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## On the existence of extremal Teichmüller mappings

LI ZHONG

### 1. Introduction

Let  $\mu$  be an orientation preserving homeomorphism of the unit circumference  $|z| = 1$  onto  $|w| = 1$  which admits a quasiconformal extension into the disk  $D: |z| < 1$ . Let  $K_0$  be the maximal dilatation of an extremal q.c. extension of  $\mu$  into  $D$ , while  $H$  denotes the dilatation of  $\mu$ , i.e. the infimum of the dilatations of all the extensions of  $\mu$  into arbitrarily small annuli  $1 - \varepsilon < |z| \leq 1$ ,  $\varepsilon > 0$ . If  $H < K_0$ , then there exists a unique extremal Teichmüller mapping associated with a quadratic differential of finite norm ([9], [10]). To estimate  $H$  one can introduce a local dilatation  $H(\zeta)$  of  $\mu$  at a point  $\zeta \in \partial D$ : It is the infimum of the maximal dilatations of all extensions of  $\mu$  into arbitrarily small neighborhoods of  $\zeta$  with respect to  $\bar{D}$  (see [5]). R. Fehlmann recently showed that  $H = \max_{|\zeta|=1} H(\zeta)$ . In the present paper we use this result to give an upper estimate of  $H$  which, together with a lower estimate of  $K_0$ , allows the conclusion  $H < K_0$ . (So far, in the literature, this has been done only in the case  $H = 1$ , see [9] and [10].)

In order to carry out the program, two quantities are introduced in sections (2) and (3) respectively. The first one is motivated by the maximal dilatation  $q(\rho)$  of the extremal selfmapping of the upper halfplane with boundary values  $\mu(x) = x$  for  $x \geq 0$  and  $\mu(x) = \rho x$ ,  $\rho \geq 1$  for  $x < 0$ , which is easily computed from [7]:

$$q(\rho) = 1 + \frac{1}{2\pi^2} \log^2 \rho + \frac{1}{\pi} (\log \rho) \sqrt{1 + \frac{1}{4\pi^2} \log^2 \rho}.$$

The second one (section (3)) is obtained in the usual way (see [1]) as the ratio of the moduli of certain quadrilaterals in  $|z| < 1$  and  $|w| < 1$  respectively.

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## 2. An estimate of the local dilatation of $\mu$

In what follows, we will write the given boundary correspondence  $\mu$  in a real form:  $\vartheta = h_\mu(\theta)$ , where  $\vartheta = \arg w$  and  $\theta = \arg z$ : namely  $h_\mu(\theta) := \arg \mu(e^{i\theta})$ . Then  $\vartheta = h_\mu(\theta)$  is a  $\rho$ -quasisymmetric homeomorphism of  $\mathbf{R}$  onto itself. Assume  $\mu$  is absolutely continuous and satisfies the condition:

$$m \leq h'_\mu(\theta) \leq M, \quad \text{almost everywhere,} \quad (1)$$

where  $m$  and  $M$  are constant,  $0 < m < M < \infty$ . We introduce the functions:

$$q(\lambda, \omega) := \frac{1}{2}(\omega + \omega^{-1}) + \frac{\omega}{2\pi^2} \log^2 \lambda + \frac{\omega}{2} \sqrt{\left[ (\omega - 1)^2 + \frac{1}{\pi^2} \log^2 \lambda \right] \left[ (\omega + 1)^2 + \frac{1}{\pi^2} \log^2 \lambda \right]}, \quad (2)$$

$$\lambda(\theta) := \lim_{\varepsilon \rightarrow 0} \sup_{0 < |t| < \varepsilon} \left\{ \frac{h_\mu(\theta + t) - h_\mu(\theta)}{h_\mu(\theta) - h_\mu(\theta - t)} \right\}, \quad (3)$$

and

$$\omega(\theta) := \lim_{\varepsilon \rightarrow 0^+} \max \left( \frac{M(\theta, \theta + \varepsilon)}{m(\theta, \theta + \varepsilon)}, \frac{M(\theta - \varepsilon, \theta)}{m(\theta - \varepsilon, \theta)} \right) \quad (4)$$

where

$$m(x, y) := \operatorname{ess\,inf}_{(x,y)} \{h'_\mu\}, \quad M(x, y) := \operatorname{ess\,sup}_{(x,y)} \{h'_\mu\}.$$

