Comment. Math. Helvetici **57** (1982) 41–59

0010–2571/82/000141–19$01.50+0.20/0

Combinatorics and intersections of Schubert varieties

HOWARD HILLER

Let G be a semi-simple, simply connected algebraic group over an algebraically closed field k and P_θ a parabolic subgroup of G corresponding to a subset θ of the simple roots Σ. The Bruhat decomposition of G/P_θ yields a poset (= partially ordered set) W^θ of Schubert varieties. Actually, this poset can be defined group theoretically in terms of the Weyl group W (and more generally for any, not necessarily finite, Coxeter group). The combinatorial study of W^θ has been initiated in the work of Verma [26], Deodhar [8] (computation of Möbius functions), Stanley [23] (Sperner properties, rank unimodality), Proctor [17], and Björner and Wachs [3] (shellability).

The goal of this paper is to explain an interesting connection between counting "paths" in the poset W^θ and the intersection theory on the variety G/P_θ. This observation is related to recent work of Seshadri [20] describing a standard monomial theory for representations of G. Indeed, his work immediately yields an interpretation of the zeta polynomial of W^θ and intervals contained in it. In particular, it gives a combinatorial interpretation of Demazure's Weyl dimension formula for the Schubert varieties [7].

In section 1, we record some basic combinatorial definitions and introduce some important lattices. In section 2, the Chow ring of G/B and G/P_θ is described and the poset W^α is introduced. As an example, we indicate how the hook formula in the representation theory of symmetric groups makes its appearance in the Schubert calculus.

In section 3, we discuss the notion of a miniscule weight ω_α from several different points of view. In particular, we see that this condition implies that the intersection theory on the corresponding G/P_α is multiplicity-free. We also explain the connection between Seshadri's work and the zeta polynomial of W^α.

In section 4, we turn to the analysis of the miniscule weight ω_n in B_n. (In some sense, this is the only interesting case). This leads us to a notion of shifted Young tableaux and we can invoke a formula of Schur to solve our problem. Similarly, in section 5 we consider the weight ω_n in C_n and get an analogous result.

It is a pleasure to thank Richard Stanley for his helpful correspondence and Robert Proctor for a copy of his thesis.

§1. Combinatorics

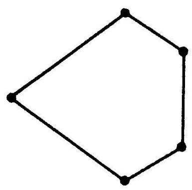
We recall some basic combinatorial language. A good reference is [1]. Let $(P, <)$ be a finite poset. If $p, q \in P$, we say q *covers* p (notation: $p \to q$) if $p < q$ and whenever $p < x \leq q$ then $x = q$. A *chain* of *length* $n-1$ from p to q is a sequence $p = p_1 \leq \cdots \leq p_n = q$ in P. The chain is said to be *maximal* if $p_i \to p_{i+1}$, $1 \leq i \leq n-1$ and we call a maximal chain a *path*. Suppose our poset has a least element 0 and a greatest element 1. We define (see [1, p. 143]) the *zeta polynomial* of P by

$$Z(P, n) = \# \{\text{chains from 0 to 1 of length } n\}$$

where $\#$ denotes cardinality. We also define similarly the *kappa polynomial*

$$K(P, n) = \# \{\text{paths from 0 to 1 of length } n\}.$$

A *rank function* for a poset p is a function $r: P \to \mathbf{N}$ with $r(0) = 0$ and if $p \to q$ then $r(q) = r(p) + 1$. Clearly, P admits a rank function if and only if all maximal chains from p to q have the same length (and that length is $r(q) - r(p)$). It rules out subposets of the form



so, in particular, the poset is decomposed into *levels* $P_n = \{p \in P : r(p) = n\}$. We call the formal power series

$$PS(P, t) = \sum_{n=0}^{\infty} (\#P_n) t^n = \sum_{p \in P} t^{r(p)}$$

the *generating function* (or Poincaré series) of the poset P. The *height* of a ranked poset P is $H(P) = \max_{p \in P} r(p) = r(1)$. If $p \leq q$ and $[p, q] = \{x \in P : p \leq x \leq q\}$ is the *interval* between them we define

$$\kappa(p, q) = K([p, q], r(q) - r(p)).$$

Notice that for a ranked poset the kappa polynomial degenerates to a single number since it only makes sense at one argument. We abbreviate $\kappa(0, q)$ by $\kappa(q)$. The following result is immediate.

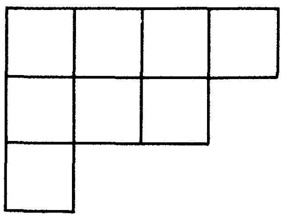
LEMMA 1.1. *If P is a ranked poset, p, $q \in P$, then*

$$\kappa(p, q) = \sum_{\substack{q' \to q \\ p \leq q'}} \kappa(p, q')$$

In particular, $\kappa(q) = \sum_{q' \to q} \kappa(q')$.

An *ideal* I in a poset P is a subposet satisfying: if $p < q \in I$, then $p \in I$. Clearly, $\{p : p \leq q\}$ is an ideal in P and is called the *principal* ideal generated by q.

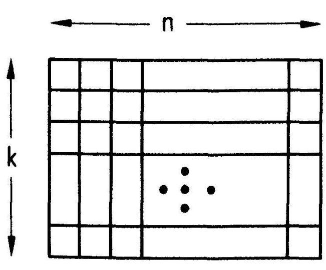
We introduce an important lattice. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ denote an infinite sequence in $\mathbf{N}$ which is eventually zero. Define $\lambda \leq \lambda'$ if $\lambda_i \leq \lambda'_i$ for all $i \geq 1$. We call the poset of such sequences the *Young lattice* $\mathcal{Y}$ [1, p. 17]. The rank function is the obvious one $r(\lambda) = \sum_{i=1}^{\infty} \lambda_i$. In particular, $\lambda \to \lambda'$ if for exactly one i, $\lambda'_i = \lambda_i + 1$; all other values unchanged. One can view λ as a partition of $r(\lambda)$ and represent it diagrammatically by its *shape*, e.g. $\lambda = (4 \geq 3 \geq 1)$ has shape



