{U_1} is to just add some standard kernel over R, so that the element in $\Gamma_{n+3}(\mathscr{F}_{m,ab})$ that it represents is unchanged.

Now let K be the image of the connecting homorphism

$$H_{k+2}(P,U_1;R)\xrightarrow{\partial_R}H_{k+1}(U_1;R).$$

---

[21] Observe that this is not implied by any of the general results on homology equivalence of [CS2].

Then as in [W, 5.7] for example, γ_1 and μ_1 vanish on K, i.e., K satisfies (PS1) of [CS2, p. 286]. To check (PS2), first note that, by 1.4 of [CS2], $H_{k+2}(P, U_1; Z) = H_{k+2}(P, U_1; R) \otimes_R Z$. Therefore, it suffices to check that the image of

$$H_{k+2}(P, U_1; Z) \xrightarrow{\partial_Z} H_{k+1}(U_1; Z)$$

is a summand of half the rank. To see this, consider the sequence (with Z coefficients)

$$0 \to H_{k+2}(P, U_1) \xrightarrow{\partial} H_{k+1}(U_1) \to H_{k+1}(P) \to K_{k+1}(P, U_1) = 0$$

On the left, this sequence is exact because $H_{k+2}(P) \cong H^{k+1}(P, \partial P)$, which is in turn isomorphic to $H^{k+1}(P, U_1) = 0$. This latter isomorphism follows from the exact sequence of the triple $U_1 \subset \partial P \subset P$, excision, and the fact that $(U, \partial U)$ has the same homology as $\#_m (S^1 \times D^{n+2}, S^1 \times S^{n+1})$. Similarly, $H_{k+1}(P) \cong H^{k+2}(P, \partial P) \cong H^{k+2}(P, U_1)$. Thus $H_{k+1}(P) = \mathrm{Hom}_Z\,(P, U_1); Z)$ is free and so the sequence splits; it also follows that rank $H_{k+1}(U_1) = 2$ rank $H_{k+2}(P, U_1)$. Thus α_{U_1} represents zero in $\Gamma_{n+3}(\mathcal{F}_{m,ab})$.

Next let $Q = W \cup_X U_1$. This ∂Q is diffeomorphic to the double of X_0. Let $h: X_* \times I \to T^m$ induce the abelianization map on fundamental groups. Because $H \mid X: X \to X_*$ is a homology equivalence and ν_X is trivial ($X \subset S^{n+2}$), the target bundle of B is also trivial. (Homology equivalences induce isomorphisms of stable linear bundle theory, as BSO is a simple space, or, alternatively, by the Atiyah-Hirzebruch spectral sequence.) Therefore let $\hat{h}$ be a (stable) bundle map from the target of B, covering h. Since $X \subset U$ induces isomorphisms of homology groups (and so also of bundle theories again), the bundle map c may be chosen with $c \mid X = \hat{h} \circ B$. Then one also has $c_1 \mid X = \hat{h} \circ B$.

Thus Q admits the normal map $(h \circ H, \hat{h} \circ B) \cup (g_1, c_1)$ to T^m. Let γ_Q be the intersection form on $H_{k+1}(Q; R)$, and let μ_Q be the self-intersection form defined by this normal map; it is also defined on $H_{k+1}(Q; R)$ because $H_i(T^m; R) = 0$, all i. We assert that

$$\alpha_Q = (H_{k+1}(Q, R), \gamma_Q, \mu_Q)$$

represents an element of $L_{n+3}(\pi_1 T^m) = L_{n+3}(F_{m,ab})$.

To see this claim, let $f = h \circ H \cup g_1$. Then $H_{k+1}(Q, R) = H_{k+2}(f; R)$, as $H_i(T^m; R) = 0$. Since H is $(k+1)$-connected, $H_i(h \circ H, R) \cong H_i(h; R)$ for $i \leq$