Because the function  $h_\mu$  is  $\rho$ -quasisymmetric, there always exists the limit (3). We denote by  $H(\theta)$  the local dilatation of  $\mu$  at the point  $e^{i\theta}$ . We have

**THEOREM 1.** *Under the above assumption we have*

$$H(\theta) \leq q(\lambda(\theta), \omega(\theta)), \quad \text{for all } \theta. \quad (5)$$

*Proof.* Without loss of generality, we only look at  $\theta = 0$  and assume that  $h_\mu(0) = 0$ . We are going to show the estimate (5) for the point  $\theta = 0$ . Applying the mapping  $\zeta = \log z$ , we pass to the strip  $\Sigma$ :

$$\Sigma := \{\zeta = \xi + i\eta : 0 < \eta < \pi\} \quad (6)$$

which is the image of the upper half-plane  $U$ . The boundary correspondence  $h_\mu$

of  $U$  becomes a boundary correspondence of  $\Sigma$  as follows:

$$\begin{cases} \xi \rightarrow \log(h_\mu(e^\xi)), & \text{for the lower boundary,} \\ \xi + \pi i \rightarrow \log(-h_\mu(-e^\xi)) + \pi i, & \text{for the upper boundary.} \end{cases} \quad (7)$$

We construct a function  $f(\zeta)$  in  $\Sigma$  with the boundary correspondence (7):

$$f(\zeta) := \left(1 - \frac{\eta}{\pi}\right) \log(h_\mu(e^\xi)) + \frac{\eta}{\pi} \log(-h_\mu(-e^\xi)) + i\eta. \quad (8)$$

It is easily seen that  $f$  is a 1-1 mapping of  $\Sigma$  onto itself. Since  $h_\mu$  is absolutely continuous,  $f$  is absolutely continuous along every line  $\xi = \text{const.}$  or  $\eta = \text{const.}$  in  $\Sigma$ . A simple computation shows that

$$2f_{\bar{\zeta}} = E(\zeta) - 1 + iL(\zeta), \quad (9)$$

and

$$2f_{\zeta} = E(\zeta) + 1 - iL(\zeta) \quad (10)$$

where  $E$  and  $L$  are real functions in  $\Sigma$ :

$$E(\zeta) := \left(1 - \frac{\eta}{\pi}\right) \frac{e^\xi h'_\mu(e^\xi)}{h_\mu(e^\xi)} + \frac{\eta}{\pi} \cdot \frac{e^\xi h'_\mu(-e^\xi)}{-h_\mu(-e^\xi)} \quad (11)$$

and

$$L(\zeta) := \frac{1}{\pi} \log \frac{-h_\mu(-e^\xi)}{h_\mu(e^\xi)}. \quad (12)$$

Therefore we have

$$k(\zeta)^2 = \left| \frac{f_{\bar{\zeta}}}{f_{\zeta}}(\zeta) \right|^2 = \frac{(E(\zeta) - 1)^2 + L(\zeta)^2}{(E(\zeta) + 1)^2 + L(\zeta)^2} = 1 - \frac{4E(\zeta)}{(E(\zeta) + 1)^2 + L(\zeta)^2}.$$

