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Embedding theorems and Kahlerity for 1-convex spaces

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Unless the contrary is explicitly stated, all ℂ-analytic spaces considered here are assumed to be non compact, countable at infinity and of bounded Zariski dimension.

Within the past three decades, the algebraicity of certain Moishezon spaces (see Definition 1) has been characterized by Kahlerian structures; namely one has:

THEOREM A [2]. *Any 2-dimensional Moishezon manifold is projective algebraic. (i.e. it can be embedded biholomorphically into some* $\mathbb{P}_M$).

THEOREM B [4][8a]. *There exist non-Kahlerian 3-dimensional Moishezon manifolds.*

THEOREM C [8b]. *A Moishezon manifold is projective algebraic if and only if it is Kahlerian.*

THEOREM D [8d]. *There exist 2-dimensional Kahlerian Moishezon spaces which are not projective algebraic.* (see Definition 4 below for the notion of Kahlerian ℂ-analytic spaces).

Those beautiful and profound results provide us sharp and accurate characterization of the global algebraic structure of Moishezon spaces, dimensionwise and singularitywise. On the other hand, recent investigation [1] [5] [10a, b] indicates that there exists a parallelism between the global analytic structure of Moishezon spaces (resp. projective algebraic spaces) and 1-convex spaces (resp. embeddable 1-convex spaces). (see Definition 3).

This paper belongs to a program in which we attempt to establish a non

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compact version of the previous 4 results within the framework of 1-convex spaces.

In section 1, a strong analogue to Theorem A is proved for 2-dimensional 1-convex spaces. In section 2, we shall construct a 3-dimensional Kahlerian 1-convex space which cannot be embedded in any $\mathbb{C}^N \times \mathbb{P}_M$.

1. Embedding Theorems

DEFINITION 1. Let S be a compact irreducible $\mathbb{C}$-analytic space and let $\mathcal{M}(S)$ be the field of global meromorphic functions on S. Then S is said to be Moishezon if

Transcendental degree of $\mathcal{M}(S) = \dim_{\mathbb{C}} S$

DEFINITION. 2 [3]. Let S be a compact analytic subvariety in a $\mathbb{C}$-analytic space X. Then S is said to be an *exceptional set* if

(i) $\dim S_x > 0$ for any $x \in S$

(ii) there exist a $\mathbb{C}$-analytic space Y, some finite set $T \subset Y$ and a proper, surjective and holomorphic map $\pi : X \to Y$ inducing a biholomorphism $X \backslash S \simeq Y \backslash T$

(iii) $\pi_* O_X \simeq O_Y$

DEFINITION 3 [3] [10a]. Let X be a $\mathbb{C}$-analytic space with its exceptional set S. Then X is said to be 1-*convex* if Y is Stein.

Furthermore, a 1-convex space X is called "*embeddable*" if X can be realized as closed analytic subvariety of some $\mathbb{C}^N \times \mathbb{P}_M$.

We are now in a position to establish the main result of this section.

THEOREM 1. *Any 2-dimensional 1-convex space X, with purely 1-dimensional exceptional set S, is embeddable.*

Proof. Let I be the ideal sheaf determined by S. One has $I \subset O_X$ and $O_S \simeq O_X/I$. Now let us consider the following commutative diagram where the rows are exact:

$$\begin{array}{ccccccccc}
\longrightarrow & H^1(X, O_X) & \xrightarrow{\tilde{e}} & H^1(X, O_X^*) & \xrightarrow{\alpha} & H^2(X, Z) & \longrightarrow & 0 \\
 & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle r_2} & & \\
\longrightarrow & H^1(S, O_S) & \xrightarrow{e} & H^1(S, O_S^*) & \xrightarrow{c_1} & H^2(S, Z) & \longrightarrow & \\
 & \downarrow & & & & \downarrow{\scriptstyle\gamma} & & \\
 & 0 & & & & H^3(X, S; Z) & &
\end{array}$$

Notice that r_1 and r_2 are restriction maps. Since by hypothesis X has no compact 2-dimensional irreducible components and since $\dim_{\mathbb{C}} X = 2$, $H^2(X, I) = H^2(X, O_X) \simeq 0$; (see e.g. [9]); consequently α and β are surjective. Similarly one can check that the singular relative cohomology group $H^3(X, S; \mathbb{Z})$ is purely torsion (see e.g. [7]).

