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Compactification of the space of vector bundles on a singular curve

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Let X be a singular, integral, projective curve of genus greater than one over an algebraically closed field k. It has been verified by Narasimhan and Newstead [N] that the method of [S] extends to construct a projective moduli space for semi-stable torsion free $\mathcal{O}_X$ modules of rank n and degree d which we denote by $\bar{M}(n, d)$. The points of $\bar{M}(n, d)$ corresponding to vector bundles form an open, irreducible subset. The object of this article is to prove the

THEOREM. *If X is embeddable in a smooth surface Z then $\bar{M}(n, d)$ is irreducible.*

To prove irreducibility it suffices (and is equivalent) to verify

(0.1) Given a torsion free $\mathcal{O}_X$-module N there is an $\mathcal{O}_{X\times \operatorname{Spec} k[\![t]\!]}$-module $\mathcal{L}$ with $\mathcal{L}/t\cdot\mathcal{L} \approx N$ and $\mathcal{L}\otimes k((t))$ a vector bundle on $X\times \operatorname{Spec} k((t))$.

If the singularities of X are not all planar then we have verified in [R] that there are rank one modules not deformable to a line bundle, hence $\bar{M}$ cannot be irreducible in that case. The case $n=1$ was first established in [A] using Iarrobino's calculation of the dimension of the Punctual Hilbert Scheme of ideals in $k[\![x, y]\!]$ of colength m,

(0.2) $\dim \operatorname{Hilb}_0^m (k[\![x, y]\!]) \leq m-1.$

In [R] we gave a self contained proof of the irreducibility of $\bar{M}(1, d)$ by induction on the multiplicity of the singular points and derived (0.2) as a consequence. The case of rank greater than one does not follow "module theoretically" from the rank one result except for very simple plane singularities for which modules split locally into a direct sum of rank one modules. By [B] this happens only when the multiplicity of each singular point is less than or equal to two.

An important ingredient in the arguments of [A] and [R] was the fact that every component of $\operatorname{Hilb}^m (X)$ is of dimension greater than or equal to m. This follows from the observation that $\operatorname{Hilb}^m (X)$ is (locally) the zero set of a section of a rank m vector bundle on the $2m$ dimensional space $\mathrm{Hilb}^m(Z)$. For $n>1$ we work with $\mathrm{Quot}^m(n, Z)$, the space of quotients of length m of a fixed free sheaf on Z of rank n. However $\mathrm{Quot}^m(n, Z)$ is singular and $\mathrm{Quot}^m(n, X) \hookrightarrow \mathrm{Quot}^m(n, Z)$ does not have a simple description as a subscheme. There is thus no way of extending the ideas of [A] to the case of $n>1$. In [R] we use the fact that $\mathrm{Hilb}^m(Z)$ is at least irreducible for Z a smooth connected surface. Again, we have no à priori proof that $\mathrm{Quot}^m(n, Z)$ is irreducible for $n>1$ and this result is deduced below as a corollary of the main theorem.

When $n=1$ the irreducibility of $\mathrm{Hilb}^m(Z)=\mathrm{Quot}^m(1, Z)$ follows from the Hilbert–Schaps' lemma "codim 2+cohen-macaulay$\Rightarrow$smoothable," where the matrices defining the presentation of the codimension 2 ideal are deformed. As the quotients in $\mathrm{Quot}^m(n, Z)$ are also defined by two term complexes it would be interesting to obtain a proof of the irreducibility of $\mathrm{Quot}^m(n, Z)$ along these lines. The main difficulty here is that for $n>1$ the matrices cannot be deformed "arbitrarily" as $\mathrm{Quot}^m(n, Z)$ is singular.

We are unable to prove that $\bar{M}(n, d)$ is reduced for $n>1$. It would suffice to know that $\mathrm{Quot}^m(n, X)$ is reduced. In the case when X has only ordinary double points Seshadri has recently proved that $\bar{M}$ is reduced. He writes down the completion of the local rings of $\bar{M}$ in determinantal form so that they can be described by available techniques. The general case is completely open.

No use is made here of the analogue of the scheme E introduced in [R] and we are able to avoid the somewhat precise (see (3.1.2.) to (3.1.7.) of [R]) dimension calculations used there. The analogue of (0.2) follows from the main theorem, as in the case of rank one, but as we have no applications details are omitted.