By the condition (1) it is easy to check that  $\|k(\zeta)\|_\infty < 1$  and hence  $f$  is a quasiconformal mapping of  $\Sigma$  onto itself. It is easily seen that

$$\begin{aligned} H(0) &\leq \sup_{\text{Re } \zeta < l} \frac{1 + k(\zeta)}{1 - k(\zeta)} \\ &= \sup_{\text{Re } \zeta < l} \frac{1}{2E} (1 + E^2 + L^2 + \sqrt{[(E + 1)^2 + L^2][(E - 1)^2 + L^2]}) \end{aligned} \quad (13)$$

for any real number  $l$ . On the other hand, a simple computation shows that

$$\frac{1}{\omega(0)} + o(1) \leq E(\zeta) \leq \omega(0) + o(1), \quad \text{as } \operatorname{Re} \zeta \rightarrow -\infty \tag{14}$$

and

$$|L(\zeta)| \leq \frac{1}{\pi} \log \lambda(0) + o(1), \quad \text{as } \operatorname{Re} \zeta \rightarrow -\infty. \tag{15}$$

Setting  $l \rightarrow -\infty$ , from (13), (14) and (15) we prove the inequality (5) for  $\theta = 0$ .

### 3. An estimate of the smallest maximal dilatation $K_0$ from below

Apply the mapping  $g : z \mapsto i(1-z)/(1+z)$  to the disk  $|z| < 1$  and a fractional linear transformation  $G$  to the disk  $|w| < 1$ , which maps  $|w| < 1$  onto a upper half-plane with  $G(\mu(-1)) = \infty$ . Then the boundary homeomorphism  $G \circ \mu \circ g^{-1}$  of  $\mathbf{R}$  onto  $\mathbf{R}$  is a  $\rho$ -quasisymmetric function, namely, there is a number  $\rho$  such that  $G \circ \mu \circ g^{-1}$  satisfies the  $\rho$ -condition. The infimum of all such numbers  $\rho$  is denoted by  $\rho_0$ . Denote by  $U(z_1, z_2, z_3, z_4)$  the quadrilateral formed by the upper half-plane and the vertexes  $z_1, z_2, z_3$ , and  $z_4$ . We introduce a function

$$p(\rho) := M\{U(\infty, -1, 0, \rho)\}, \quad \text{for } \rho > 0, \tag{16}$$

where  $M\{U(\infty, -1, 0, \rho)\}$  is the modulus of  $U(\infty, -1, 0, \rho)$ , the  $a$ -side and  $b$ -side of which are chosen such that  $p(\rho)$  is an increasing function of  $\rho$ . It is known that  $p(\rho) = 1 + r(\rho) \log \rho$ , where  $r(\rho)$  is a monotone function of  $\rho$  and

$$0.2284 \cdots < r(\rho) < \frac{1}{\pi}. \tag{17}$$

(See [1] by A. Beurling and L. Ahlfors.)

We are now going to show the inequality:

$$p(\rho_0) \leq K_0. \tag{18}$$

Obviously, for any  $x \in \mathbf{R}$  and  $t > 0$ ,  $M\{U(\infty, x-t, x, x+t)\} = 1$  and

$$M\{U(\infty, \tilde{\mu}(x-t), \tilde{\mu}(x), \tilde{\mu}(x+t))\} = M\{U(\infty, -1, 0, B(x, t))\} = p(B(x, t)), \tag{19}$$

where  $\tilde{\mu} := G \circ \mu \circ g^{-1}$  and

$$B(x, t) := \frac{\tilde{\mu}(x+t) - \tilde{\mu}(x)}{\tilde{\mu}(x) - \tilde{\mu}(x-t)}. \tag{20}$$

On the other hand, if  $f_0$  is an extremal mapping of  $|z| < 1$  onto  $|w| < 1$  with the given boundary correspondence  $\mu$ , then the mapping  $G \circ f_0 \circ g^{-1}$  is an extremal mapping of  $U$  onto itself with the boundary correspondence  $\tilde{\mu} = G \circ \mu \circ g^{-1}$  and hence we have

$$p(B(x, t)) = \frac{M\{U(\infty, \tilde{\mu}(x-t), \tilde{\mu}(x), \tilde{\mu}(x+t))\}}{M\{U(\infty, x-t, x, x+t)\}} \leq K_0. \tag{21}$$

Similarly, one can show that the inequality (21) is true for  $t < 0$ . Noting that  $\rho_0$  is the supremum of  $B(x, t)$  for all  $x \in \mathbf{R}$  and  $t \neq 0$ , we get the estimate (18).