Let $\xi \in H^1(S, O_S^*)$. Since $c_1(\xi^N) = N \cdot c_1(\xi)$ and since $\operatorname{Im} \gamma$ is purely torsion, hence, for $N \gg 0$, there exists an element $a \in H^2(X, \mathbb{Z})$ such that $r_2(a) = c_1(\xi^N)$.

In view of the surjectivity of α, there exists an $E \in H^1(X, O_X^*)$ such that $\alpha(E) = a$. Hence $c_1(r_1(\bar{E}^1) \otimes \xi^N) = 0$. Consequently, there exists an element $b \in H^1(S, O_S)$ such that $e(b) = r_1(\bar{E}^1) \otimes \xi^N$.

In view of the surjectivity of α, there exists an element $c \in H^1(X, O_X)$ such that $\beta(c) = b$. Now let $F := \bar{e}(c)$ and let $L := E \otimes F$; one can check easily that $L \mid S \simeq \xi^N$.

Since S is projective algebraic, ξ can be chosen to be an ample line bundle on S. The previous argument tells us that there exists a line bundle L on X such that $L \mid S \simeq \xi^N$ is ample. The main result in [10a] implies that the global sections of L^k, for $k \gg 0$, embed X biholomorphically into some $\mathbb{C}^N \times \mathbb{P}_M$. *Q.E.D.*

Remark. A weaker version than Theorem 1 appeared in [10b]. Another proof of Theorem 1 has been given by C. Banica is "Sur les fibres infinitésimales d'un morphisme propre d'espaces complexes" Seminaire F. Norguet, Fonctions de plusieurs variables complexes IV Springer-Verlag Lec. notes in Math. #807 (1980). However his proof contains a gap.

At this point we would like to mention a special case to Theorem 1 and to provide an alternate proof for it.

COROLLARY 2. *Let X be a connected 2-dimensional 1-convex manifold. Let us assume that its exceptional set S is non singular. Then X is embeddable.*

Proof. Let I be the invertible ideal sheaf in O_X determined by S and let L be the line bundle associated to I. Following [3] $L \mid S \simeq N^*_{S/X}$ is ample where $N_{S/X}$ is the normal bundle of S in X. Hence the main result in [10a] tells us that X is embeddable. *Q.E.D.*

The crucial fact we would like to point out here is the following: The non singular hypothesis of both X and S really imposes a constraint on the analytic structure of both X and S as the following result will show us (see [10c] for complete proof).

THEOREM 3. *Let X be a 1-convex manifold and let S be its exceptional set.*

(i) If S is non singular, then S is projective algebraic.

(ii) If S is non singular and codim $S=1$, then X is embeddable.

Remarks. (a) Certainly Theorem 3 is false if either X or S acquires some singularities. [1] [10a].

(b) Theorem 3(i) provides a negative answer to the following question posed by Remmert.

Question. Can any Moishezon space S be realized as exceptional set of some $\mathbb{C}$-analytic *manifold* M?

Meanwhile Theorem 3(ii) partially answers Problem 1 in [10a].

(c) Although Theorem 3 asserts that, for a given 1-convex manifold X, its non singular exceptional set S is projective algebraic, one should be aware that the normal bundle $N_{S/X}$ of S in X, in general, is not weakly negative in the sense of Grauert [3] (see [3] [6]); this is in sharp contrast with the case where dim $X=2$, as shown in the proof of Corollary 2. We refer the reader to [10c] for further detailed discussions on this topic.

2. Kahlerian 1-convex spaces

DEFINITION 4 [8d] (see also [3]). Let X be a $\mathbb{C}$-analytic space. We say that there is a Kahler metric on X if there exist an open covering $\{U_i\}$ of X and strongly plurisubharmonic functions $\phi_i \in C^{\infty}_{\mathbb{R}}(U_i)$ such that $\phi_i - \phi_j$ is pluriharmonic (i.e. locally the real part of some holomorphic function) on $U_i \cap U_j$. Now X is called Kahlerian if there exists some Kahler metric on it.

In the special case where X is non singular, one can check that Definition 4 coincides with the standard notion of Kahler manifold (see [3] [8d]).

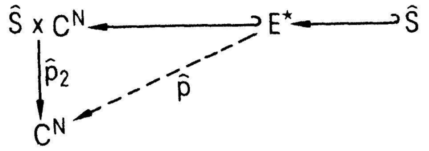
Certainly, any subspace of a Kahlerian space is also Kahlerian; in particular any embeddable 1-convex space is Kahlerian. However the converse is not true in general as the result below will convince us. Following closely an idea in [5] we have.