$k+1$, e.g., by 1.4 of [CS2] and a suitable long exact sequence. Similarly, as $\pi_i X = 0$ for $1 < i \leq k-1$ and θ is an isomorphism (so that $H \mid X$ is k-connected, $H_i(f \mid X; R) \cong H_i(h; R)$ for $i \leq k$. Since g_1 is also $(k+1)$-connected, it now follows from the Meyer-Vietoris sequence that $H_i(f; R) = 0$ for $i \leq k+1$. Furthermore, for any R-module M, $H^{k+3}(h; M) = 0$. For $H^{k+3}(h; M) = H^{k+2}(Q, M) \cong H_k(Q, \partial Q; M)$; the first isomorphism follows from the exact sequence and the vanishing of $H_i(T^m; M)$ for $i \geq k > m$, the 2nd by Poincare duality. But $H_k(Q; M) \cong H_{k+1}(f; M) = 0$; the first isomorphism holds because $m < k$ again, the 2nd from what has already just been noted (and 1.4 of [CS2]). Further, ∂Q is the double of X_0 (i.e., $\partial(X_* \times [0, 1])$ up to homotopy), and it is easy to check in several ways (even using a cell decomposition, for example), that $H_{k-1}(\partial Q; M) = 0$. Thus $H^{k+3}(h; M) \cong H_k(Q, \partial Q; M) = 0$. Therefore by [W, 2.3], $H_{k+1}(Q, R)$ is projective. From [Ba], it is therefore stably free. Therefore, after adding copies of $S^{k+1} \times S^{k+1}$ to U_1 by connected sum, if necessary (i.e., perform trivial surgeries), it may be supposed that $H_{k+1}(Q, R)$ is free.

To see that γ_Q is unimodular, recall that its adjoint is given by the composite

$$H_{k+1}(Q, R) \xrightarrow{\ell_*} H_{k+1}(Q, \partial Q; R) \xrightarrow{D} H^{k+1}(Q, R) \xrightarrow{e} \mathrm{Hom}_R(H_{k+1})Q, R), R),$$

$e(x)(y) = \langle x, y \rangle$. Again, as ∂Q is the double of X_0, one easily sees that ℓ_* is an isomorphism, and D is by Poincare duality. Since $H_{k+1}(Q, R) = H_{k+2}(f, R)$ and $H_i(f; R) = 0$ for $i \leq k+1$, the analogue 1.4 of [CS2] implies that e is also an isomorphism. Thus γ_Q is unimodular, and α_Q represents an element of $L_{n+3}(F_{m,ab})$.

Consider now the map

$$H_{k+1}(U_1, R) \oplus H_{k+1}(W, R) \to H_{k+1}(Q, R)$$

induced by inclusions of the two subspaces of Q. Note that

$$H_{k+1}(W; R) \cong H_{k+2}(H; R) \cong H_{k+2}(H; ZF_m) \otimes_{ZF_m} R \cong H_{k+1}(W; ZF_m) \otimes_{ZF_m} R.$$

The middle isomorphism follows from connectivity of H and 1.4 of [CS2], the others from an exact sequence and the fact that X_* has no homology (or even cells) in the relevant dimensions. The above map preserves intersections and self intersections, and therefore induces a map

$$\lambda : \alpha_{U_1} \perp (\alpha_W \oplus_{ZF_m} R) \to \alpha_Q$$

of forms, where $\perp$ denotes orthogonal direct sum. Again, the assumptions on X imply that $H|X$ is (k)-connected, so that by arguments as above, $H_k(X; R)\otimes_R Z \cong H_k(X; Z)$, which vanishes by Alexander duality. Therefore, by the Mayer-Vietoris sequence, $\lambda \otimes_R Z$ is surjective.

Both domain and range of λ become unimodular when tensored over R with Z; this was shown for α_{U_1} and α_W, and α_Q is already unimodular over R. A map of *unimodular* forms is always injective. Hence $\lambda \oplus_R Z$ is an isomorphism. From this one easily sees that the elements of the type $x \oplus y \oplus \lambda(x \oplus y)$ will constitute a *presubkernel* [CS2, p. 287] for $(\alpha_{U_1} \perp \alpha_W \perp_{ZF_m} R) \perp (-\alpha_Q)$; i.e., this sum represents the trivial element in $\Gamma_{n+3}(\mathcal{F}_{m,ab})$. But so does α_{U_1}, and α_Q represents an element from $L_{n+3}(F_{m,ab})$. Thus $\tilde{j}_*(\xi)$, represented by $\alpha_W \oplus_{ZF_m} R$, is trivial, as was to be shown.