§1. Initial definitions and propositions

Let Y be a scheme over k and $V=\mathcal{O}_Y^n$. The functor of $\mathcal{O}_Y \otimes \mathcal{O}_T$ submodules of $V \otimes \mathcal{O}_T$, N_T, satisfying "$V \otimes \mathcal{O}_T / N_T$ is a locally free $\mathcal{O}_T$ module of rank m" is represented by a projective scheme denoted by $\mathrm{Quot}^m(n, Y)$. In the sequel Y will usually be a smooth surface or a curve on a smooth surface. Note that if W is a subscheme of Y we have a closed immersion $\mathrm{Q}^m(n, W) \hookrightarrow \mathrm{Q}^m(n, Y)$ where $N \in \mathrm{Q}^m(n, W)$ iff $\mathcal{I}_W \cdot V \subset N$, where $\mathcal{I}_W$ is the defining ideal of W.

PROPOSITION 1.1. *Let Z be a smooth surface. Then $\mathrm{Q}^m(n, Z)$ is singular for $n>1$, $m>1$.*

Proof. The tangent space at a point corresponding to $N \subset V$ is canonically identified with Hom $(N, V/N)$. Suppose V/N is supported at m distinct points of Z. We claim V/N defines a smooth point of Quot. To see this first compute the tangent space. Since it is a local question is suffices to fix a local ring $\mathcal{O}$ of Z with maximal ideal $\mathfrak{M}$ and suppose $N_0 = \mathfrak{M} \oplus \mathcal{O} \oplus \cdots \oplus \mathcal{O} \subset \mathcal{O}^n = V$. Then Hom $(N_0, V/N_0)$ has dimension $(n+1)$. If V/N is supported at m distinct points its tangent space has rank $m(n+1)$. We now check that Quot^m has dimension $m(n+1)$ at V/N. Again it suffices to prove that at $N_0 = \mathfrak{M} \oplus \mathcal{O} \cdots \oplus \mathcal{O} \subset V$ $\text{Quot}^1(n, Z)$ has dimension $(n+1)$. Note that N_0 defines a point of $\mathbf{P}(V/\mathfrak{M} \cdot V) \hookrightarrow \text{Quot}^1$. For each point of Z we thus obtain a $\mathbf{P}^{n-1} \subset \text{Quot}^1$ of quotients supported at that point. As $\dim Z = 2$ we find $\dim \text{Quot}^1 \geqslant (n-1)+2 = n+1$. By the tangent space computation $\dim \text{Quot}^1 = (n+1)$.

We denote by U^m the smooth open subset of Quot^m defined by quotients supported at m distinct points. To see that Quot^m is singular for $m \geqslant 2$, $n \geqslant 2$ we pick a point in the closure of U^m which has a tangent space of rank greater than $(n+1)m$. One such point is defined by the module $N \subset V$ of colength 1 at $(m-2)$ points and of the type $\mathfrak{M} \oplus \mathfrak{M} \oplus \mathcal{O} \oplus \cdots \oplus \mathcal{O} \subset V$ at one point. It is clear how to deform this quotient so that it has support at m points. If x, y are generators of $\mathfrak{M}$ just take the $k[\![t]\!]$ deformation $((x+t, y) \oplus \mathfrak{M} \cdots \oplus \mathcal{O}) \subset V \otimes k[\![t]\!]$. This shows that V/N is in the closure of U^m. However its tangent space has rank equal to $(n+1)(m-2)+2(n+2)$ which is greater than $(n+1) \cdot m$. This proves the proposition.

PROPOSITION 1.2. *Let X be a projective integral curve with singular points $P_1 \cdots P_r$ and N a torsion free $\mathcal{O}_X$-module of rank n. Then N is deformable (over* Spec $k[\![t]\!]$) *to a vector bundle on $X \times$* Spec $k((t))$ *if and only if N_{P_i} is deformable to a projective module over $\mathcal{O}_{X,P_i} \otimes k[\![t]\!]$ $\forall i$.*

Proof. One way is clear so suppose N_{P_i} is deformable to a projective module $\forall i$ and let $N_{P_i}[t]$ be the $\mathcal{O}_{P_i} \otimes k[\![t]\!]$ modules representing these deformations. Choose imbeddings $q_i \colon N_{P_i}[t] \subset \mathcal{O}^n_{P_i} \otimes k[\![t]\!]$ and observe that (coker q_i) is a finite $k[\![t]\!]$ module iff it is not supported at any height one maximal ideals. In any case there is an $N'_{P_i}[t]$ $\forall i$ with

$$N_{P_i}[t] \subset N'_{P_i}[t] \underset{q_i'}{\hookrightarrow} \mathcal{O}^n_{X,P_i} \otimes k[\![t]\!]$$

with (coker q_i) a finite free $k[\![t]\!]$ module and $N'_{P_i}[t]$ specializes to N_{P_i}. Let $U_i = X - (\bigcup_{j \neq i} P_j)$ and increasing the number of P_i's if necessary we can assume N is trivial over $U_i \cap U_j$ $\forall i, j$. Then the q_i''s define sheaves $\mathcal{N}_i$ on $U_i \times \text{Spec}\, k[\![t]\!]$ which are vector bundles outside $(P_i) \times (0)$ and trivial on $(U_i - (P_i)) \times \text{Spec}\, k[\![t]\!]$.