We will be concerned with certain ideals in $\mathcal{Y}$. Define:

$$\mathcal{Y}_{k,n} = \{\lambda \in \mathcal{Y} : \lambda_1 \leq n \text{ and } \lambda_i = 0, i > k\}.$$

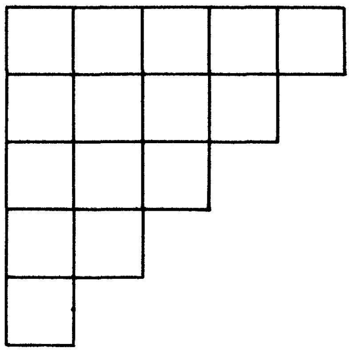
It is easy to see $\mathcal{Y}_{k,n}$ is the ideal generated by the "rectangular" partition $n \geq \cdots \geq n \geq 0 \ldots$, (with k non-zero terms)



The generating function of $\mathcal{Y}_{k,n}$ is the Gaussian polynomial; namely

$$PS(\mathcal{Y}_{k,n}, t) = \begin{bmatrix} n+k \\ k \end{bmatrix}(t) = \frac{(1-t^{n+1}) \cdots (1-t^{n+k})}{(1-t) \cdots (1-t^k)}.$$

Let $\tilde{\mathcal{Y}}$ denote the sublattice of *strict* sequences $\lambda=(\lambda_1\geq\lambda_2\geq\cdots)$ satisfying $\lambda_i>\lambda_{i+1}$ unless $\lambda_i=0$. We also write $\tilde{\mathcal{Y}}_{k,n}=\mathcal{Y}_{k,n}\cap\tilde{\mathcal{Y}}$. This is the principal ideal generated by $(n>n-1>n-2>\cdots)$ with k non-zero entries. In particular, $\tilde{\mathcal{Y}}_n=\tilde{\mathcal{Y}}_{n,n}$ is generated by $(n>n-1>\cdots>1)$, so the shape is



for $n=5$. The generating function of $\tilde{\mathcal{Y}}_n$ can also be computed

$$PS(\tilde{\mathcal{Y}}_n, t)=\begin{cases}\dfrac{(1-t^{n+2})(1-t^{n+4})\cdots(1-t^{2n})}{(1-t)(1-t^3)\cdots(1-t^{n-1})} & \text{if} \quad n\equiv 0(2)\\[2ex] \dfrac{(1-t^{n+1})(1-t^{n+3})\cdots(1-t^{2n})}{(1-t)(1-t^3)\cdots(1-t^{n})} & \text{if} \quad n\equiv 1(2).\end{cases}$$

As we will see later, these lattices occur naturally in the geometry of certain homogeneous spaces.

§2. *G/B*

We recall here some basic facts about intersection theory on the flag variety G/B [2], [7]. We begin with a barrage of notation:

G = split, simple, simply-connected algebraic group over a field $k=\bar{k}$
B = Borel subgroup
T = maximal torus $\subset B$
$X(T)$ = character group on T
$V=\mathbf{R}\otimes_{\mathbf{Z}}X(T)$
Δ = root system in V
$\Sigma=\{\alpha_1,\ldots,\alpha_l\}$ a set of simple roots $\subset\Delta$
Δ^+ = positive roots, $\Delta^-=-\Delta^+$.
$\Sigma^\vee$ = coroots $\{\alpha_1^\vee,\ldots,\alpha_l^\vee\}$ where $\alpha_i^\vee=2(\alpha_i,\alpha_i)^{-1}\alpha_i$
ω_i = i^{th} fundamental weight, satisfying $(\omega_i,\alpha_j^\vee)=\delta_{ij}$
W = Weyl group generated by simple reflections $S=\{s_\alpha:\alpha\in\Sigma\}$ with length function $l(w)$ and longest word w_0; so that $l(w_0)=|\Delta^+|$ and $w_0(\Delta^+)=\Delta^-$.
$A^i(\cdot)$ = Chow group of codimension i cycles up to rational equivalence.

It is a consequence of the Bruhat decomposition for G that G/B possesses a "cell-decomposition" given by the B-orbits $B_w = BwB/B$, $w \in W$, where B_w is isomorphic to an affine variety of dimension $l(w)$. We let X_w denote the (Schubert) class in $A^{l(w)}(G/B)$ corresponding to the closure $\bar{B}_{w_0 w}$ (*Schubert variety*). This gives a $\mathbf{Z}$-basis $\{X_w\}_{w \in W}$ for the Chow ring $A^*(G/B)$. In order to complete the description of $A^*(G/B)$ we must compute intersection multiplicities. The first reduction is that every Schubert class is a polynomial in the X_{s_α}'s, $\alpha \in \Sigma$. For example, if $N = l(w_0)$, then

$$X_{w_0} = \frac{1}{N!}\Big(\sum_{\alpha \in \Sigma} X_{s_\alpha}\Big)^N \quad [2, \text{p. } 17].$$

The other polynomials are obtained by applying appropriate polynomial operators to X_{w_0} [2, p. 15]. (These results are like the Giambelli (or determinantal) formula of the Schubert calculus.) The upshot of this is that it suffices to compute $X_w \cdot X_{s_\alpha}$, $\alpha \in \Sigma$, $w \in W$. We have the following Pieri-type formula.

THEOREM 2.1 (Chevalley [5]). *If $w \in W$, $\alpha \in \Sigma$ then*

$$X_w \cdot X_{s_\alpha} = \sum (\beta^\vee, \omega_\alpha) X_{w s_\beta}$$

where $\beta \in \Delta^+$ satisfies $l(ws_\beta) = l(w) + 1$.

This range of summation gives us our definition of the Bruhat order on the Weyl group W. Namely, the covering relation $w \to w'$ requires that there exist a reflection s_β, $\beta \in \Delta^+$, so that $w' = ws_\beta$ and $l(w') = l(w) + 1$. The Bruhat order $<$ is the transitive closure of this relation. This algebraic definition is equivalent to the geometric condition $B_w \subset \bar{B}_{w'}$.