§8. Proof of Theorem 1

First note that one has the commutative diagram

$$\begin{array}{ccc}
\tilde{\Gamma}_{n+3}(\mathcal{F}_1)^m & \xrightarrow{\gamma\times\cdots\times\gamma} & C(n,1)^m = C_n \times \cdots \times C_n \\
\downarrow & & \downarrow{\scriptstyle\phi} \\
\tilde{\Gamma}_{n+3}(\mathcal{F}_m) & \xrightarrow{\gamma} & C(n,m)
\end{array}$$

Here ϕ was defined in the introduction, and the unlabeled map is given in §6 (it is induced on the i^{th} component by mapping Z to the subgroup of F_m generated by x_i). Therefore, from 7.2 (see also 7.3) one obtains the diagram $(\mathcal{F}_1 = \mathcal{F}_{1,ab})$

$$\begin{array}{ccc}
C(n)^m & \xrightarrow{(\varepsilon,\Delta_1)^m} & \tilde{\Gamma}_{n+3}(\mathcal{F}_1)^m \oplus mP_{n+1} \\
\downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle k_*\oplus id} \\
\operatorname{Im}\delta & \xrightarrow{(\varepsilon,\Delta_1)} & \tilde{\Gamma}_{n+3}(\mathcal{F}_{m,ab}) \oplus mP_{n+1}
\end{array}$$

where k_* is induced on the i^{th} summand by mapping Z to be subgroup of $F_{m,ab}$. generated by the image of x_i. It is easily seen that k_* is a monomorphism, by considering the various projections $\mathcal{F}_{m,ab} \to \mathcal{F}_1$ (i.e. $F_{m,ab} \to Z$). (In [CS, §13], it is shown that $(\varepsilon, \Delta,)^m$ is an isomorphism.)

PROPOSITION 8.1. *The map* (ε, Δ_1) *induces a surjective map*

$$\varepsilon_*: \operatorname{Im}\delta / C(n)_m \to \tilde{\Gamma}_{n+3}(\mathcal{F}_{m,ab}) / _{\tilde{\Gamma}_{n+3}(\mathcal{F}_1)^m}$$

Theorem 1 will now be proven by showing that the target of ε_* is *not* finitely generated.

First of all, since $L_{n+3}(Z^m)$ and $L_{n+3}(Z)$ are finitely generated (e.g. by [S1]), it sufficies to prove that the quotient $\Gamma_{n+3}(\mathcal{F}_{m,ab}/\Gamma_{n+3}(\mathcal{F}_1)^m$ is not finitely generated. Let $\psi:\Gamma_{n+3}(\mathcal{F}_{m,ab})\to\Gamma_{n+3}(\mathcal{F}_1)^m$ be induced by the projections of $F_{m,ab}=Z\oplus\cdots\oplus Z$ to the various summands. Then it suffices to show that the kernel of ψ is not finitely generated, for $m\geq 2$.

Case 1. $n+3\equiv 0(\mathrm{mod}\,4)$

Let $s,t\in F_{m,ab}$ be two independent generators. The map to the complex numbers that sends s and t to $e^{2\pi i/p}$ and the other generators (if any) to 1 transforms a form α over $Z[F_{m,ab}]=R$, representing an element of $\Gamma_{n+3}(\mathcal{F}_{m,ab})$, into a Hermitian form α_p over the complex numbers. As in [CS5] the assignment

$$\alpha\to\text{signature of }\alpha_p$$

induces a homomorphism

$$c_p:\Gamma_{n+3}(\mathcal{F}_{m,ab})\to Z$$

Now consider the forms over R (on a rank 2 free module) given by the matrices

$$\alpha(N)=\begin{pmatrix} s+s^{-1}-2 & 1\\ 1 & N(t+t^{-1}-2)\end{pmatrix},$$

$N\geq 1$ an integer. Then $\alpha(N)$ (with the μ-form given by $s-1$ and $N(t-1)$ on generators) represents $\xi_N\in\Gamma_{n+3}(\mathcal{F}_{m,ab})$. The components of $\psi(\xi_N)$ are easily seen to be represented by forms of the type

$$\begin{pmatrix}0 & 1\\ 1 & *\end{pmatrix}\quad\text{or}\quad\begin{pmatrix}* & 1\\ 1 & 0\end{pmatrix},$$

with μ of the appropriate basis element also zero; i.e. $\psi(\xi_N)=0$. It is an exercise to check that

$$c_p(\xi_N)=\begin{cases}0 & \text{if}\quad N(e^{2\pi i/p}+e^{-2\pi i/p}-2)^2-1<0\\ -2 & \text{if}\quad N(e^{2\pi i/p}+e^{-2\pi i/p}-2)^2-1>0\end{cases}$$