Now N can be defined by matrices in $Gl_n(\mathcal{O}_{U_i \cap U_j})$. Lifting these matrices to elements of $Gl_n(\mathcal{O}_{U_i \cap U_j} \otimes k[\![t]\!])$ defines an $\mathcal{O}_{X \times \operatorname{Spec} k[\![t]\!]}$ module $\mathcal{N}$ which is generically a vector bundle and specializes to N. This proves the proposition.

PROPOSITION (1.2.0). *Let X be a smooth irreducible curve; then* Quot^m (n, X) *is irreducible. In particular, for any irreducible curve, the open subset of* Quot *supported at smooth points is irreducible.*

Proof. Write $Q = \text{Quot}^m\,(n, X)$ and recall we have an exact sequence

$$(1.1.1) \quad 0 \to \mathcal{N} \hookrightarrow \mathcal{O}^n_{X \times Q} \to H \to 0$$

where $\mathcal{N}$ is a rank n vector bundle on $X \times Q$ and $p_{2*}H$ is a rank m vector bundle on Q. The determinant defines a map $d: \bigwedge^n \mathcal{N} \to \mathcal{O}_{X \times Q}$ with cokernel finite of rank m over Q. Hence we get a morphism p from Q to $\text{Hilb}^m\,(X) = \text{Quot}^m\,(1, X)$ with fibres representing quotients which are supported at the "determinantal cycle." Consider the subset $U^m(n, X) \subset \text{Quot}^m\,(n, X)$ of quotients supported at m distinct points. As in the proof of Proposition (1.1) we see that the fibres of p are m-fold products of $\mathbf{P}^{n-1}$'s. Since $\text{Hilb}^m\,(X)$ is irreducible and the image of $U^m(n, X)$ is dense open we find $U^m(n, X)$ is irreducible. It remains to prove that $U^m(n, X)$ is dense. But any $N \subset \mathcal{O}^n_X$ is locally free as X is smooth so the arguments used in the proof of Proposition (1.1) show that N can be deformed over $k[\![t]\!]$ so that the quotient is supported at m distinct points. This proves the proposition.

Remark (1.2.1). Since any finite set of points on a smooth irreducible projective variety Z can be joined by a smooth irreducible curve X, given two points on $U^m(n, Z)$ we can find an X with $U^m(n, X)$ containing them. Hence $U^m(n, Z)$ is irreducible.

PROPOSITION (1.3) [D'Souza]. *Let N be a torsion free module of rank n over a one dimensional Gorenstein ring $\mathcal{O}$. Let $A' \to A$ be a surjective map of complete local k-algebras (with residue field k). Given an embedding $N_A \hookrightarrow \mathcal{O}^n \otimes_k A$ with $\mathcal{O}^n \otimes A/N_A$ a flat A module and a flat deformation $N_{A'}$ of N over A', lifting N_A, there is an embedding $N_{A'} \hookrightarrow \mathcal{O}^n \otimes A'$ so that the diagram*

$$\begin{array}{ccc} N_{A'} & \hookrightarrow & \mathcal{O}^n \otimes A' \\ \downarrow & & \downarrow \\ N_{A'} \otimes_{A'} A \approx N_A & \hookrightarrow & \mathcal{O}^n \otimes A \end{array}$$

is commutative.

Proof. [O–S, Appendix].

(1.4) From now on fix a singular local ring $\mathcal{O}$ of an integral Gorenstein curve X and write $F=\mathcal{O}^n$, $\bar{\mathcal{O}}$ the normalization of $\mathcal{O}$, K the quotient field of $\mathcal{O}$, $\bar{F}=\bar{\mathcal{O}}^n$, $\delta=\text{length}(\bar{\mathcal{O}}/\mathcal{O})$, $C\subset\mathcal{O}$ the conductor of $\mathcal{O}$ in $\bar{\mathcal{O}}$. Let N be a torsion free $\mathcal{O}$ module of rank n. Write $\bar{N}=N\cdot\bar{\mathcal{O}}=N\otimes\bar{\mathcal{O}}/\text{Torsion}$ and as $\bar{N}$ is torsion free over a P.I.D. it is free. Choose n elements in N which generate $\bar{N}$ over $\bar{\mathcal{O}}$. These define an imbedding $F\hookrightarrow N$ so that $F\cdot\bar{\mathcal{O}}=N\cdot\bar{\mathcal{O}}=\bar{F}$. Thus every isomorphism class of $\mathcal{O}$ modules is represented by one between F and $\bar{F}$.