Remark [6]. If $V_{\mathbf{Z}} = X(T)$, then there is a map

$$C : S(V_{\mathbf{Z}}) \to A^*(G/B)$$

where S denotes polynomial ring over $\mathbf{Z}$. This is obtained by taking the first Chern class of the line bundle L_χ associated to a character χ of T. This map satisfies

(i) Ker(c) is the ideal generated by the positive W-invariants, i.e. $\bigoplus_{j>0} S_j(V_{\mathbf{Z}})^W$.

(ii) Coker(c) is finite and annihilated by $\# W$. For example, if $\omega_\alpha \in S_1(V_{\mathbf{Z}})$ is the fundamental weight dual to $\alpha^\vee$, then $c(\omega_\alpha) = X_{s_\alpha}$.

It is possible to "restrict" this Schubert calculus description of G/B to G/P, P a parabolic in G. We usually suppose P is a maximal parabolic P_α corresponding to a fundamental weight ω_α. It is helpful to recall the following (see [4]).

LEMMA 2.2. (i) *If* $W^\alpha = \bigcap_{\beta\in\Sigma-\{\alpha\}}\{w \in W : l(ws_\beta) = l(w)+1\}$, *then* W^α *is a set of minimal length left coset representatives of* W^α *in* W. (W_α *is the subgroup of* W *generated by* $\{s_\beta : \beta \in \Sigma - \{\alpha\}\}$).

(ii) *If* $w \in W$, *then there exist unique elements* $w^\alpha \in W^\alpha$, $w_\alpha \in W_\alpha$ *such that* $w = w^\alpha \cdot w_\alpha$. *Furthermore,* $l(w) = l(w^\alpha) + l(w_\alpha)$.

From this fact and a computation of the action of W on $A^*(G/B)$ one finds $A^*(G/P_\alpha)$ is $\mathbf{Z}$-free on $\{X_w : w \in W^\alpha\}$. Hence the projection $G/B \xrightarrow{\pi_\alpha} G/P_\alpha$ induces an inclusion $A^*(G/P_\alpha) \hookrightarrow A^*(G/B)$. Observe that the unique codimension one class is $H = X_{s_\alpha} \in A^1(G/P_\alpha)$.

Remark [2]. Under the map π_α^* one can actually identify $A^*(G/P_\alpha)$ with $A^*(G/B)^{W_\alpha}$, where the superscript denotes invariants.

EXAMPLE. If $G = SL_{n+k}$, $\alpha = e_k - e_{k+1} \in \Sigma$, then $W = \Sigma_{n+k}$, $W^\alpha = \Sigma_k \times \Sigma_n$ and $W^\alpha = \{\sigma \in \Sigma : 1 \leq \sigma(1) < \cdots < \sigma(k) \leq n+k$ and $\sigma(k+1) < \cdots < \sigma(k+n)\}$. Associate to $\sigma \in W^\alpha$, the non-increasing k-tuple $(a_1, \ldots, a_k)$ where $a_i = \sigma(k-i+1) - (k-i+1)$. This is a bijection and $(a_1, \ldots, a_k)$ is a partition or, in the notation of §1, an element of $\mathcal{Y}$; actually $\mathcal{Y}_{k,n}$ as one can check. It is not difficult to see that this bijection is a poset isomorphism. Hence the Chevalley formula (2.1) becomes in $A^*(G/P_\alpha)$

$$X_\lambda \cdot H = \sum_{\lambda \to \lambda'} X_{\lambda'}$$

when one computes the coefficients $(\beta^\vee, \omega_\alpha)$. (It is actually possible to derive the full Pieri formula in this framework [14].)

In particular, one gets $H^{nk} = \kappa(n, n, \ldots, n)$. The number on the right counts the number of standard Young tableaux on a rectangular shape, so is given by the hook formula [1, p. 132]. Hence

$$H^{nk} = \frac{(nk)!}{1^1 2^2 \cdots k^k \cdots (n+1)^{k-1} \cdots (n+k-1)^1}$$

a fact that was observed in [13]. This leads one to the following general question.

PROBLEM. 2.3. If P_α is a maximal parabolic corresponding to a fundamental weight ω_α, $H \in A^1(G/P_\alpha)$ is the class of the unique codimension one subvariety, compute the number

$$H^d \in A^d(G/P_\alpha) \approx \mathbf{Z}$$

where $d = \dim_k (G/P_\alpha)$.

In the next section we determine which weights are reasonable to handle and in §4 we analyze these cases.

§3. Miniscule weights

In this section, we introduce the notion of a miniscule weight. The main fact is that these weights can be characterized abstractly, or from the points of view of representation theory or intersection theory.

Let $\Delta, \Sigma, \ldots$ be as in §2. Let $Q = \sum_{i=1}^{l} \mathbf{Z}\alpha_i$ denote the lattice of roots and similarly for $Q^\vee$. Recall that if Λ is a lattice in V, then $\Lambda^* = \{x \in V : (\lambda, x) \in \mathbf{Z}\ \forall \lambda \in \Lambda\}$ is the *dual lattice*. The *weight lattice* $P = \sum_{i=1}^{l} \mathbf{Z}\omega_i$ is, by definition, dual to $Q^\vee$ and $P \supseteq Q$. We let C denote the Weyl chamber $\{x \in V : (x, \alpha^\vee) > 0\}$ and $P_+ = P \cap \bar{C}$ is the set of *dominant* weights. The weights are ordered by $\lambda \leq \lambda'$ if $\lambda' - \lambda$ is a non-negative sum of simple roots.

DEFINITION 3.1. A set $S \subset P$ is *saturated* if whenever $\lambda \in S$, $\alpha \in \Delta$, $0 \leq i \leq (\lambda, \alpha^\vee)$, then also $\lambda - i\alpha \in S$.

A typical saturated set arises in the representation theory of the complex, simple Lie algebra $\mathfrak{g} = \text{lie}\,(G_{\mathbf{C}})$. If $\lambda \in P_+$, we let V_λ denote the corresponding finite-dimensional irreducible representation with highest weight λ and $P(\lambda)$ the weights that occur in the weight-space decomposition:

$$V_\lambda = \sum_{\mu \in P(\lambda)} V_\lambda^\mu \quad [15].$$

The set $P(\lambda)$ is saturated.

