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## Hölder continuity of conformal mappings and non-quasiconformal Jordan curves

JOCHEN BECKER and CHRISTIAN POMMERENKE

Let  $\Gamma \subset \mathbf{C}$  be a quasiconformal curve (quasicircle), that is the image of the unit circle under a quasiconformal mapping of the plane. Let  $f$  and  $f^*$  denote conformal mappings of the unit disk  $\mathbf{D}$  onto  $\text{int } \Gamma$  and  $\text{ext } \Gamma$  respectively.

It is well-known that  $f$  and  $f^*$  have quasiconformal extensions onto the plane [3], p. 98. Because of the Hölder continuity of quasiconformal mappings ([3], p. 71),  $f$  and  $f^*$  as well as their inverse mappings  $f^{-1}$  and  $f^{*-1}$  satisfy Hölder conditions:

$$\begin{aligned} |f(z_1) - f(z_2)| &\leq K |z_2 - z_1|^\alpha & (z_1, z_2 \in \mathbf{D}), \\ |f^{-1}(w_1) - f^{-1}(w_2)| &\leq L |w_1 - w_2|^\beta & (w_1, w_2 \in \text{int } \Gamma) \end{aligned} \tag{1}$$

and similarly for  $f^*$  where one has to use the spherical metric in  $\text{ext } \Gamma$ . The Hölder exponents  $\alpha$  ( $0 < \alpha \leq 1$ ) and  $\beta$  ( $1 \leq \beta < 2$ ) depend only on  $\Gamma$  [4], [5], [8], p. 287, 289, 347.

By means of a simple geometrical characterization of (1) (Theorem 1), we construct a non-quasiconformal Jordan curve (Theorem 2) such that nevertheless  $f$ ,  $f^*$ , and also  $f^{-1}$ ,  $f^{*-1}$  remain Hölder continuous. This shows that Hölder continuity of all these conformal mappings is only a necessary, but not a sufficient condition for quasicircles.

Let  $G \subsetneq \mathbf{C}$  be a simply connected domain and let  $h(w_1, w_2)$  denote the hyperbolic distance of the points  $w_1, w_2 \in G$  defined by

$$h(w_1, w_2) = \log \frac{|1 - \bar{z}_1 z_2| + |z_2 - z_1|}{|1 - \bar{z}_1 z_2| - |z_2 - z_1|} \quad (w_i = f(z_i), i = 1, 2) \tag{2}$$

where  $f$  is a conformal mapping of  $\mathbf{D}$  onto  $G$ . Let  $\delta(w) = \text{dist}(w, \partial G)$  denote the (Euclidean) boundary distance.

**THEOREM 1.** *If  $G$  is a bounded simply connected domain,  $w_0 \in G$ , and  $f$  a conformal mapping of  $\mathbf{D}$  onto  $G$ , then (1) is satisfied if and only if*

$$\limsup_{\delta(w) \rightarrow 0} [h(w_0, w) + \frac{1}{\alpha} \log \delta(w)] < +\infty. \tag{3}$$

*Proof.* It is well-known that (1) is equivalent to

$$|f'(z)| \leq M(1 - |z|^2)^{\alpha-1} \quad (|z| < 1) \quad (4)$$

(see for instance [2], p. 361–363). Introducing the non-Euclidean length element

$$\rho(w) |dw| = \frac{2 |dz|}{1 - |z|^2} \quad (w = f(z))$$

corresponding to (2) we see that (4) means

$$h(f(0), f(z)) \leq \frac{1}{\alpha} \log \left[ \frac{M}{2} (1 + |z|)^{2\alpha} \rho(f(z)) \right].$$

Since  $G$  is bounded we have  $\delta(f(z)) \rightarrow 0$  if and only if  $|z| \rightarrow 1$ . Thus we obtain as an equivalent condition

$$\limsup_{\delta(w) \rightarrow 0} \left[ h(f(0), w) - \frac{1}{\alpha} \log \rho(w) \right] < +\infty. \quad (5)$$

Because of

$$\frac{1}{2} \leq \rho(w) \delta(w) \leq 2$$

[8], p. 22 it follows by the triangle inequality for  $h(w_1, w_2)$  that (5) is the same as (3) which proves Theorem 1.

The following corollary gives a convenient sufficient condition for Hölder continuity.