DEFINITION–PROPOSITION (1.5). *The functor of $\mathcal{O}$-submodules of $\bar{F}$ with colength d is denoted by $E(d)$. It is represented by a closed subset of a Grassmanian.*

(1.6) By the above $\bigcup_{d\leqslant n\delta}E(d)$ 'contains' every isomorphism class of $\mathcal{O}$ modules. We claim $E(n\delta)$ contains an open subset of free $\mathcal{O}$ modules which has dimension $\delta\cdot n^2$. Openness is immediate. Now let $F_1,F_2\in E(n\delta)$, $F_1\overset{\varphi}{\approx}F_2\approx F$. Then φ yields an element of $\text{Aut}(K^n)$ which preserves $\bar{F}$ i.e. an element of $Gl_n(\bar{\mathcal{O}})$. Thus $Gl_n(\bar{\mathcal{O}})$ acts transitively on this open subset of $E(n\delta)$ so to obtain its dimension we just calculate the isotropy at any one point, say, F. This is clearly $Gl_n(\mathcal{O})$ and the coset space $Gl_n(\bar{\mathcal{O}})/Gl_n(\mathcal{O})$ has dimension equal to $n^2\cdot\text{length}(\bar{\mathcal{O}}/\mathcal{O})=\delta n^2$. For X rational with one singular point this open subset defines all vector bundles trivial on $\tilde{X}$. However the space of stable vector bundles should be $\delta\cdot n^2-(n^2-1)$. This is accounted for by the fact that $PGl_n(k)$ operates freely at the generic module in $E(n\delta)$ and the moduli is got generically by taking a quotient.

(1.7) Take N, $F\subset N\subset\bar{F}$. Then

$$(1.7.1)\quad N^*=\left\{(n_i^*)\in K^N\;\middle|\;\sum n_i^*\cdot n_i\in\mathcal{O}\;\forall(n_i)\in N\right\}$$

is canonically identified with $\text{Hom}(N,\mathcal{O})$. Note that for N as above $N^*\subset F$ and $C\cdot F\subset N^*$. By reflexivity

(1.7.2) $\text{length}(F/N^*)=\text{length}(N/F)$.

(*Remark.* It is a standard fact that rank one torsion free modules over $\mathcal{O}$ are reflexive. For higher rank just use induction on the rank and the vanishing of $\text{Ext}^1(N,\mathcal{O})$ for N torsion free.)

PROPOSITION (1.7.2). (a). *Every module N can be represented by $C\cdot F\subset N\subset F$.*
(b) *If $N\subset F$, $C\cdot F\not\subset N$ then there is an $N'\approx N$ with $C\cdot F\subset N'\subset F$ satisfying*

(1.7.4) $\text{length}(F/N')\underset{\neq}{<}\text{length}(F/N)$

Proof. Writing $N = P^*$, $F \subset P \subset \bar{F}$, (a) is clear by reflexivity.

To prove (b) use (a) to get N' with $C \cdot F \subset N' \subset F$ and extend the isomorphism $\varphi : N' \approx N$ to an isomorphism $N' \otimes K \approx N \otimes K = K^n$ so $\varphi \in Gl_n(K)$. As $\varphi(C \cdot F) \subset N \subset F$ all the entries of φ are in $\bar{\mathcal{O}}$ so $\varphi \in M_n(\bar{\mathcal{O}})$. It is easy to verify

(1.7.5) $\quad \text{length}(\bar{F}/\varphi(\bar{F})) = \text{length}(\bar{\mathcal{O}}/\det \varphi).$

It follows that length $(F/N') = \text{length}\,(F/N) - \text{length}\,(\bar{\mathcal{O}}/\det(\varphi))$. Suppose $\det(\varphi)$ is unit in $\bar{\mathcal{O}}$ so $\varphi \in Gl_n(\bar{\mathcal{O}})$. Then as $C \cdot F \subset N'$, $C \cdot F \subset \varphi(C \cdot F) \subset \varphi(N') = N$ which contradicts our assumption. So φ is not in $Gl_n(\bar{\mathcal{O}})$ and hence length $(\bar{\mathcal{O}}/\det(\varphi)) > 0$. This proves the proposition.

§2.

In this section the curve X will be assumed to be embedded in a smooth surface Z. We first prove.