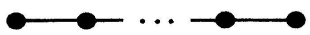
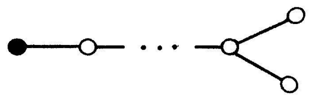
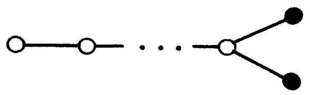
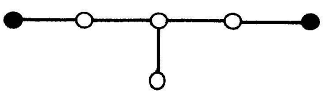
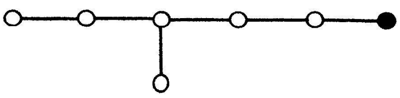
PROPOSITION–DEFINITION 3.2. A dominant weight λ is *miniscule* if one of the following equivalent conditions hold:

(i) The W-orbit $W\lambda$ is saturated
(ii) λ is minimal, i.e. if $\mu \in P_+$ and $\mu \leq \lambda$ then $\mu = \lambda$
(iii) $P(\lambda) = W\lambda$
(iv) $(\beta^\vee, \lambda) = 0$ or 1, for all $\beta \in \Delta^+$.

Remark. According to the formula of Chevalley (2.1), condition (iv) precisely says that intersections with H in $A^*(G/P_\alpha)$ are multiplicity-free. (see [20]).

We call P/Q the *fundamental group* of G. It is a finite group of order equal to the determinant of the Cartan matrix. Every non-zero coset contains a non-zero miniscule weight. Hence the number of non-zero miniscule weights is $|P/Q|-1$. If $\tilde{\alpha}$ is the highest root of Δ and $\tilde{\alpha}^\vee = \sum n_i\alpha_i^\vee$ then the number of miniscule weights is $\#\{i : n_i = 1\}$. The following table lists all miniscule weights and information about the associated poset W^α.

Miniscule weights

G	Dynkin diagram	Miniscule weights	$\#\lvert W^\alpha\rvert$	$H(W^\alpha) = \dim_{\mathbf{C}}(G/P_\alpha)$
A_{n+k-1}		$\omega_l,\ 1 \le l \le n+k-1$	$\binom{n+k}{k}$	kn
B_n		ω_n	2^n	$\binom{n+1}{2}$
C_n		ω_1	$2n$	$2n$
D_n		ω_1	$2n$	$2n+1$
D_n		ω_{n-1}, ω_n	2^{n-1}	$\binom{n}{2}$
E_6		ω_1, ω_6	27	16
E_7		ω_7	56	27

Remark. The vertex representations of the affine Lie algebras [9] play a role analogous to that of the miniscule representations in the classical theory. One new feature is that the action of the Weyl group must be replaced by that of that of the affine Weyl group plus an appropriate infinite-dimensional Heisenberg subalgebra.

The following result combines the ideas of §§1 and 2.

COROLLARY 3.3. *If ω_α is a miniscule weight then in $A^*(G/P_\alpha)$*

$$X_w \cdot H = \sum_{\substack{w \to w' \\ w' \in W^\alpha}} X_{w'} \qquad w \in W^\alpha$$

Proof. According to (3.2iv) we need only check: if $\beta \in \Delta^+$, $l(ws_\beta) = l(w)+1$ and $w \in W^\alpha$, $ws_\beta \in W^\alpha$ then $(\beta^\vee, \omega_\alpha) \neq 0$. This is a consequence of the following more general result.

PROPOSITION 3.4. *If* $w \in W^\alpha$ *and* $l(ws_\beta) = l(w)+1$ *then* $ws_\beta \in W^\alpha$ *if and only if* $(\beta^\vee, \omega_\alpha) \neq 0$. (We are no longer assuming ω_α is miniscule.)

Proof. Suppose $ws_\beta \in W^\alpha$ and $(\beta^\vee, \omega_\alpha) = 0$. Then if β is written as a non-negative sum of simple roots, α does not appear. Hence $s_\beta \in W^\alpha$. Since $w \in W^\alpha$, by (2.2ii) $l(ws_\beta) = l(w) + l(s_\beta)$, so $l(s_\beta) = 1$. Then $\beta \in \Sigma - \{\alpha\}$ and this contradicts $ws_\beta \in W^\alpha$.

The other direction is a consequence of (2.1) and the fact that $A^*(G/P_\alpha)$ is a subalgebra of $A^*(G/B)$, (see also [23, 2.2] or construct an elementary argument).

Concretely, (3.4) says that any class that can occur in the intersection with H does occur. We now have:

COROLLARY 3.5. *If* ω_α *is a miniscule weight, then in* $A^*(G/P_\alpha)$

$$X_w \cdot H^d = \sum \kappa(w, w')X_{w'} \qquad w \in W^\alpha$$

where the summation ranges over $w' \in W^\alpha$, $w < w'$, $l(w') = l(w) + d$. *In particular, if* $d = \dim_k (G/P_\alpha) = H(W^\alpha)$ *and* $w = 1$, *then*

$$H^d = K(W^\alpha, d) = \kappa(w_0^\alpha)$$

where w_0^α *is the longest word in* W^α.

Proof. Combine (3.3), (1.1) and an induction argument,

We can now use Poincaré duality on G/P_α to write $\kappa(w, w')$ as a triple intersection product.

COROLLARY 3.6. *If* ω_α *is a miniscule weight then in* $A^*(G/P_\alpha)$

$$\kappa(w, w') = X_{w_0 w' w_0^\alpha w_0} \cdot X_w \cdot H^d$$

under the usual identification.

Proof. One need only check that the map $w \to w_0 w w_0^\alpha w_0$ on W^α induces the Poincaré duality map.

Miniscule weights also appear naturally in the work of Seshadri [20]. He shows that if ω_α is miniscule weight then the induced representation $H_\alpha^m = H^0(G/P_\alpha, L_{\omega_\alpha}^{\oplus m})$ admits a k-basis of "standard monomials" parametrized by chains $0 = w_1 \leq w_2 \leq \cdots \leq w_{m+2} \leq 1$ in W^α of length $m+1$. In characteristic zero, H_α^m is the irreducible G-module with highest weight $mi(\omega_\alpha)$ where i is the Weyl involution. This implies, in the language of §1,

PROPOSITION 3.7. *If* ω_α *is a miniscule weight*

$$Z(W^\alpha, m+1) = \dim_k (H_\alpha^m).$$

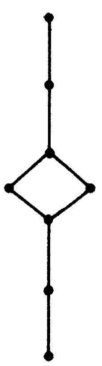
In particular, $\dim_k (V_{i(\omega_\alpha)}) = \# W^\alpha$.

