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A remark on two dimensional periodic potentials

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1. Introduction

Let $A = a_1\mathbf{Z}\oplus a_2\mathbf{Z}$ be the lattice generated by the linearly independent vectors $a_1, a_2 \in \mathbf{R}^2$. Suppose q is a bounded, real-valued function periodic with respect to A. It is known that (see [2] and [5]) the spectrum of the self-adjoint operator $-\Delta+q$ acting on $L^2(\mathbf{R}^2)$ is purely absolutely continuous and is the union of the bands $B_n(q)$ $(n \geqslant 1)$. The nth band $B_n(q) \subset \mathbf{R}^1$ $(n \geqslant 1)$ is by definition the image of the function $\lambda_n(\cdot, q)$ defined on $\mathbf{R}^2$, where $\lambda_n(k, q)$ $(k \in \mathbf{R}^2)$ is the nth eigenvalue (counting with multiplicity) of the boundary value problem

$$(-\Delta+q)f = \lambda f,$$
$$f(x+a_j) = e^{ik\cdot a_j}f(x), \qquad j = 1, 2, \ldots$$

Let $|\mathfrak{F}|$ be the area of a fundamental cell $\mathfrak{F}$ for the lattice B dual to A.

THEOREM. *There exist positive constants $c(q)$, $C(q)$ such that for all $n \geqslant N(q)$*

$$\left[\frac{|\mathfrak{F}|}{\pi}n - cn^{1/4}, \frac{|\mathfrak{F}|}{\pi}n + cn^{1/4}\right] \subset B_n(q) \subset \left[\frac{|\mathfrak{F}|}{\pi}n - Cn^{1/3}, \frac{|\mathfrak{F}|}{\pi}n + Cn^{1/3}\right].$$

Suppose $q(x)$ is a bounded, real-valued, periodic function on $\mathbf{R}^1$. Then the spectrum of $-(d^2/dx^2)+q(x)$ acting on $L^2(\mathbf{R}^1)$ is purely absolutely continuous and its complement generically consists of an infinite number of intervals tending to $+\infty$ [see [1], p. 161]; i.e., there are infinitely many gaps. In two dimensions, however, we have the following immediate consequence of the Theorem.

COROLLARY. *The spectrum of $-\Delta+q$ contains a ray $[\lambda^*, \infty)$. In particular, there are at most a finite number of gaps.*

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The corollary has been announced in [3] for the special case of rational lattices.

2. Proofs

Let $N_B(r, k)$ be the number of lattice points of B in the *open* disk of radius r centered at $k \in \mathbf{R}^2$. Denote by c, C generic positive constants.

LEMMA 1. *For every $\delta > 0$ there is a constant $c = c(\delta) > 0$ such that*

$$\sup_{k \in \mathbf{R}^2} N_B(r, k) \geq \frac{\pi}{|\mathfrak{F}|} r^2 + c r^{1/2},$$

$$\inf_{k \in \mathbf{R}^2} N_B(r, k) \leq \frac{\pi}{|\mathfrak{F}|} r^2 - c r^{1/2}.$$

for all $r \geq \delta$.

Proof. Let h_r be the characteristic function of the open disk $\{k \in \mathbf{R}^2 : |k| < r\}$ and set $E_B(r, k) = N_B(r, k) - (\pi/|\mathfrak{F}|)r^2$. If $a \in A \setminus \{0\}$ we have

$$\begin{aligned} \int_{\mathfrak{F}} E_B(r, k) e^{-ik.a}\, dk &= \int_{\mathfrak{F}} \left(-\pi \frac{r^2}{|\mathfrak{F}|} + \sum_{b \in B} h_r(k - b) \right) e^{-ik.a}\, dk \\ &= \int_{\mathbf{R}^2} h_r(k) e^{-ik.a}\, dk \\ &= \frac{2\pi r}{|a|} J_1(r\,|a|) \end{aligned}$$

where J_1 is the Bessel function of order 1. A similar calculation yields

$$\int_{\mathfrak{F}} E_B(r, k)\, dk = 0.$$

It follows that for all $a \in A \setminus \{0\}$

$$\begin{aligned} \int_{\mathfrak{F}} |E_B(r, k)|\, dk &\geq \max \left(\left| \int_{\mathfrak{F}} E_B(r, k) e^{-ik\cdot a}\, dk \right|, \left| \int_{\mathfrak{F}} E_B(r, k) e^{-2ik\cdot a}\, dk \right| \right) \\ &\geq 2\pi r \max \left(\frac{|J_1(r\,|a|)|}{|a|}, \frac{|J_1(2r\,|a|)|}{2\,|a|} \right) \\ &\geq (8\pi r)^{1/2} \max \left(\frac{|\sin(r\,|a| - \pi/4)|}{|a|^{3/2}}, \frac{|\sin(2r\,|a| - \pi/4)|}{|2a|^{3/2}} \right) \\ &\quad - C r^{-1/2} |a|^{-5/2} \\ &\geq c r^{1/2} |a|^{-3/2} - C r^{-1/2} |a|^{-5/2}. \end{aligned}$$

The asymptotic estimate

$$J_1(x)=\left(\frac{2}{\pi x}\right)^{1/2}\sin\left(x-\frac{\pi}{4}\right)+O(x^{-3/2})$$

was used in the third line and

$$\inf_{t\in\mathbf{R}}\max\left(\left|\sin\left(t-\frac{\pi}{4}\right)\right|,\frac{|\sin(2t-\pi/4)|}{\sqrt{8}}\right)>0$$

in the fourth. Consequently, for any fixed $a^*\in A\setminus\{0\}$ there is an r^* such that

$$\int_{\mathfrak{F}}|E_B(r,k)|\,dk\geq cr^{1/2}$$

for all $r\geq r^*$. Moreover, for all $0<\delta<1$ there is a $c(\delta)$ for which

$$\int_{\mathfrak{F}}|E_B(r,k)|\,dk\geq c(\delta)r^{1/2}$$

when $\delta<r<r^*$. This is achieved by picking an $a\in A$ with $|a|$ sufficiently large. Therefore,

$$\int_{\mathfrak{F}}E_B^+(r,k)\,dk=\int_{\mathfrak{F}}E_B^-(r,k)\,dk>cr^{1/2}.$$

Here, $E_B^+=\max(E_B,0)$ and $E_B^-=\min(-E_B,0)$.

