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An elementary formula for the Fenchel–Nielsen twist

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A deformation of hyperbolic Riemann surfaces was investigated in the Fenchel–Nielsen manuscript. The surface is cut along a simple closed hyperbolic geodesic. The two sides are separated, rotated relative to each other and glued. The purpose of this note is to present a formula for the first variation under the twist deformation of the length of a closed geodesic. The formula is local in nature and entails the angles at the intersections of the geodesic and the cut.

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1. A variational formula

Let $f_t : R_0 \to R_t$ be a smooth deformation. Denote by g the hyperbolic line element on R_t. Choose a closed curve γ on R_0 and denote by $l(t)$ the length of the unique g geodesic freely homotopic to $f_t(\gamma)$ on R_t. Then

$$\frac{d}{dt} l(t) = \int_{\gamma_0} \frac{d}{dt} g_t \tag{1.1}$$

where $g_t = f_t^* g$ and γ_0 is the g geodesic on R_0.

Proof. Consider the lift $\tilde{f}_t$ of f_t to the universal covers, each represented as the upper half plane. Now $\tilde{f}_t$ is smooth in t and z, thus for A a deck transformation corresponding to γ_0 the Möbius transformation $A_t = \tilde{f}_t A \tilde{f}_t^{-1}$ varies smoothly in t. The endpoints on $\mathbb{R}$ of the axis of A_t (the fixed points of A_t) are algebraic in the matrix entries of A_t. Consequently the axes of the A_t vary smoothly in t. Now $\tilde{\gamma}_t = \tilde{f}_t^{-1}$ (axis of A_t) form a smooth 1-parameter family of curves. Denote by γ_t the projection to R_0 of $\tilde{\gamma}_t$. Then

$$\frac{d}{dt} l(t) = \frac{d}{dt} \int_{\gamma_t} g_t = \lim_{t \to 0} \frac{1}{t} \left(\int_{\gamma_t} g_t - \int_{\gamma_0} g_0 \right)$$

$$= \lim_{t \to 0} \frac{1}{t} \left(\int_{\gamma_t} g_t - \int_{\gamma_0} g_t + \int_{\gamma_0} g_t - \int_{\gamma_0} g_0 \right).$$

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Expanding in t, $g_t = g_0 + t\dot{g} + O(t^2)$ where $\dot{g}^2$ is a symmetric 2-tensor. We then have

$$\frac{d}{dt} l(t) = \lim_{t\to 0} \frac{1}{t}\left(\int_{\gamma_t} g_0 - \int_{\gamma_0} g_0\right) + \lim_{t\to 0}\left(\int_{\gamma_t} \dot{g} - \int_{\gamma_0} \dot{g}\right) + \int_{\gamma_0} \frac{d}{dt} g_t.$$

The curve γ_0 is a periodic geodesic and consequently the first variation of its length is zero; the second limit vanishes by continuity of the integral.

2. The Fenchel–Nielsen twist

Let $z \in H$, the upper half plane and $\theta = \arg z$. Choose $\phi(\theta)$ smooth with compact support in $(0, \pi)$, $\varphi \geq 0$ and $\int_0^\pi \varphi\, d\theta = \frac{1}{2}$. Define $\Phi(\theta) = \int_0^\theta \varphi\, d\theta$. The formula

$$w = z \exp(2t\Phi(\theta)) \tag{2.1}$$

defines a quasiconformal self map of H realizing the Fenchel–Nielsen twist deformation for the cyclic group stabilizing the imaginary axis. The boundary values of w are independent of the particular choice of φ. We recall that the boundary values determine the deformation as a point of the Teichmüller space. Now

$$w_z = \exp(2t\Phi)(1 - it\varphi), \qquad w_{\bar{z}} = \exp(2t\Phi)(it\varphi)$$

and the Beltrami differential of w is

$$\mu = w_{\bar{z}}/w_z = \frac{it\varphi}{1 - it\varphi}\frac{z}{\bar{z}}$$

The first variation of μ with respect to t is the tangent vector to the deformation.

Let Γ be a finitely generated Fuchsian group containing a hyperbolic element A stabilizing the imaginary axis. We assume A is primitive and that the imaginary axis projects to a simple closed geodesic. The infinitesimal Fenchel–Nielsen twist deformation of Γ about A is

$$\mu_\Gamma = \sum_{B \in \langle A\rangle\backslash\Gamma} B^*\mu,$$

for $\langle A\rangle$ the cyclic group generated by A, [2]. We emphasize that μ_Γ as a tangent to the deformation space is independent of the particular choice of φ.