One can easily prove that the estimate is sharp in the sense that for any  $K_0$ , there is a boundary correspondence  $\mu$  such that the equality in (18) holds.

#### 4. The main theorem

For a given homeomorphism  $\mu$  of  $|z|=1$  onto  $|w|=1$ , we call a boundary point  $z = e^{i\theta}$  an essential boundary point if  $H(\theta) = K_0$ . R. Fehlmann proved that if there is a degenerating Hamilton sequence, then there exists an essential boundary point on the circle  $|z|=1$  (See [2], p. 567). On the other hand, K. Strebel proved that if there is no degenerating Hamilton sequence for a complex dilatation of an extremal mapping  $f_0$  then  $f_0$  is a Teichmüller mapping associated with a quadratic differential of finite norm. Therefore one can conclude that if there is no essential boundary point, then the extremal Teichmüller mapping exists. By the inequality (18) and Theorem 1 we see that if  $q(\lambda(\theta), \omega(\theta)) < p(\rho_0)$  for all  $\theta$ , then there is no essential boundary point. We have proved

**THEOREM 2.** *Let  $\mu$  be an orientation preserving homeomorphism of  $|z|=1$  onto  $|w|=1$  which admits a quasiconformal extension into the disk  $|z| < 1$ . Suppose that  $\mu$  is absolutely continuous and satisfies the condition (1). If*

$$q(\lambda(\theta), \omega(\theta)) < p(\rho_0), \quad \text{for all } \theta, \tag{22}$$

*then there exists an extremal Teichmüller mapping associated with a quadratic differential of finite norm.*

## 5. Applications

Applying Theorem 1 and Theorem 2 to the special case that the given boundary homeomorphism  $\mu$  is piecewise smooth, we get some interesting results. In this case the condition (1) requires

$$h'_+(\theta) \neq 0 \quad \text{and} \quad h'_-(\theta) \neq 0 \quad \text{for all } \theta, \quad (23)$$

where  $h'_+(\theta)$  and  $h'_-(\theta)$  are the right-derivative and the left-derivative of  $h_\mu$ , respectively. It is easily seen that

$$\lambda(\theta) = \max \left\{ \frac{h'_+(\theta)}{h'_-(\theta)}, \frac{h'_-(\theta)}{h'_+(\theta)} \right\} \quad \text{and} \quad \omega(\theta) = 1. \quad (24)$$

By Theorem 1, we have

$$H(\theta) \leq q(\lambda(\theta), 1) := q(\lambda(\theta)). \quad (25)$$

Moreover, by a normal family argument, one can prove that the equality  $H(\theta) = q(\lambda(\theta))$  for every point. Here the function  $q(\lambda)$  is given by the expression

$$q(\lambda) = 1 + \frac{1}{2\pi^2} \log^2 \lambda + \frac{1}{\pi} (\log \lambda) \sqrt{1 + \frac{1}{4\pi^2} \log^2 \lambda}. \quad (26)$$

From (24), (25) and Theorem 2, we have

**THEOREM 3.** *If the given boundary homeomorphism  $\mu$  is assumed as above and*

$$\frac{1}{q^{-1} \circ p(\rho_0)} < \frac{h'_+(\theta)}{h'_-(\theta)} < q^{-1} \circ p(\rho_0), \quad \text{for all } \theta, \quad (27)$$

*then there exists an extremal Teichmüller mapping.*

**COROLLARY.** *If there are three points  $z_1, z_2,$  and  $z_3$  on the circle  $|z| = 1$  such that the cross ratio  $D(-1, z_1, z_2, z_3) = -1$ ,  $D(\mu(-1), \mu(z_1), \mu(z_2), \mu(z_3)) = -A$ ,  $A \geq 1$ , and*

$$\max(5^{-1}, A^{-1/2}) \leq \frac{h'_+(\theta)}{h'_-(\theta)} \leq \min(5, A^{1/2}), \quad \text{for all } \theta, \quad (28)$$