LEMMA 2.0. *$\bar{M}$ is irreducible* $\Leftrightarrow$ $\text{Quot}^m(n, X)$ *is irreducible for every m.*

Proof. Let N be an $\mathcal{O}_X$ module of rank n, $N \subset \mathcal{O}_X^n$ with finite cokernel of length m. Suppose $\text{Quot}^m(n, X)$ is irreducible so N can be deformed to $N(t) \subset \mathcal{O}_X^n \otimes k[[t]]$ with the quotient supported at m distinct $k[[t]]$ rational primes and none of them singular. Then clearly $N(t) \otimes k((t))$ is locally free on $X \times \operatorname{Spec} k((t))$.

Conversely, let $\bar{M}$ be irreducible and suppose $\text{Quot}^m(n, X)$ is irreducible for $m \leq m_0 - 1$. Let $N \subset \mathcal{O}_X^n$ define a point in Quot^{m_0}. Recall that as X is irreducible the quotients $\mathcal{O}_X^n/N$ supported at m_0 distinct smooth points form an irreducible open subset $U = U^{m_0}(n, X)$. Also if $\mathcal{O}_X^n/N$ is supported at smooth points of X it lies in the closure of U as may be verified by treating $\mathcal{O}_X^n/N$ as a sheaf on the normalization of X. Suppose $\mathcal{O}_X^n/N$ is supported at $y_1, y_2, \ldots, y_s$, $s > 1$. Then for every i we have $s_i = \text{length}\,(\mathcal{O}_{X,y_i}^n/N_{y_i}) < m_0$. Since each quotient $\mathcal{O}_{X,y_i}^n/N_{y_i}$ defines a point in $\text{Quot}^{s_i}(n, X)$ which is in the closure of $U^{s_i}(n, X)$ there exists a deformation of $\mathcal{O}_X^n/N$ generically having support at $m_0 = \sum s_i$ distinct smooth points. We may therefore assume that $\mathcal{O}_X^n/N$ and all its small deformations are supported at one point $x \in X$. This means that the Punctual Quot scheme $\text{Quot}_x^{m_0}(n, X)$ contains a component W of $\text{Quot}^{m_0}(n, X)$ and the map $\text{Quot}_x^{m_0}(n, X) \hookrightarrow \text{Quot}^{m_0}(n, X)$ is bijective in a neighbourhood of $\mathcal{O}_X^n/N$. Since $\bar{M}$ is irreducible N can be deformed to a locally free $\mathcal{O}_X$ module. Let $N[t]$ be an $\mathcal{O}_X \otimes k[[t]]$ module defining this deformation. If we localize around x then we can use (1.3) to lift the given imbedding $N_x \hookrightarrow \mathcal{O}^n_{X,x}$ to $N_x[t] \hookrightarrow \mathcal{O}^n_{X,x} \otimes k[\![t]\!]$ with cokernel a free $k[\![t]\!]$ module of rank m_0. Now the imbedding $N_x[t] \hookrightarrow \mathcal{O}^n_{X,x} \otimes k[\![t]\!]$ is the restriction of an inclusion $N'[t] \hookrightarrow \mathcal{O}^n_X \otimes k[\![t]\!]$ with the same cokernel and $N'[t]$ is generically a vector bundle specializing to N on X. As $\mathrm{Quot}^{m_0}_x(n, X)$ is bijective with $\mathrm{Quot}^{m_0}(n, X)$ in a neighbourhood of $\mathcal{O}^n_X/N$, $\mathcal{O}^n_X \otimes k[\![t]\!]/N'[t]$ is supported at $(x) \times \mathrm{Spec}\, k[\![t]\!]$ and so there are points of W in every neighbourhood of $\mathcal{O}^n_X/N$ defined by vector bundles. We will derive a contradiction.

Let $h_1, \ldots, h_n \in \mathcal{O}^n_{X,x}$ define a free $\mathcal{O}^n_{X,x}$ module P with $\mathcal{O}^n_{X,x}/P$ having length m_0. Then the deformation $(h_i + t)$ is a flat deformation that is not supported at x. But then $\mathrm{Quot}^{m_0}(n, X)$ cannot be bijective with $\mathrm{Quot}^{m_0}_x(n, X)$ in a neighbourhood of $\mathcal{O}^n_X/P$ and the proposition is proved.

Remark 2.1. The proof of the above proposition yields the fact that if $N \subset \mathcal{O}^n_X$ is deformable over $k[\![t]\!]$ to a vector bundle then the given injection lifts to a (possibly different) deformation that is supported generically at m distinct points where $m = \mathrm{length}\,(\mathcal{O}^n_X/N)$. We will use this remark later.