**COROLLARY.** *Let  $f$  be a bounded univalent function defined in  $\mathbf{D}$ ,  $G = f(\mathbf{D})$ , then  $f$  satisfies (1) if there are positive numbers  $M, \delta_0, \delta_1$  with the property that, for every  $w \in G$  with  $\delta(w) < \delta_0$ , there exists  $w_1 \in G$  with  $\delta(w_1) \geq \delta_1$  and a connecting arc  $C \subset G$ , such that*

$$\int_C \frac{|d\omega|}{\delta(\omega)} \leq M + \frac{1}{2\alpha} \log \frac{1}{\delta(w)}. \quad (6)$$

*Proof.* We choose  $w_0 = f(0)$ . Then we have

$$\begin{aligned} h(w_0, w) &\leq h(w_0, w_1) + \int_C \rho(\omega) |d\omega| \\ &\leq \log \frac{1 + |f^{-1}(w_1)|}{1 - |f^{-1}(w_1)|} + 2 \int_C \frac{|d\omega|}{\delta(\omega)}. \end{aligned}$$

From (6) and  $\delta(w_1) \geq \delta_1 > 0$  it follows that (3) is satisfied which implies (1) by Theorem 1.

We are now ready to construct a Jordan curve with the desired properties.

**THEOREM 2.** *There is a non-quasiconformal Jordan curve  $\Gamma$  such that the conformal mappings  $f$  and  $f^*$  of  $\mathbf{D}$  onto  $\text{int } \Gamma$  and  $\text{ext } \Gamma$  respectively, and also their inverse mappings are Hölder continuous.*

*Proof.* We set  $Q = \{w = u + iv : |u| < 1, |v| < 1\}$  and  $R_n = \{w = u + iv : 0 \leq u - 1 < a_n, |v - v_n| < \epsilon_n\}$  where we choose  $v_n = 1/n, \epsilon_n = 2^{-n}, a_n = -\epsilon_n \log \epsilon_n, n = 2, 3, \dots$ . We consider the domain  $G = Q \cup \bigcup_{n=2}^{\infty} R_n$ . Since  $\Gamma = \partial G$  is a locally connected continuum without cut points it is a Jordan curve [8], p. 281. But it is not a quasicircle, because the Ahlfors criterion [1] is not satisfied. For we have

$$\frac{\text{diam } R_n}{2\epsilon_n} \geq \frac{a_n}{2\epsilon_n} \rightarrow +\infty (n \rightarrow \infty).$$

On the other hand, we shall show below that, for every  $w \in G, \delta(w) < \frac{1}{2}$ , there exists  $w_1 \in G, \delta(w_1) \geq \frac{1}{2}$ , and a connecting arc  $C$  such that

$$\int_C \frac{|d\omega|}{\delta(\omega)} \leq 2 \log \frac{1}{\delta(w)}. \tag{7}$$

Thus (6) is satisfied with  $\alpha = \frac{1}{4}$  which shows that  $f$  is  $\alpha$ -Hölder-continuous for some  $\alpha \geq \frac{1}{4}$ .

The Hölder continuity of  $f^{-1}, f^*, f^{*-1}$  follows easily from results by Näkki and Palka [6], [7]:

Let  $d(w_1, w_2) = \inf_C \int_C |dw|$  denote the inner distance of  $w_1, w_2 \in G$  where the infimum is taken over all connecting arcs  $C \subset G$ . Then, obviously, there exists a constant  $A$  such that

$$d(w_1, w_2) \leq A |w_2 - w_1| (w_1, w_2 \in G).$$

It follows from [6] that  $f^*$  and  $f^{-1}$  are Hölder continuous.

Considering the inner distance  $d^*(w_1, w_2)$  with regard to  $G^* = \text{ext } \Gamma$  one easily sees that, for every  $\beta$  ( $0 < \beta < 1$ ), and  $R > 0$ , there exists a constant  $B$  such that

$$d^*(w_1, w_2) \leq B |w_2 - w_1|^\beta \quad (w_i \in G^*, |w_i| \leq R, i = 1, 2)$$

This implies, by another result of Näkki and Palka [6], [7], that  $f^{*-1}$  is also Hölder continuous.

Hence it remains to prove (7). We have to consider the following cases.