It is now possible to use the Weyl dimension formula to get a product expansion for this zeta polynomial. Notice also that the graded object $\oplus_m H_\alpha^m$ can be interpreted as the coordinate algebra of G/P under an appropriate projective embedding. Hence, its Poincaré series can be described by the Weyl character formula. (See final comment before remarks in §4).

Seshardri has a relative version of his result. If X_w is a Schubert variety one can compute the character of $H^\circ(X_w, L_{\omega_\alpha}^{\oplus m} \mid X_m)$. The dimension of this representation is now related, as in (3.7) to the zeta polynomial of the interval $[1, w]$ in W^α. Demazure [7] also has an abstract Weyl dimension formula for the Schubert variety that one can invoke in this situation.

The picture that emerges is that chains in W^α are connected to the representation theory of G, while in a similar way the paths in W^α are tied to the intersection theory of G/P_α. Can one explain this relation between representations and intersections in an intrinsic way? Finally, we remark that Proctor [17] has proven the persuasive result that W^θ is a distributive lattice precisely when ω_α is a miniscule weights and $W^\theta = W^\alpha$ (excluding the trivial case of G_2).

Let us look at what the table of miniscule weights tells us about the problem (2.3). The case A_n is a classical and analyzed in §2. The poset corresponding to the pair (C_n, ω_1) is a simple chain, so $H^d = 1$. The poset corresponding to (D_n, ω_1) is only slightly more complicated, e.g. $n = 4$ is



so $H^d = 2$. The posets of (D_n, ω_{n-1}) or (D_n, ω_n) are both actually identical to (B_{n-1}, ω_{n-1}). The cases of E_6 and E_7 are covered in the penultimate remark of §4. So it remains to consider the case (B_n, ω_n) which we turn to now.

§4. Orthogonal groups

Let V denote a real vector space of dimension n equipped with the standard Euclidean inner product (,). We recall the usual realization of the root system of type B_n [4]. If $\{e_1, \ldots, e_n\}$ denotes the standard basis of V, then Δ is the set of vectors

$$\{\pm e_i \pm e_j : 1 \leq i \leq j \leq n\} \cup \{\pm e_i : 1 \leq i \leq n\}.$$

a basis $\Sigma = \{\alpha_1, \ldots, \alpha_n\}$ of simple roots is obtained by letting

$$\alpha_i = \begin{cases} e_i - e_{i+1} & 1 \leq i < n \\ e_n & i = n \end{cases}$$

so that the positive roots Δ^+ *are*

$$\{e_i - e_j : 1 \leq i < j \leq n\} \cup \{e_i + e_j : 1 \leq i < j \leq n\} \cup \{e_i : 1 \leq i \leq n\}.$$

The Weyl group W is the semi-direct product $\sum_n \times \mathbf{Z}_2^n$ where the symmetric group $\sum_n$ acts in the obvious way. W has a natural integral representation as signed permutation matrices; it is the symmetry group of an n-dimensional cube (the hyperoctahedral group). We write a typical element $w \in W$ as a pair (σ, ε), $\sigma \in \sum_n$, $\varepsilon \in \mathbf{Z}_2^n$. It is not hard to show, by induction on $l(w)$, that

LEMMA 4.1. *If* $w = (\sigma, \varepsilon) \in W$, *then*

$$l(w) = \tilde{l}(\sigma) + \sum_{\varepsilon_j = -1} (2d_j + 1)$$

where $d_j = d_j(\sigma) = \#\{x > j : \sigma(x) < \sigma(j)\}$ *and* $\tilde{l}$ *is the length function on* $\sum_n$ *with respect to* $s_1, \ldots, s_{n-1}$.

We also recall that $\tilde{l}(\sigma) = \sum_{j=1}^{n-1} e_j$, where $e_j = e_j(\sigma) = \{x > j : \sigma(x) < \sigma(j)\}$. This yields

COROLLARY 4.2. *If* $w = (\sigma, \varepsilon) \in W$, *then*

$$l(w) = \sum_{\varepsilon_j = -1} (n + 1 - j) + \sum_{\varepsilon_j = -1} d_j + \sum_{\varepsilon_j = +1} e_j.$$

Proof. Clearly $d_j + e_j = n - 1$ so

$$\begin{aligned} l(w) &= \tilde{l}(\pi) + \sum_{\varepsilon_j=-1} (2d_j + 1) \\ &= \sum_{j=1}^{n-1} e_j + \sum_{\varepsilon_j=-1} d_j + \sum_{\varepsilon_j=-1} d_j + \sum_{\varepsilon_j=-1} 1 \\ &= \sum_{\varepsilon_j=-1} (n-j) + 1 + \sum_{\varepsilon_j=1} e_j + \sum_{\varepsilon_j=-1} d_j. \end{aligned}$$

Our first task is to explicitly identify W^α, where $\alpha = \alpha_n$ (see also [23]). We begin with

COROLLARY 4.3. *If $w = (\sigma, \varepsilon) \in W$, $i < n$, $s_i = s_{\alpha_i}$, then $l(ws_i) = l(w) + 1$ if and only if $\varepsilon_{i+1}\sigma(i) < \varepsilon_i\sigma(i+1)$.*

Proof. The length goes up by one if and only if $w(e_i - e_{i+1}) = \varepsilon_i e_{\sigma(i)} - \varepsilon_{i+1} e_{\sigma(i+1)} \in \Delta^+$. The argument is finished by checking the four possible cases.

If $\{x_1 < \cdots < x_k\}$ is a subset of $\{1, 2, \ldots, n\}$ arranged in increasing order, let $\{y_1 > \cdots > y_{n-k}\}$ be the complementary subset arranged in decreasing order. Define an element $\langle x_1, \ldots, x_k \rangle$ of W by

$$\pi(i) = \begin{cases} x_i & i \le k \\ y_{i-k} & i > k \end{cases} \qquad \varepsilon_i = \begin{cases} 1 & i \le k \\ -1 & i > k. \end{cases}$$

We now have

PROPOSITION 4.4. *The set $W^\alpha = \{\langle x_1, \ldots, x_k \rangle\}$ where $x_1 < \cdots < x_k$ varies over the 2^n subsets of $\{1, 2, \ldots, n\}$.*

Proof. Each $\langle x_1, \ldots, x_k \rangle \in W^\alpha$ by (2.2i) and (4.3). But $|W^\alpha| = 2^n$ and the result follows.