COROLLARY 2.2. *If $\bar{M}$ is irreducible then*

(2.2.0) $\dim \mathrm{Quot}^m_x(n, X) \leq n \cdot m - 1$

for all $x \in X$ and $m \geq 1$.

Proof. Since Quot^m_x is a proper closed subset of Quot^m and $\dim U^m(n, X) = n \cdot m$ the result follows.

From now on X is an irreducible and reduced curve on a smooth surface Z and C is the conductor of $\mathcal{O}_X$ in its normalization.

LEMMA 2.3. *If ν is the multiplicity of $\mathcal{O}_{X,x}$ and $\mathfrak{M}$ the maximal ideal then*

(2.3.1) $C \subset \mathfrak{M}^{\nu-1}, \qquad C \not\subset \mathfrak{M}^{\nu}$

Proof. The conductor is defined by the set of curves $g = 0$, $g \in \mathcal{O}_Z$ with multiplicity greater than or equal to $(\mathrm{mult}\, \mathcal{O}_{X,x} - 1)$ at x as well as at all infinitely near points. Hence $C \subset \mathfrak{M}^{\nu-1}$. Recall that if $\mathcal{O}'$ is the blow up of $\mathcal{O} = \mathcal{O}_{X,x}$ then $\mathfrak{M}^{\nu-1}$ is the conductor of $\mathcal{O}$ in $\mathcal{O}'$ and $C = C_1 \cdot \mathfrak{M}^{\nu-1}$ where C_1 is the conductor of $\mathcal{O}'$ in $\mathcal{O}$. Also by the definition of blowing up there is a z in $\mathfrak{M}$ satisfying $z \cdot \mathcal{O}' = \mathfrak{M} \cdot \mathcal{O}'$ so that $\mathfrak{M}^{\nu-1} \cdot \mathcal{O}' = z^{\nu-1} \cdot \mathcal{O}'$. Assume that $C \subset \mathfrak{M}^{\nu}$; we will derive a contradiction. We have

$$
\begin{aligned}
& C \subset \mathfrak{M}^{\nu} \\
\Rightarrow\ & C_1 \cdot \mathfrak{M}^{\nu-1} \subset \mathfrak{M}^{\nu} \\
\Rightarrow\ & C_1 \subset \operatorname{Hom}(\mathfrak{M}^{\nu-1}, \mathfrak{M}^{\nu}) = \operatorname{Hom}(\mathfrak{M}^{\nu-1}\mathcal{O}', \mathfrak{M}^{\nu}\mathcal{O}') \\
& = \operatorname{Hom}(z^{\nu-1} \cdot \mathcal{O}', z^{\nu-1} \cdot z \cdot \mathcal{O}') \\
& = \operatorname{Hom}(z^{-1} \cdot \mathcal{O}', \mathcal{O}') = z \cdot \mathcal{O}'.
\end{aligned}
$$

This says that $z^{-1} \cdot C_1 \subset \mathcal{O}'$, z a non unit in $\bar{\mathcal{O}}$ and contradicts the definition of C_1 as the largest $\bar{\mathcal{O}}$ ideal in $\mathcal{O}'$. The lemma is thereby proved.

Remark 2.4. In characteristic zero a polar of the equation of X in Z at x gives an element of C not in $\mathfrak{M}^{\nu}$ since a derivative has lower order than that of the equation. (see Coolidge – A Treatise on Algebraic Plane Curves). In general we refer to any $g \in C - \mathfrak{M}^{\nu}$ as a polar.

THEOREM (2.5). $\operatorname{Quot}^m(n, X)$ *is irreducible for all* m.

Proof. Since the problem is local around the singular points we use induction on the multiplicity of one singular point, $x \in X$. Assume the result true for a curve with multiplicity less than $\nu = \operatorname{mult}(\mathcal{O}_{X,x})$. By Lemma 2.3 there is a $g \in \mathcal{O}_Z$ with g of order $(\nu - 1)$ and g defining an element of C. We have

(2.5.1) $\operatorname{Quot}^m(n, \mathcal{O}_X/C) \hookrightarrow \operatorname{Quot}^m_x(n, \mathcal{O}_Z/(g))$.

By adding a general element of $\mathcal{O}_Z$ of high order to g we can assume that $g = 0$ defines (locally) a reduced curve in Z irreducible in a neighbourhood of x. Now induction and Cor. 2.2 gives

(2.5.2) $\dim \operatorname{Quot}^m(n, \mathcal{O}_X/C) \leq \dim \operatorname{Quot}^m_x(n, \mathcal{O}_Z/(g)) \leq n \cdot m - 1$.

If $x_1, \ldots, x_r$ are the singular points of X and N a torsion free rank n $\mathcal{O}_X$ module then to show that N is deformable to a vector bundle it suffices to know this for N_{x_i}, $\forall i$. We may therefore assume that X has one singular point, x.