(a)  $w = u + iv \in Q$ , and, if  $u \geq v \geq 0$ ,  $|v - v_n| \geq \epsilon_n$ ,  $n = 2, 3, \dots$ . Hence  $\delta(w) = \min(1 - |u|, 1 - |v|)$ . Choose  $w_1 = 0$  and  $C = [w_1, w_2]$  (segment with end points  $w_1, w_2$ ). Then

$$\int_C \frac{|d\omega|}{\delta(\omega)} \leq \int_{\delta(w)}^1 \frac{\sqrt{2} dt}{t} = \sqrt{2} \log \frac{1}{\delta(w)}.$$

(b)  $w = u + iv \in Q$ ,  $u \geq v \geq 0$ ,  $|v - v_n| < \epsilon_n$  for some  $n$  (hence  $\delta(w) > 1 - u$ ). Choose, for  $\delta(w) < \frac{1}{2}$ ,  $w_1 = \frac{1}{2} + iv$ ,  $C = [w_1, w]$ . Because of

$$\delta(w) = \sqrt{((1 - x)^2 + (\epsilon_n - |v - v_n|)^2)}, \quad \omega = x + iv \in C,$$

we obtain

$$\begin{aligned} \int_C \frac{|d\omega|}{\delta(\omega)} &= \log \frac{\frac{1}{2} + \sqrt{\frac{1}{4} + (\epsilon_n - |v - v_n|)^2}}{1 - u + \sqrt{((1 - u)^2 + (\epsilon_n - |v - v_n|)^2)}} \\ &\leq \log \frac{1 + \sqrt{(1 + 4\epsilon_n^2)}}{2\delta(w)} < 2 \log \frac{1}{\delta(w)}. \end{aligned}$$

(c)  $w = u + iv \in R_n$  for some  $n$ ,  $0 \leq u - 1 \leq a_n - \epsilon_n$  hence  $\delta(w) = \epsilon_n - |v - v_n|$ . Choose  $w_1 = \frac{1}{2} + iv_n$  and  $C = [w_1, u + iv_n] \cup [u + iv_n, w]$ . Then

$$\begin{aligned} \int_C \frac{|d\omega|}{\delta(\omega)} &= \int_{1/2}^1 \frac{dx}{\sqrt{((1 - x)^2 + \epsilon_n^2)}} + \int_1^u \frac{dx}{\epsilon_n} + \int_0^{|v - v_n|} \frac{dy}{\epsilon_n - y} \\ &< \frac{a_n}{\epsilon_n} - \log(\epsilon_n - |v - v_n|) \leq 2 \log \frac{1}{\delta(w)}. \end{aligned}$$

(d)  $w = u + iv \in R_n$  for some  $n$ ,  $a_n - \epsilon_n < u - 1$  hence  $\delta(w) = \min(\epsilon_n - |v - v_n|, 1 + a_n - u)$ . Choose  $w_1 = \frac{1}{2} + iv_n$ ,  $C = [w_1, 1 + a_n - \epsilon_n + iv_n] \cup [1 + a_n - \epsilon_n + iv_n, w]$ .

Then

$$\int_C \frac{|d\omega|}{\delta(\omega)} \leq \int_{1/2}^1 \frac{dx}{\sqrt{(1-x)^2 + \varepsilon_n^2}} + \int_0^{a_n - \varepsilon_n} \frac{dx}{\varepsilon_n} + \int_{\delta(w)}^{\varepsilon_n} \frac{\sqrt{2} dt}{t}$$

$$< -\log \varepsilon_n + \frac{a_n}{\varepsilon_n} + \sqrt{2} \log \frac{\varepsilon_n}{\delta(w)} \leq 2 \log \frac{1}{\delta(w)}.$$

Thus we have proved that (7) holds in each case which completes the proof of Theorem 2.

*Remark.* The Hölder continuity of  $f^*$  could have also been shown by this method.

#### REFERENCES

- [1] L. V. AHLFORS, *Quasiconformal reflections*, Acta Math. 109 (1963), 291–301.
- [2] G. M. GOLUZIN, *Geometric Theory of Functions of a Complex Variable*, (Moscow 1952). German transl.: Deutscher Verlag der Wissenschaften, Berlin 1957.
- [3] O. LEHTO and K. I. VIRTANEN, *Quasiconformal Mappings in the Plane*, Springer, Berlin–Heidelberg–New York 1973.
- [4] F. D. LESLEY, *Hölder continuity of conformal mappings at the boundary via the strip method*, to appear.
- [5] R. NÄKKI and B. PALKA, *Quasiconformal circles and Lipschitz classes*, Comment. Math. Helvetici 55 (1980), 485–498.
- [6] R. NÄKKI and B. PALKA, *Lipschitz conditions, b-arcwise connectedness and conformal mappings*, to appear.
- [7] R. NÄKKI and B. PALKA, *Lipschitz classes and quasiconformal mappings*, Indiana J. Math., to appear.
- [8] CH. POMMERENKE, *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen 1975.

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