We now compute the length function restricted to W^α.

PROPOSITION 4.5. *If $\langle x_1, \ldots, x_k \rangle \in W^\alpha$, then*

$$\langle x_1, \ldots, x_k \rangle = \sum_{j=1}^{k} x_j + (n+1)\left(\frac{n}{2} - k\right).$$

Proof. By (4.2), we have

$$\begin{aligned} l\langle x_1,\ldots,x_k\rangle &= \sum_{j=k+1}^{n}(n+1-j)+\sum_{j=k+1}^{n} d_j+\sum_{j=1}^{k} e_j \\ &=(n+1)(n-k)-\sum_{j=k+1}^{n} j+0+\sum_{j=1}^{k}(x_j-j) \\ &=(n+1)(n-k)-\frac{n(n+1)}{2}+\sum_{j=1}^{k} x_j \end{aligned}$$

and the result follows.

We would like a notation for the elements of W^α so that the length function has a simpler form. Associate to the symbol $(x_1>\cdots>x_k)$ the element $\langle y_i<\cdots<y_{n-k}\rangle\in W^\alpha$, where $y_1,\ldots,y_{n-k}$ is an ordered enumeration of the complement to the set $\{(n+1)-x_i:1\le i\le k\}$. Then we get

PROPOSITION 4.6. *If* $n\ge x_1>\cdots>x_k\ge 1$, *then*

$$l(x_1,\ldots,x_k)=\sum_{i=1}^{k} x_i.$$

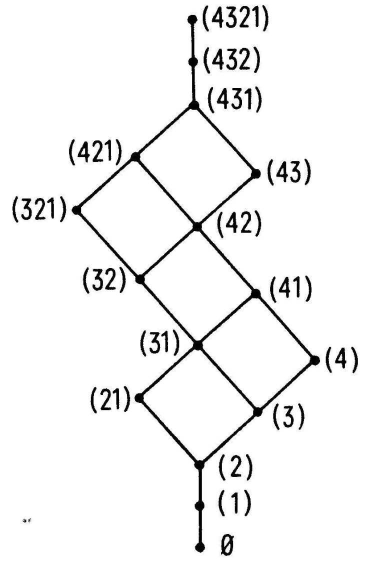
Proof. By (4.5) and the definition of ( ),

$$\begin{aligned} l(x_1,\ldots,x_k) &= \sum_{j=1}^{n-k} y_j+(n+1)\left(\frac{n}{2}-(n-k)\right) \\ &=\frac{n(n+1)}{2}-\sum_{i=1}^{k}(n+1-x_i)-\frac{n(n+1)}{2}+k(n+1) \\ &=\sum_{i=1}^{k} x_i. \end{aligned}$$

Hence we view $(x_1>\cdots>x_k)$ as a natural notation for elements of W^α. Clearly, $(n, n-1,\ldots,1)$ is the unique element of maximal length $\binom{n+1}{2}$. It remains to understand the Bruhat order restricted to W. If we view $(x_1>\cdots>x_k)$ as a strict partition in the sense of §1, we get:

PROPOSITION 4.7. *There is an isomorphism of posets* $W^\alpha\leftrightarrow\tilde{\mathscr{Y}}_n$, *where the latter poset is the ideal of* $(n, n-1,\ldots,1)$ *in* $\tilde{\mathscr{Y}}$ (as in §1).

Here is a picture of the Hasse diagram of $\tilde{\mathcal{Y}}_4$.



In order to understand the intersection multiplicities in the d-fold self-intersection of $H = X_{s_n}$, we must compute the function $(x_1 > \cdots > x_k)$. Fortunately, this combinatorial problem has been solved by Schur (see also Thrall [25]). We record the result:

PROPOSITION 4.8. (Schur [19]). *The number of paths from* ϕ *to* $(x_1 > \cdots > x_k)$ *in* $\tilde{\mathcal{Y}}$ *is given by*

$$\kappa(x_1 > \cdots > x_k) = \frac{(x_1 + \cdots + x_k)!}{x_1! \cdots x_k!} \prod_{1 \le i < j \le k} \frac{x_i - x_j}{x_i + x_j}.$$

Remark. According to Schur [19] the irreducible projective characters of $\sum_n$ are parametrized by the n^{th} level of $\tilde{\mathcal{Y}}$, but there are two corresponding irreducible projective characters if $\sum_{i=1}^{k} (x_i - 1)$ is odd, in the notation of (4.8). The formula of Schur above can be thought of a projective version of the hook formula discussed in §2.

We can now solve the remaining case of problem (2.3) for miniscule weights.

COROLLARY 4.9. *The intersection* $H^d = K_n \cdot X_{(n,n-1,\ldots,1)}$ *in* $A^d(G/P_\alpha)$ *where* $d = \dim_k (G/P_\alpha) = \binom{n+1}{2}$ *and*

$$K_n = \begin{cases} \dfrac{d!\, 2!\, 4! \cdots (n-2)!}{(n+1)!\, (n-3)! \cdots (2n-1)!} & n = 0(2) \\[2ex] \dfrac{d!\, 2!\, 4! \cdots (n-1)!}{n!\, (n+2)! \cdots (2n-1)!} & n \equiv 1(2). \end{cases}$$

So, for example, in $A^*(SO_{13}/U_6)$, $H^{21} = 33{,}592$ times the class $X_{w_0^\alpha}$.

Remark. We use the notation of the remark following (2.1). If G is the group of type B_n, $\alpha = \alpha_n$, there is a map

$$c : S(V)^{\Sigma_n} \to A^*(G/P_\alpha).$$

It is possible to compute this map explicitly; namely

$$c(\sigma_j) = 2X_{(j)}$$

where $(j) = s_{n+1-j} \cdots s_{n-1}s_n$ in terms of the fundamental reflections. (The coefficient 2 arises because the index of torsion for G is 2 [6].) These Schubert classes $X_{(j)}$ play the same role as the special Schubert cycles in the classical Schubert calculus (see [14]). We hope to write down a Pieri formula for $j > 1$ in a future paper ($j = 1$ is (3.3)); the result is complicated by the multiplicities.