3. The cosine formula

Let $C\in\Gamma$ correspond to a closed geodesic γ on $R=H/\Gamma$. The first variation of the length of γ under a Fenchel–Nielsen twist about the geodesic α corresponding to A is

$$\sum_{p\in\{\alpha\}\cap\{\gamma\}} \cos\beta_p.$$

The sum is over the intersection of the geodesics α and γ; β_p is the angle between the tangents at p of α and γ. The angle β_p is defined by the segment between 0 and p on the imaginary axis and that segment of the axis of C in the right half plane. The angle is well defined; note the expression $\cos\beta_p$ is anti-symmetric in the roles of α and γ. In particular for a twist about γ the contribution at p to the derivative of the length of α is $\cos(\pi-\beta_p)$.

We begin the proof by considering the solution w_t of the Beltrami equation $w_{\bar z}=\mu_\Gamma w_z$. The solution was constructed in [2]. Indeed, in a neighborhood of the imaginary axis w_t coincides with the map defined by (2.1). Now in the notation of section 1

$$g=|dw|/\operatorname{Im} w \quad \text{and}$$

$$\begin{aligned} g_t&=|w_z|\,|dz+\mu_\Gamma\, d\bar z|/\operatorname{Im} w\\ &=|1-it\varphi|\,|dz+\mu_\Gamma\, d\bar z|/\operatorname{Im} z.\end{aligned}$$

Our calculations only require an expression for g_t in a neighborhood of the imaginary axis, where $\mu_\Gamma=it\varphi e^{2i\theta}+O(t^2)$. Furthermore, φ will approximate $\frac{1}{2}$ the Dirac delta at $\pi/2$; consequently

$$\begin{aligned} g_t&=|dz-it\varphi\, d\bar z|/\operatorname{Im} z+O(t^2,\ \text{support}\ \varphi)\\ &=((dx-t\varphi\, dy)^2+(dy-t\varphi\, dx)^2)^{1/2}/\operatorname{Im} z+O(t^2,\ \text{support}\ \varphi).\end{aligned}$$

Now differentiating

$$\frac{d}{dt}((dx-t\varphi\, dy)^2+(dy-t\varphi\, dx)^2)^{1/2}/\operatorname{Im} z\big|_{t=0}=-2\varphi\, dx\, dy/(dx^2+dy^2)^{1/2}\operatorname{Im} z.$$

We are now ready to apply formula (1.1). Let s be the arc length parameter along γ. Assume increasing s corresponds to moving to the left. The tangent of γ

at p is

$$\left(\frac{dx}{ds}, \frac{dy}{ds}\right)_p = \operatorname{Im} z(-\sin \beta_p, \cos \beta_p)_p.$$

The contribution to the integral is then $\int 2\varphi \sin \beta_p \cos \beta_p \, ds$. At p we have $\operatorname{Im} z \, d\theta = -dx = \operatorname{Im} z \sin \beta_p \, ds$ and finally the integral

$$\int 2\varphi \cos \beta_p \, d\theta.$$

Now the contribution of p to the integral (1.1) tends to $\cos \beta_p$ as φ approximates $\frac{1}{2}$ the Dirac delta. A check of the calculations shows the integral is indeed unchanged if s is replaced by $-s$. Finally each intersection of α and γ contributes such a term.

4. Remarks

Our calculations suggest that the appropriate Beltrami differential representing a Fenchel–Nielsen twist is a distribution. This is consistent with the point of view of the Fenchel–Nielsen twist found in the theory of amalgamation of Fuchsian groups.

Finally, we give an alternative formula for the cosine. Let the axis of two hyperbolic transformations A and B intersect at a point p. Let δ be the angle at p formed by the segments along the axes of A and B to their respective attractive fixed points. Then

$$\cos \delta = \frac{-\operatorname{sgn}(\operatorname{tr} A \operatorname{tr} B)(\operatorname{tr} A \operatorname{tr} B - 2 \operatorname{tr} AB)}{(\operatorname{tr}^2 A - 4)^{1/2}(\operatorname{tr}^2 B - 4)^{1/2}}$$

where tr denotes the trace of a matrix and sgn denotes the sign of a real number.

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