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Many different disk knots with the same exterior

L. R. HITT⁽¹⁾ and D. W. SUMNERS

§1. Introduction

Much of codimension-two knot theory is concerned with finding and computing topological invariants of knot exteriors in order to distinguish between the knots themselves. It is well-known ([G], [L-S], [B]) that there are at most two inequivalent smooth *n*-sphere knots with the same exterior $(n \ge 2)$, and examples of two inequivalent *n*-knots with the same exterior have recently been discovered ([C-S], [Go]). We show that the corresponding theory for (n+1)-disk knots is more complicated. Let Y denote the bounded exterior of a smooth (n+1)-disk knot. The *indeterminacy index* $\zeta(Y)$ is the number of inequivalent (n+1)-disk pairs having exteriors diffeomorphic to Y. We show that there exist disk knots with large indeterminacy indices (bigger than two, in particular). We then show that $\zeta(Y) \le 2|\pi'|$, where $|\pi'|$ denotes the cardinality of π' , the commutator subgroup of $\pi = \pi_1(\partial Y)$. This yields as a corollary a new and easy proof of the well-known fact that $\zeta(X) \le 2$, where X is the exterior of an *n*-sphere knot, and $\zeta(X)$ its indeterminacy index.

§2. The indeterminacy index

For convenience, we work in the smooth category (the same results hold in the locally flat PL situation). We let S^n and D^{n+1} denote the standard *n*-sphere and (n+1)-disk, respectively. An *n*-sphere knot (or just *n*-knot) is the pair (S^{n+2}, kS^n) where $k: S^n \to S^{n+2}$ is an embedding. The exterior X of an *n*-knot is the complement in S^{n+2} of an open trivial 2-disk bundle neighborhood of the submanifold kS^n . An (n+1)-disk knot is the pair (D^{n+3}, gD^{n+1}) where $g: D^{n+1} \to D^{n+3}$ denotes a proper embedding, one in which the submanifold gD^{n+1} intersects ∂D^{n+3} transversely in $g(\partial D^{n+1})$. We let Y denote the (n+1)-disk knot exterior. Two knots are equivalent if there is a diffeomorphism of the ambient space throwing one submanifold onto the other (we disregard orienta-

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tions), and the *indeterminacy index* ζ is the number of inequivalent knots determined by a given knot exterior.

We will now produce examples to show that $\zeta(Y)$ can be large. The reason for this is that ∂Y contains the exterior X of the boundary sphere pair, and X can be very complicated. Recall the example of Kato [Ka 2, Theorem 4.9]:

Let $n \ge 3$, and M^{n+2} be a contractible manifold such that $\pi_1(\partial M)$ is the binary icosohedral group $G = \langle a, b | a^5 = b^3 = (ab)^2 \rangle$ [Ke]. Let $Y^{n+3} = S^1 \times M^{n+2}$; we will show that Y is the exterior of at least three inequivalent (n+1)-disk knots. Then by modifying the construction, we will show that the indeterminacy index of a disk knot exterior can be at least as large as six.

Let H be a group. A weight element of H is an element whose normal closure is all of H. The automorphism class of an element of H is the orbit of the element under the automorphism group of H. Two elements of H are algebraically distinct if they are in different automorphism classes.

We are interested in finding different automorphism classes of weight elements in the group $\pi_1(\partial Y) \cong Z \times G \cong \langle t, a, b | a^5 = b^3 = (ab)^2, ta = at, tb = bt \rangle$ where Z denotes the infinite cyclic group generated by t. An element of the form $t^n g$, for $g \in G$, is a weight element of $Z \times G$ if and only if t^n is a weight element of Z and g is a weight element of G, which forces $n = \pm 1$. To determine the weight elements of G, note that $\{1\} \triangleleft \{1, (ab)^2\} \triangleleft G$ is a composition series for G, since $\langle a, b | a^5 = b^3 = (ab)^2 = 1 \rangle$ is a presentation of the simple group A_5 . The center of G is $C(G) = \{1, (ab)^2\}$, the cyclic group of order 2. Any element of G which is not in C(G) is a weight element of G. The set of algebraically distinct weight elements of G is $\{a, a^2, b, b^2, ab\}$. That they are algebraically distinct follows from their different orders: 10, 5, 6, 3 and 4, respectively.