In the case of groups of type A_n, a path in W^α admitted an interpretation as a standard Young tableaux. We give a similar notion for the poset $\tilde{\mathcal{Y}}_n$.

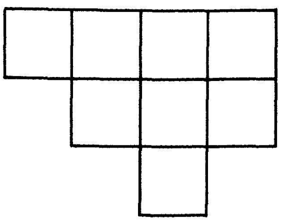
DEFINITION 4.10. A *Strict Young tableau* on a strict partition $x = (x_1 > \cdots > x_k)$ is an assignment of the numbers $1, \ldots, r(x) = x_1 + \cdots + x_k$ to the boxes of the shape of x so that entries in each row and antidiagonal increase.

For example, $\begin{smallmatrix}1&2\\3&\end{smallmatrix}$ is a strict Young tableau, but $\begin{smallmatrix}1&3\\2&\end{smallmatrix}$ is not. Notice the definition forces that the entries increase in each column so a strict Young tableau is a standard Young tableau, but not conversely as our example shows. It is trivial to check

PROPOSITION 4.11. *There is a bijection*:

$$\left\{\begin{array}{l}\textit{paths in } W^\alpha\\ \textit{from } \phi \textit{ to } x = (x_1 > \cdots > x_k)\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\textit{strict Young tableaux}\\ \textit{on the shape of } x\end{array}\right\}.$$

Now suppose we take a strict shape and shift each row over to the right by one box relative to the row above it. For example, the shape of $(4 > 3 > 1)$ corresponds to the *shifted shape*



We now observe

PROPOSITION 4.12. *There is a bijection*

$$\left\{\begin{array}{l}\textit{strict Young tableaux}\\ \textit{on the shape of } x\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\textit{standard Young tableaux}\\ \textit{on the shifted shape of } x\end{array}\right\}.$$

The objects on the right-hand side of the bijection of (4.12) are called *shifted Young tableaux* and have been studied extensively by students of R. Stanley [11], [12], [18]. Schur's formula (4.9) counts these objects. Indeed, it possible to assign a shifted hook-length to each box of the shifted shape, so that (4.9) has the form of the usual Frame–Robinson–Thrall hook formula (see [16, p. 135]). For example, the shifted hook-lengths for $(4>3>1)$ are indicated

7	5	4	2
	4	3	1
		1	

It seems to be an open problem to compute the relative function $\kappa(w, w')$ in this case. In the case of the lattice $\mathcal{Y}_{k,n}$ (i.e. groups of type A_n) such a skew-hook formula is known. Indeed, it made its first appearance in an 1891 computation of H. Schubert in enumerative geometry and was rediscovered in this century by W. Feit in the context of representations of Σ_n.

According to Seshadri's theory (see the end of §3) the chains of length $m+1$ in W^α will parametrize a k-basis of the representation $V_{m\omega_\alpha}$. Observe that $m=1$ is the spinor representation of dimension 2^n. A chain in W determines a *shifted plane partition.* But Stanley [24] shows that such an object is equivalent to a *column strict plane partition* (see [21] for definitions). By writing down a specialization of the Weyl character formula one can derive the generating function for these objects (see [17, 4.2]).

We conclude with two remarks. The first completes the solution of problem (2.3) for the remaining exceptional groups. The second gives an interpretation of $X_w \cdot H^d$ in terms of the degree of a Schubert variety.

Remarks. 1. In the Chow ring of the homogeneous varieties E_7/E_6 and E_6/D_5 one can try to compute the multiplicity of the highest self-intersection of $H(2.3)$. Fortunately, there is a picture of the respective Bruhat orders in [17]. So we can just count and get

$$H^{16}=78 \quad \text{in} \quad A^*(E_6/D_5)$$
$$H^{27}=13,188 \quad \text{in} \quad A^*(E_7/E_6).$$

2. (Geometric application). There is a projective embedding of G/P_α into a large enough projective space $\mathbf{P}^N$ (coming from the ample line bundle L_{ω_α}). For $G=GL_{n+k}$ this is the classical Plücker embedding of the Grassmannian into $\mathbf{P}^N$, $N=\binom{n+k}{k}-1$. We show how to compute the classical *degree* of Schubert varieties in G/P_α, with ω_α a miniscule weight. This amounts to successively cutting X_w with a hyperplane until one is reduced to counting points. By (3.5)

$$X_w \cdot H^{d-l(w)} = \kappa(w, w_0^\alpha)\cdot X_{w^\alpha} \qquad w\in W^\alpha$$

where $d=\dim_{\mathbf{C}}(G/P_\alpha)$. So if $\mathcal{P}_\alpha$ denotes Poincaré duality for G/P_α, then

$$\deg(X_w)=\kappa(\mathcal{P}_\alpha(w)).$$

For example, referring to the Hasse diagram of $\tilde{\mathcal{Y}}_4$ for SO_9/U_4

$$\deg(X_{(3)})=\kappa(421)=7$$

since Poincaré duality is given by complementation.

§5. Sympletic groups

Let $\omega_\alpha=\omega_n$ denote the "right-most" fundamental weight in the root system of type C_n. The corresponding homogeneous space G/P_α is homeomorphic to Sp_n/U_n. Let $H\in A^1(Sp_n/U_n)$ denote the unique codimension one class. We show how to solve problem (2.3) for this non-miniscule weight by extending the technique of §4.