Assume all $\mathcal{O}_X$ modules N which have an embedding $N \hookrightarrow \mathcal{O}_X^n$ with length $(\mathcal{O}_X^n/N) < m_0$ can be deformed to locally free modules. By Remark (2.1) this is equivalent to the assumption that $U^m(n, X)$ is dense in $\operatorname{Quot}^m(n, X)$ for $m < m_0$ and in particular $\operatorname{Quot}^m(n, X)$ is irreducible for $m < m_0$. We want to show that $\operatorname{Quot}^{m_0}(n, X)$ is irreducible. By Lemma 2.0 and induction on m_0 this would yield the theorem.

Suppose $\mathcal{O}_X^n/N$ is supported at $y_1, y_2, \ldots, y_s$, $s>1$. Then for every i we have $s_i = \text{length}\,(\mathcal{O}_{X,y_i}^n/N_{y_i}) < m_0$.

Since each quotient $\mathcal{O}_{X,y_i}^n/N_{y_i}$ defines a point in $\text{Quot}^{s_i}(n, X)$ which is the closure of $U^{s_i}(n, X)$ there exists a deformation of $\mathcal{O}_X^n/N$ generically having support at m_0 distinct smooth points. Hence $\mathcal{O}_X^n/N$ lies in the closure of $U^{m_0}(n, X)$ so that N is deformable to a vector bundle. Therefore suppose $\mathcal{O}_X^n/N$ is supported at one point $y \in X$. If $y \neq x$, i.e. if y is a smooth point then N is actually locally free since $N_x = \mathcal{O}_{X,x}^n$ and N_y is free when $\mathcal{O}_{X,y}$ is a discrete valuation ring.

We can thus restrict ourselves to quotients in $\text{Quot}_x^{m_0}(n, X)$. The above discussion says that the open set $U = \text{Quot}^{m_0}(n, X) - \text{Quot}_x^{m_0}(n, X)$ has $U^{m_0}(n, X)$ as a dense (irreducible) subset. If $\text{Quot}^{m_0}(n, X)$ is reducible then $\text{Quot}_x^{m_0}(n, X)$ must contain a component W of $\text{Quot}^{m_0}(n, X)$. We will prove the theorem by deriving a contradiction.

Let $Q \in \text{Quot}_x^{m_0}(n, X) \subset \text{Quot}^{m_0}(X)$ be a general point of W so that if $\mathcal{O}_X^n/N$ represents Q every small deformation of $\mathcal{O}_X^n/N$ is supported at x. By Remark (2.1), N cannot be deformed to a locally free module. If $C \cdot \mathcal{O}_X^n \not\subset N$ then by Proposition (1.7.3) there is an N' with $N'_x \approx N_x$ and with $C \cdot \mathcal{O}_X^n \subset N' \subset \mathcal{O}_X^n$. Further

$$\text{(2.5.3)} \quad \text{length}\,(\mathcal{O}_X^n/N') \lneqq \text{length}\,(\mathcal{O}_X^n/N) = m_0$$

and so by assumption N' is deformable to a vector bundle. But for $y \neq x$ $N'_y = \mathcal{O}_{X,y}^n$ and $N'_x \approx N_x$ so that Proposition (1.2) implies that N is also deformable to a vector bundle. This contradicts Remark (2.1) and hence we may assume $C \cdot \mathcal{O}_X^n \subset N$. As $\mathcal{O}_X^n/N$ is a general point of W the natural map $\text{Quot}^{m_0}(n, \mathcal{O}_X/C) \to \text{Quot}^{m_0}(n, X)$ is a bijection in a neighbourhood of P. Of course, as W is a component, $\text{Quot}^{m_0}(n, X)$ is bijective with $\text{Quot}_x^{m_0}(n, X)$ in a neighbourhood of P.