Therefore we have ta, ta^2 , tb, tb^2 , and tab as weight elements of $Z \times G$. However ta and ta^2 are in the same automorphism class in $Z \times G$, as are tb and tb^2 (e.g., the automorphism θ , induced by $\theta(t) = t(ab)^2$, $\theta(a) = a^7$, and $\theta(b) = ba^8 b$ sends ta to ta^2). So our list of possibly algebraically distinct weight elements is shortened to ta, tb, and tab. That these three elements are algebraically distinct follows from the fact that the center of $Z \times G$ is $Z \times \{1, (ab)^2\}$, so $Z \times G$ modulo its center is $A_5 \cong \langle a, b \mid a^5 = b^3 = (ab)^2 = 1 \rangle$. But the center is a characteristic subgroup, so any automorphism of $Z \times G$ induces one on A_5 . Since a, b, and ab have different orders in A_5 , their counterparts in $Z \times G$ must be algebraically distinct.

Let $\{\sigma_i \mid 1 \le i \le 3\}$ denote smooth embeddings of S^1 in ∂Y representing the homotopy class in ∂Y of each of the above weight elements of $Z \times G$. Choose a trivialization of the normal bundle of each σ_i , and attach 2-handles to form the manifolds $Y \cup_{\sigma_i} h^2$. The cocore or transverse disk of each 2-handle is an (n+1)-disk, and $(Y \cup_{\sigma_i} h^2$, cocore $(h^2) \approx (D^{n+3}, g_i d^{n+1})$, where $g_i : D^{n+1} \to D^{n+3}$ is a

proper smooth embedding. This is because $Y \cup_{\sigma_i} h^2$ is contractible, with simplyconnected boundary, and $n+3 \ge 6$. However, no two of the three disk pairs $(Y \cup_{\sigma_i} h^2, g_i D^{n+1})$ are equivalent, because any diffeomorphism of pairs between them would restrict to a diffeomorphism on Y, inducing an isomorphism on $\pi_1(\partial Y)$ taking one of the weight elements of $Z \times G$ to another, or its inverse.

In [S], it is shown that (n+1)-disk pairs $(n \ge 2)$ can be constructed with an arbitrarily prescribed Alexander polynomial in a single dimension p $(2 \le p \le n)$, and trivial Alexander polynomials elsewhere. Moreover, these disk pairs have the property that $\pi_1(Y) \cong \pi_i(\partial Y) \cong \pi_i(S^1)$ for i < p. Thus, by taking the boundary connected sum of the above examples with these disk pairs, one obtains infinitely many distinct (n+1)-disk exteriors, each with indeterminacy index $\zeta \ge 3$. This proves

THEOREM 2.1. For each $n \ge 3$, there exist infinitely many homeomorphically distinct (n+1)-disk knot exteriors Y_i , each with indeterminacy index $\zeta(Y_i) \ge 3$.

Remark. The analogue of Theorem 2.1 for n = 2 can be done in the topological category (non-PL embeddings). One takes $Y = S^1 \times (c * \Sigma^3)$, where $c * \Sigma^3$ is the cone on Σ^3 , the Poincare' 3-sphere. Then Y is a topological manifold [Ca], and arguments of Scharlemann [Sc] can be used to prove that the various handle attachments give rise to different non-PL disk pairs (D^5, gD^3) .