Since Weyl$(B_n)=$Weyl(C_n), the relevant poset W^α is identical to $\tilde{\mathcal{Y}}_n$ of §4. But by computing inner products $(\beta^\vee,\omega_n)$ [4, p. 254] one gets

$$H\cdot(x_1>\cdots>x_k)=2\sum_{x_i+1<x_{i-1}}(x_1,\ldots,x_i+1,\ldots,x_k)+(1-\delta_{x_k,1})(x_1,\ldots,x_k,1).$$

Let us write $\bar{x}$ for $(x_1,\ldots,x_k)$ and define $\tilde{\kappa}(\bar{x})$ by the equation

$$H^j=\sum_{l(\bar{x})=j}\tilde{\kappa}(\bar{x})(x_1,\ldots,x_k). \tag{5.0}$$

The $\tilde{\kappa}$-function can easily be computed in terms of the κ-function of §4. We have

PROPOSITION 5.1. If $\bar{x} \in w^{\alpha}$, *then* $\tilde{\kappa}(\bar{x}) = 2^{I(\bar{x})}\kappa(\bar{x})$, *where* $I(\bar{x}) = \sum_{i=1}^{k}(x_i - 1)$.

First we leave it as an exercise to check

LEMMA 5.2. $I(\bar{x}(i)) = I(\bar{x}) - 1 + \delta_{x_k,1}\delta_{i,k}$ *where* $\bar{x}(i) = (x_1, \ldots, x_i - 1, \ldots x_k)$ if $x_i - 1 > x_{i+1}$.

Proof of (5.1). We induct on $l(\bar{x})$.

$$\tilde{\kappa}(\bar{x}) = 2\sum_{i \neq k} \tilde{\kappa}(\bar{x}(i)) + 2^{1-\delta_{x_k,1}}\tilde{\kappa}(\bar{x}(k))$$

$$= 2\sum_{i \neq k} 2^{I(\bar{x}(i))}\kappa(\bar{x}(i)) + 2^{1-\delta_{x_k,1}+I(\bar{x}(k))}\tilde{\kappa}(\bar{x}(k))$$

$$= 2^{I(x)}\Big(\sum_{i \neq k} \kappa(\bar{x}(i)) + \kappa(\bar{x}(k)) = 2^{I(x)}\kappa(\bar{x})\Big)$$

by (5.0), (5.2) and (1.1).

COROLLARY 5.3. *If* $H \in A^1(Sp_n/U_n)$, *then* $H^{\binom{n+1}{2}} = 2^{\binom{n}{2}}K_n$, *where* K_n *is as in* (4.9).

EXAMPLE. For Sp_5/U_5, $H^{15} = 2^{10} \cdot 286 = 292864$, so the degree of the symplectic variety grows much faster than its orthogonal counterpart.

REFERENCES

[1] M. Aigner, *Combinatorial theory*, Springer Verlag, Berlin, 1971.

[2] I. Bernstein, I. Gelfand and S. Gelfand, *Schubert cells and cohomology of the spaces G/P*, Russian Math. Surveys *28* (1973) 1–26.

[3] A. Björner and M. Wachs, Bruhat order or Coxeter group and shellability, research announcement.

[4] N. Bourbaki, *Groupes et algèbres de Lie*, Ch. IV, V, VI, Hermann Paris, 1968.

[5] C. Chevelley, Sur les décompositions cellulaires des espaces *G/B*, unpublished manuscript, 1958.

[6] M. Demazure, *Invariants symétriques des groupes de Weyl et torison*, Inv. Math. *21* (1973), 287–301.

[7] ——, *Désingularisation des varietes de Schubert généralisées*, Ann. Scient. Éc. Norm. Sup. 7 (1974), 53–88.

[8] V. Deodhar, *Some characterizations of Bruhat ordering on a Coxeter grôup and determination of the relative Möbius function*, Inv. Math. *39* (1977), 187–198.

[9] I. Frenkel and V. Kac, *Basic representation of affine Lie algebras and dual resonance models*, Inv. Math. *62* (1980), 23–66.
[10] E. Gansner, *Matrix correspondence and the enumeration of plane partitions*, Ph.D. thesis, M.I.T., 1978.
[11] ——, *The Hillman-Grassl correspondence and the enumeration of reverse plane partitions*, to appear.
[12] I. Gessel, *Determinants and plane partitions*, preprint.
[13] H. Hiller, *On the height of the first Stiefel–Whitney class*, Proc. A.M.S. *79* (1980), 495–498.
[14] ——, *Schubert calculus of a Coxeter group*, to appear in L'Enseignement Math.
[15] J. Humphreys, *Introduction to Lie algebras and representation theory*, Springer verlag, Berlin, 1972.
[16] I. Macdonald, *Symmetric functions and Hall polynomials*, Oxford University Press, Oxford, 1979.
[17] R. Proctor, *Interaction between combinatorics, Lie theory and algebraic geometry via the Bruhat orders*, Ph.D. thesis, M.I.T., 1981.
[18] B. Sagan, *Partially ordered seats with hook-lengths – an algorithmic approach*, Ph.D. thesis, M.I.T., 1979.
[19] I. Schur, *Über die Darstellung der symmetrischen und der alternierenden Gruppen durch gebrochene lineare Substitutionen*, J. für Math. 139 (1911), 155–250.
[20] C. Seshadri, *Geometry of G/P, I*, in C.P. Ramanujan – A tribute, pp. 207–239, Studies in Math. No. 8, Tata Press, Bombay, (1978).
[21] R. Stanley, *Theory and application of plane partitions*, Studies in Appl. Math. I, *50* (1971), 167–188 and II, 50 (1971), 259–279.
[22] ——, *Some combinatorial aspects of the Schubert calculus*, Lecture Notes in Mathematics, pp. 217–251, Springer Verlag, Berlin, 1977.
[23] ——, *Weyl groups, the hard Lefschetz theorem and the Sperner property*, SIAM J. Alg. and Discr. Meth, *1* (1980), 168–184.
[24] ——, *The character generator of SU(n)*, J. Math. Phys. 21 (9) (1980), 2321–2326.
[25] R. Thrall, *A combinatorial problem*, Michigan Math. J. *1* (1952), 81–88.
[26] D. Verma, *Möbius inversion for the Bruhat ordering on a Weyl group*, Ann. scient. Éc. Norm. Sup. 4 (1971) 393–398.
[27] V. Lakshmibai and C. S. Seshadri, *Geometry of G/P, II* (The work of De Concini and Procesi and the basic conjectures), Proc. Indian Acad. Sci. A *87* (1978), 1

Department of Mathematics
Yale University
New Haven, CT 06520
USA

Received March 10/December 7, 1981