THEOREM 4. *There exist Kahlerian 1-convex spaces which are not embeddable.*

Proof. Let S be a Kahlerian Moishezon space which is not projective algebraic (see [8d]). Certainly dim $S \geq 2$. Following [8c] there exist a projective algebraic manifold $\hat{S} \subset \mathbb{P}_M$ and a modification morphism $\pi: \hat{S} \rightarrow S$. Let $\mathbb{H}$ be the hyperplane bundle on $\mathbb{P}_M$ and let $E := \mathbb{H} \mid \hat{S}$. Since $\mathbb{H}$ (and hence E) is ample, one has the following global resolution on $\hat{S}$

$$O_{\hat{S}}^N \to E \to 0 \tag{†}$$

where $N := \dim \Gamma(\hat{S}, E)$. Let $L(\mathbf{E})$ be the linear fibre space associated to the locally free sheaf $\mathbf{E}$ in the sense of [3], then, in view of (†), one has the following embedding $L(\mathbf{E}) \hookrightarrow \hat{S} \times \mathbb{C}^N$. Notice that $E^* \simeq L(\mathbf{E})$, so one has the following diagram

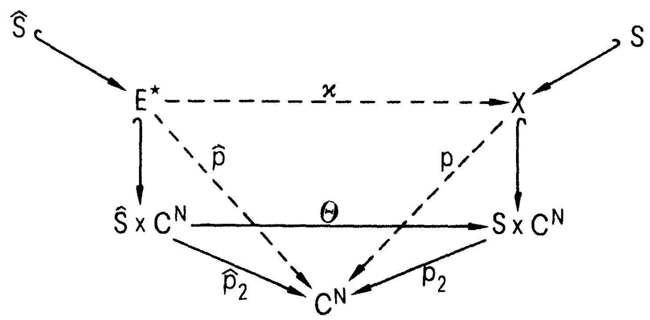


where we identify $\hat{S}$ with the zero section of E^* and $\hat{p}_2$ is the natural projection. Now one can check that

(i) the proper morphism $\hat{p} := \hat{p}_2 \mid E^*$ maps $E^* \backslash \hat{S}$ injectively into $\mathbb{C}^N \backslash \{0\}$.

(ii) $\hat{p}(\hat{S}) = 0 \in \mathbb{C}^N$.

In particular E^* is a 1-convex manifold admitting $\hat{S}$ as its maximal compact subvariety in the sense of [3]. Let us consider the following diagram



where $\Theta := \pi \times \mathrm{id}_{\mathbb{C}^N}$ and p_2 is the natural projection. Since Θ is proper, $X := \Theta(E^*)$ is a closed analytic subvariety in $S \times \mathbb{C}^N$. Now let $\chi := \Theta \mid E^*$ and $p := p_2 \mid X$. One can check that

$$p(S) = 0 \tag{§}$$

and

$$\hat{p} = p \circ \chi \tag{††}$$

Claim. S is a maximal compact subvariety of X.

In fact for any positive dimensional compact irreducible subvariety $T \subset X \hookrightarrow S \times \mathbb{C}^N$, one has either $T \cap S = \varnothing$ or $T \subset S$.

If $T \cap S = \varnothing$, then (§) tells us that the point $\alpha := p(T) \neq 0 \in \mathbb{C}^N$. In view of (††), $\hat{T} := \hat{p}^{-1}(\alpha) = \chi^{-1}(T)$ is a compact analytic subvariety in E^* such that $\dim \hat{T} > 0$ and $\hat{T} \cap \hat{S} = \varnothing$; this contradicts the fact that $\hat{S}$ is maximal. Hence $T \subset S$ i.e. S is maximal and the claim is proved.

It is clear that X is holomorphically convex. A result in [3] tells us that X is 1-convex admitting S as its exceptional set. Since by hypothesis S is Kahlerian (resp. non projective algebraic), hence X is Kahlerian (resp. non embeddable). *Q.E.D.*

In conclusion, in view of our Theorems 1 and 2 above and the main result in [10a] (namely the construction of a 3-dimensional non Kahlerian 1-convex manifold), in order to achieve our program, the following problem remains to be settled:

PROBLEM. Let X be a 1-convex manifold with $\dim X \geq 3$. Is the Kahlerity of X sufficient (necessary of course!) for X to be embeddable?

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