We can modify the above construction to increase the lower bound for the indeterminacy index. Consider the group

$$G \times G \times G = \langle a, b, c, d, e, f \mid a^5 = b^3 = (ab)^2, c^5 = d^3 = (cd)^2, e^5 = f^3 = (ef)^2,$$

 $ac = ca, ad = da, bc = cb, bd = db, ae = ea, af = fa, be = eb, bf = fb,$
 $ce = ec, cf = fc, de = ed, df = fd \rangle.$

Now $G \times G \times G$ is finitely presented, and $H_1(G \times G \times G) = H_2(G \times G \times G) = 0$, so by Kervaire [Ke], for $n \ge 4$ there exists a contractible manifold M^{n+2} with $\pi_1(\partial M) \cong G \times G \times G$. As before, $Y = S^1 \times M$, and $\pi_1(\partial Y) \cong Z \times G \times G \times G$. Since the center of $Z \times G \times G \times G$ is the product of the center of each factor, we see that $Z \times G \times G \times G$ modulo its center is $A_5 \times A_5 \times A_5$. Then, as before, *tacf*, *tacef*, *tbdef*, *tace*, *tbdf*, and *tabcdef* are all algebraically distinct since their projections modulo the center have orders 15, 10, 6, 5, 3, and 2, respectively. We have the following

COROLLARY 2.2. For each $n \ge 4$, there exist infinitely many homeomorphically distinct (n+1)-disk knot exteriors Y_i , each with indeterminacy index $\zeta(Y_i) \ge 6$.

§3. An upper bound for the indeterminacy index

Now that we have seen that in some cases the lower bound of ζ can be large, we are interested in finding upper bounds. Along these lines, we have the following

THEOREM 3.1. Let Y^{n+3} be an (n+1)-disk knot exterior $(n \ge 2)$. Then $\zeta(Y) \le 2 |\pi'|$, where $|\pi'|$ denotes the cardinality of the commutator subgroup π' of $\pi = \pi_1(\partial Y)$.

Proof. Consider the disk pair (D^{n+3}, gD^{n+1}) . Choose a trivialization $G: D^2 \times D^{n+1} \to N(gD^{n+1})$ of the tubular neighborhood of the submanifold; thus $G(\{0\} \times y) = g(y)$ for $y \in D^{n+1}$. We have that the exterior $Y = D^{n+3} - G(\mathring{D}^2 \times D^{n+1})$. Regarding $N(gD^{n+1})$ as a 2-handle attached to Y via the meridian attaching curve $G(\partial D^2 \times \{0\})$, we have $(D^{n+3}, gD^{n+1}) \approx (Y \cup_G h^2, \operatorname{cocore}(h^2))$. We now wish to study the number of different ways it is possible to attach a 2-handle to Y to produce D^{n+3} . We first count the maximum number of possible isotopy classes in ∂Y of attaching curves for a 2-handle which produce a contractible manifold after handle attachment is performed. If $\pi = \pi_1(\partial Y)$, and π' is the commutator subgroup of π , we have the short exact sequence

$$1 \to \pi' \to \pi \to Z \to 1. \tag{3.2}$$

Denoting the generator of the infinite cyclic multiplicative group by t, we have a semi-direct product structure for π , and once a splitting for (3.2) is chosen, we can write each element $x \in \pi$ uniquely as $x = t^a g$ where a is an integer and $g \in \pi'$. By abuse of notation, let $t^{a}g$ represent an embedding of S^{1} in the same homotopy class, and choose a trivialization of its normal bundle. In order for $Y \cup_{t^{a_g}} h^2$ to be acyclic, we must have $a = \pm 1$, because $H_1(Y; Z)$ is infinite cyclic on the generator t. In order for $Y \cup_{t^{a_g}} h^2$ to be contractible, $i_*(t^a g)$ must be a weight element of $\pi_1(Y)$, where $i_*: \pi_1(\overline{\partial}Y) \to \pi_1(Y)$ is the inclusion homomorphism. In order for $\partial(Y \cup_{t^{a_{\alpha}}} h^2)$ to be simply-connected, $t^{a_{\alpha}}g$ must be a weight element of $\pi_1(\partial Y)$. The upper bound we are aiming at is very crude, coming just from the homology condition $(a = \pm 1)$, so we are in fact counting the ways it is possible to complete Y to obtain an integral homology disk. The set of elements of $\pi_1(\partial Y)$ producing acyclic manifolds upon handle attachment is $\{t^{\pm 1}g \mid g \in \pi'\}$. But since the sign of the exponent of t in an element of $\pi_1(\partial Y)$ is reversed by changing the orientation of the attaching curve of h^2 (or equivalently, reversing the orientation on the cocore D^{n+1}), the set of elements of π corresponding to possibly different manifold pairs is $\{tg \mid g \in \pi'\}$, a set of the cardinality of π' . Now since we are in the