Since $(e^{2\pi i/p}+e^{-2\pi i/p}-2)^2$ is a decreasing sequence tending to zero as $p\to\infty$, it follows there is a strictly increasing sequence N_p so that

$$c_p(\xi_N)=\begin{cases} 0 & \text{if} \quad N\le N_p \\ -2 & \text{if} \quad N>N_p. \end{cases}$$

This clearly implies that the elements ξ_N generate a subgroup of $\Gamma_{n+3}(\mathcal{F}_{m,ab})$ that is not finitely generated.

Case 2. $n+3\equiv 2(4)$. In this case, put

$$\alpha(N)=\begin{pmatrix} t-t^{-1} & 1 \\ -1 & N(s^{-1}-s) \end{pmatrix},$$

(with μ having values t and s^{-1} on the generators of this 2-dimensional form). This time a homomorphism to Z, for each p, is defined by sending s and t to $e^{2\pi i/p}$, and other generators to 1, multiplying by $\sqrt{-1}$ to obtain a Hermitian form, and then taking the signature. An argument similar to case I then shows again that the elements ξ_N represented by $\alpha(N)$ contain an infinite subset of linear independent elements, and that $\psi(\xi_N)=0$.

Note. One can obtain an explicit construction of non-splittable links by applying the construction of [CS2, Thm 1.8], to the forms $\alpha(N)$ and the complementary map of the trivial link ((h, c) in 1.8 of [CS2].) From the explicit form of $\alpha(N)$, it can be verified that any component of these links will have complement of the homotopy type of the trivial knot. Hence, it will be trivial, by [L] [S3]. With more work, one can arrange examples in which each proper sublink is trivial; this requires larger matrices and uses 5.1.

§9. Concluding remarks

For any m-link group π with given normal generators $x_1,\dots,x_m$ ("meridians"), one can define $C_n(\pi)$, cobordism classes of links with group π. Recall from [K] that π is an m-link group if and only if it normally generated by m elements and $H_2(\pi)=0$. An element of $C_n(\pi)$ is represented by an m-link $\Sigma^n\subset S^{n+2}$, together with a homomorphism $\pi_1(S^{n+2})-\Sigma^n\to\pi$ that sends a set of meridians to $x_1,\dots,x_m$. Similarly, the cobordism relation is defined as it was for $C_n(F_m)$.

Let $\mathcal{F}_\pi: Z\pi\to Z$ be the augmentation. Then, by analogy with 6.2, it is natural to conjecture that

$$C_n(\pi)\cong\tilde{I}_{n+3}(\mathcal{F}_\pi)\oplus m\,P_{n+1}. \tag{9.1}$$

The methods of this paper seem potentially capable (with more work) of proving this result, at least in the case that $H_i(\pi)=0$ for $i \geq 2$.

Link groups form a partially ordered set, with $\pi < \pi'$ if there is a homomorphism $\pi \to \pi'$ preserving meridians, up to congugacy. (Use the free product amalgamated along meridians to see that given π and π', there is π'' with $\pi<\pi'$, $\pi<\pi''$.) Thus, from (8.1), one would get

$$C(n, m) = \varinjlim_{\pi} (\tilde{\Gamma}_{n+3}(\mathcal{F}_\pi)/\mathcal{A}_\pi) \oplus m P_{n+1}. \tag{9.2}$$

Here $\mathcal{A}_\pi$ would denote the automorphisms of π that send meridians to conjugates of themselves. To prove (9.2), of course, it would suffice to have 9.1 for a cofinal set of π. For example, it would be especially fortunate if the set of π with $H_i(\pi)$ trivial for $i \geq 2$ were cofinal.

It would remain to study further the groups $\tilde{\Gamma}_{n+3}(\mathcal{F}_\pi)$. For example, to assert that $B(n, m) \to C(n, m)$ is surjective ("every link is cobordant to a boundary link") would amount to the assertion that the natural map $\tilde{\Gamma}_{n+3}(\mathcal{F}_m) \to \varinjlim_{\pi} (\tilde{\Gamma}_{n+3}(\mathcal{F}_\pi)/\mathcal{A}_\pi)$ is surjective.

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