The statement of Lemma 1 follows at once. □

Let $[x_n,y_n]$, $n\geq 1$, be the nth band corresponding to $q=0$.

LEMMA 2. *For*[1] $n\geq 1$

$$\sup_{k\in\mathbf{R}^2}N_B(\sqrt{x_n},k)\leq n\leq\inf_{k\in\mathbf{R}^2}N_B\left(\sqrt{y_n}+\frac{1}{\sqrt{y_n}},k\right)$$

Proof. Suppose $\sup_k N_B(\sqrt{x_n},k)\geq n+1$. Then there is a k such that $\lambda_{n+1}(k,0)<x_n$. This contradicts the definition of $\lambda_{n+1}(k,0)$. Similarly, suppose $\inf_k N_B(\sqrt{y_n}+1/\sqrt{y_n},k)\leq n-1$. Then there is a k such that $\lambda_n(k,0)>y_n$. But this contradicts the definition of $\lambda_n(k,0)$. □

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[1]Observe that $0\leq x_n<y_n$, $n\geq 1$.

We can now prove the theorem.

Combining Lemmas 1, 2 with the obvious estimate $(\sqrt{y_n}+1/\sqrt{y_n})^{1/2}>y_n^{1/4}$, we have

$$\frac{\pi}{|\mathfrak{F}|}x_n+cx_n^{1/4}\leqslant n\leqslant\frac{\pi}{|\mathfrak{F}|}\left(y_n+2+\frac{1}{y_n}\right)-c\left(\sqrt{y_n}+\frac{1}{\sqrt{y_n}}\right)^{1/2}$$

$$\leq\frac{\pi}{|\mathfrak{F}|}y_n-c'y_n^{1/4}.$$

By similar arguments it is easy to show that

$$n\leqslant\sup_k N_B\left(\sqrt{x_n}+\frac{1}{\sqrt{x_n}},k\right)\leqslant cx_n$$

and

$$n\leqslant\sup_k N_B(\sqrt{y_n},k)\geqslant cy_n.$$

Consequently,

$$x_n\leqslant\frac{|\mathfrak{F}|}{\pi}n-cn^{1/4},\qquad \frac{|\mathfrak{F}|}{\pi}n+cn^{1/4}\leqslant y_n.$$

This proves the left hand inclusion of the theorem for $B_n(0)$. The statement for general q follows from the min-max characterization of $\lambda_n(k,q)$:

$$\lambda_n(k,q)=\sup_{\Phi}\inf_{\substack{\psi\in\Phi^\perp\\ \|\psi\|=1}}\langle\psi,(-\Delta+q)\psi\rangle$$

where $\Phi=\{\Phi_1,\ldots,\Phi_{n-1}\}\subset L^2(\mathbf{R}^2)$, $\Phi^\perp$ is the orthogonal complement of the span of Φ and $\psi(x+a_j)=e^{ik\cdot a_j}\psi(x)$, $j=1,2$. It gives

$$|\lambda_n(k,q)-\lambda_n(k,0)|\leqslant\|q\|_\infty$$

which yields the results.

The right hand inclusion is verified in almost the same way. The only significant difference is that Lemma 1 is replaced by the well known estimate

$$\left|N_B(r,k)-\frac{\pi}{|\mathfrak{F}|}r^2\right|\leqslant cr^{2/3}$$

one derives from the Poisson summation formula (c is independent of k). The proof of the theorem is finished.

Remark. The overlap of $B_n(0)$ and $B_{n+1}(0)$ is uniformly bounded away from 0; i.e., $|B_n(0)\cap B_{n+1}(0)|\geqslant c>0$ for all $n\geqslant 1$. For

$$\begin{aligned} n &\geqslant \sup_k N_B\left(\sqrt{x_{n+1}}-\frac{\varepsilon}{\sqrt{x_{n+1}}}, k\right) \\ &\geqslant \frac{\pi}{|\mathfrak{F}|}\left(x_{n+1}-2\varepsilon+\frac{\varepsilon^2}{x_{n+1}}\right)+c\left(\sqrt{x_{n+1}}-\frac{\varepsilon}{\sqrt{x_{n+1}}}\right)^{1/2} \end{aligned}$$

and

$$\begin{aligned} n &\leqslant \inf_k N\left(\sqrt{y_n}+\frac{\varepsilon}{\sqrt{y_n}}, k\right) \\ &\leqslant \frac{\pi}{|\mathfrak{F}|}\left(y_n+2\varepsilon+\frac{\varepsilon^2}{y_n}\right)-c\left(\sqrt{y_n}+\frac{\varepsilon}{\sqrt{y_n}}\right)^{1/2}. \end{aligned}$$

Combining the inequalities and letting ε tend to 0 we find

$$x_{n+1}\leqslant y_n-c(y_n^{1/4}+x_{n+1}^{1/4}).$$

It is an immediate consequence that the spectrum of $-\Delta+q$ has no gaps at all when $|q|_\infty$ is sufficiently small. this reproves the result of [4].

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