dimension range $(n+2) \ge 4$ for ∂Y , homotopy of embedded one-spheres gives rise to isotopy, so the number of possible isotopy classes of attaching curves in ∂Y giving rise to acyclic manifolds is bounded above by $|\pi'|$. Now, given a represenatative of an isotopy class of attaching curves in ∂Y , there are precisely two ways to attach the 2-handle h^2 , corresponding to the $\pi_1(SO) = Z_2$ ways of choosing a trivialization of the normal bundle of the curve. Hence the number of possible handle attachments yielding acyclic manifolds is bounded above by $2|\pi'|$.

COROLLARY 3.3. Suppose that Y^{n+3} $(n \ge 2)$ is an (n+1)-disk knot exterior, and that $\pi_1(\partial Y) = Z$. Then $\zeta(Y) \le 2$, and the two possibly different disk pairs are obtained, each from the other, by re-attaching the 2-handle corresponding to the normal bundle over the submanifold via the non-trivial element of $\pi_1(SO)$.

Corollary 3.3 yields an easy proof of the well-known result that there are at most two inequivalent n-knots with the same exterior:

COROLLARY 3.4. ([B], [L-S], [Ka 1], [Sw]). Let X^{n+2} ($n \ge 3$) be an *n*-sphere exterior. Then $\zeta(X) \le 2$. Moreover, if $(X \cup_{\gamma} (D^2 \times S^n), \{0\} \times S^n)$ denotes a sphere pair obtained by sewing $D^2 \times S^n$ onto X via some trivialization of the S^n -bundle over the meridian curve $\gamma = S^1 \times \{*\} \subset \partial X$, then the possibly different sphere pair is $(X \cup_{\overline{\gamma}} (D^2 \times S^n), \{0\} \times S^n)$, where $\overline{\gamma}$ denotes the same meridian curve with different trivialization of the S^n -bundle (i.e., $D^2 \times S^n$ is sewn in with a $\pi_1(SO)$ -twist).

Proof. There is a one-to-one correspondence between *n*-sphere knots and *n*-disk knots with unknotted boundary (n-1)-sphere pair, obtained by removing an unknotted disk pair (the neighborhood of a point on the submanifold) from the sphere pair to obtain the required disk pair. An *n*-sphere knot and its corresponding *n*-disk knot have the same exterior X. But $\partial X \approx S^1 \times S^n$, and $\pi_1(\partial X) = Z$, so by Corollary 2.4, $\zeta(X) \leq 2$. That is, X (thought of as a disk exterior) determines at most two inequivalent disk pairs. Therefore, thinking of it as a sphere pair exterior, then $\zeta(X) \leq 2$ as well.

§4. Some questions

- 1. Given a positive integer N, does there exist an (n+1)-disk exterior Y with $\zeta(Y) \ge N$?
- 2. Is there an (n+1)-disk exterior Y with $\zeta(Y) = +\infty$?
- 3. If X is an *n*-sphere exterior and $\pi_1(X) = Z$, must if follow that $\zeta(X) = 1$?

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