If X' is an affine open subset of X then $\text{Quot}^m(n, X')$ is an open subset of $\text{Quot}^m(n, X)$ and if $x \in X'$ then $\text{Quot}_x^m(n, X') = \text{Quot}_x^m(n, X)$. As we will encounter only quotients supported at x and their small deformations we can restrict ourselves to an open neighbourhood of x. Let $Z' \subset Z$ be an affine open set with $X' = X \cap Z'$ satisfying $x \in X'$ and X' defined by one equation, $\{f=0\}$, $f \in \mathcal{O}_Z$. Let $g \in \mathcal{O}_Z$ define a polar of X' at x. If g is chosen sufficiently general and Z' small enough we can arrange so that $\{f=0\} \cap \{g=0\} = z \in Z$ and $\{f+tg=0\} \subset Z' \times \text{Spec}\, k[\![t]\!]$ is smooth outside $(z) \times \text{Spec}\, k[\![t]\!]$, where of course, $\mathcal{O}_{Z,z}/(f) = \mathcal{O}_{X,x}$. Write $S = \text{Spec}\, k[\![t]\!]$ and $\{f+tg=0\} = X'_S$ and $\varphi' : X'_S \to S$ the restriction of the projection map. The family φ is smooth outside $(z) \times S$ and the singular point of the generic fibre has multiplicity $\nu - 1 = (\text{order } g \text{ at } z)$, by the definition of a polar. As in the proper case we have a corresponding family $\rho' : \text{Quot}^{m_0}(n, X'_S \mid S) \to S$ where the fibres of ρ' are open subsets of the Quot schemes of the fibres of φ'. Since $f+tg \in C \subset N$, $\mathcal{O}_X^n/N$ or rather $\mathcal{O}_{X\times S}^n/N \otimes \mathcal{O}_S$, defines a section σ of ρ'. By induction on the multiplicity of the singular point the generic fibre of ρ' is irreducible of dimension $n \cdot m_0$. As the section $\sigma(S)$ passes through $P \in W' = W \cap \mathrm{Quot}^{m_0}(n, X')$, semicontinuity of dimension gives $\dim_P \mathrm{Quot}^{m_0}(n, X) \geqslant n \cdot m_0$. But we have seen that

$$\mathrm{Quot}^{m_0}(n, \mathcal{O}_X/C) \to \mathrm{Quot}_x^{m_0}(n, X) \to \mathrm{Quot}^{m_0}(n, X)$$

are bijections around P so we have

(2.5.4) $\dim_P \mathrm{Quot}_x^{m_0}(n, \mathcal{O}_X/C) \geqslant n \cdot m_0$.

However g defines an element of C and hence $\mathcal{O}_X/C$ is a quotient of $\mathcal{O}_{Z'}/(g)$. This means $\mathrm{Quot}^{m_0}(n, \mathcal{O}_X/C) \subset \mathrm{Quot}_y^{m_0}(n \cdot \mathcal{O}_{Z'}/(g))$ where $y \in \mathrm{Spec}\, \mathcal{O}_{Z'}/(g)$ maps to $z \in Z'$. As $\mathrm{Quot}^{m_0}(n, \mathcal{O}_{Z'}/(g))$ is irreducible of dimension $n \cdot m_0$ and $\mathrm{Quot}_y^{m_0}(n, \mathcal{O}_{z'}/(g))$ is a proper closed subset of $\mathrm{Quot}^{m_0}(n, \mathcal{O}_{Z'}/(g))$ we have

(2.5.5) $\dim_P \mathrm{Quot}_x^{m_0}(n, \mathcal{O}_X/C) < n \cdot m_0$.

This contradicts (2.5.4) and proves the theorem.

COROLLARY 2.6. $\mathrm{Quot}^m(n, Z)$ *is irreducible* $\forall m$.

Proof. Note that for $n = 1$ Quot^m is just $\mathrm{Hilb}^m(Z)$ which is smooth and connected of dimension $2m$. For $n > 1$ we need to use the irreducibility of $\mathrm{Quot}^m(n, X) \subset \mathrm{Quot}^m(n, Z)$ for a suitable irreducible curve X in Z. Now it is clear that if any two points of a scheme can be joined by an irreducible subscheme the scheme is irreducible. We claim that given two quotients $\mathcal{O}_Z^n/M$ and $\mathcal{O}_Z^n/N$ of length m there is an irreducible curve X with $\mathrm{Quot}^m(n, X)$ containing both the given quotients.

Let $Q_1, Q_2, \ldots, Q_s$, and $P_1, P_2, \ldots, P_t$ be the supports of $\mathcal{O}_Z^n/M$ and $\mathcal{O}_Z^n/N$ respectively. The annihilator of $\mathcal{O}_Z^n/M \oplus \mathcal{O}_Z^n/N$ is an ideal $I \subset \mathcal{O}_Z$ with $\mathcal{O}_Z/I$ supported at $(P_1, P_2, \ldots, Q_1, Q_2 \ldots)$. Let B be the semi local ring of the (P_i, Q_j) which exists as Z is projective. Then all we need to define a suitable X is to find a height one prime $\mathfrak{P} \subset I$. As B is a U.F.D. any irreducible element in I defines a $\mathfrak{P}$ and this proves the corollary.

Remark (2.7). We do not know if $\mathrm{Quot}^m(n, Z)$ is reduced for $n > 1$.

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