*then there exists an extremal Teichmüller mapping.*

*Proof.* If  $A = 1$ , then the condition (28) implies that  $\mu$  is smooth everywhere and hence the dilatation of  $\mu$  is equal to one. By the result of K. Strebel [9], there is an extremal Teichmüller mapping. We assume  $A > 1$ . A simple computation shows that  $p(\rho_0) \geq p(A)$ . By the inequality (17), we have

$$p(A) > 1 + 0.2284(\log A). \quad (29)$$

On the other hand, by (28) and (26),  $\lambda(\theta) \leq \min(5, A^{1/2})$ , and

$$\begin{aligned} q(\lambda(\theta)) &\leq 1 + \left[ \frac{1}{4\pi^2} \log 5 + \frac{1}{2\pi} \left( 1 + \frac{1}{8\pi^2} \log^2 5 \right) \right] \log A \\ &\leq 1 + 0.2052(\log A). \end{aligned} \quad (30)$$

Therefore  $q(\lambda(\theta)) < p(\rho_0)$  for all  $\theta$  and hence the corollary is proved.

#### REFERENCES

- [1] A. BEURLING and L. AHLFORS, *The boundary correspondence under quasiconformal mappings*, Acta Math. 96 (1956), 125–142.
- [2] R. FEHLMANN, *Ueber extremale quasikonforme Abbildungen*, Comment. Math. Helv. Vol. 56, Fasc. 4 (1981), 558–580.
- [3] R. S. HAMILTON, *Extremal quasiconformal mappings with prescribed boundary values*, Trans. Amer. Math. Soc. 138 (1969), 399–406.
- [4] O. LETHO and K. VIRTANEN, *Quasiconformal Mappings in the Plane*, Springer-Verlag, Berlin and New York 1973.
- [5] E. REICH, *On the relation between local and global properties of boundary values for extremal quasi-conformal mappings*, Annals of Math. Studies 79 (1974), 391–407.
- [6] E. REICH and K. STEBEL, *Extremal quasiconformal mappings with given boundary values*, in Contributions to analysis, Academic Press, 1974, 375–391.
- [7] K. STREBEL, *Zur Frage der Eindeutigkeit extremaler quasikonformer Abbildungen des Einheitskreises*, Comment. Math. Helv. 36 (1962), 306–323.
- [8] —, *Zur Frage der Eindeutigkeit extremaler quasikonformer Abbildungen des Einheitskreises II*, Comment. Math. Helv. 39 (1964), 77–89.
- [9] —, *On quadratic differentials and extremal quasiconformal mappings*, International Congress, Vancouver, 1974.
- [10] —, *On the existence of extremal Teichmüller mappings*, Journal d'analyse Math. 30 (1976), 464–480.

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## Buchanzeigen

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GOHBERG I., **Toeplitz Centennial – Toeplitz Memorial Conference in Operator Theory**, Dedicated to the 100th Anniversary of the Birth of Otto Toeplitz, Tel Aviv, May 11–15, 1981, Birkhäuser Verlag Basel–Boston–Stuttgart, 1982, 588 p., SFr. 92.–, DM 108.–

Contributions by: Albrecht, E., Arocena, R./Cotlar, M., Azoff, E./Clancey, K./Gohberg, I., Ball, J. A., Bart, H./Gohberg, I./Kaashoek, M. A., Baum, P./Douglas, R. G., Brown, L. G., Clark, D. N., Coburn, L. A., Cordes, H. O., Costabel, M., Davis, C., Dym, H./Iacob, A., Foias, C., Gauchman, H., Haller, H./Jacobs, K., Kaballo, W., Kailath, T./Lev-Ari, H., Kalman, R. E., Kaper, H. G., Lax, P. D./Phillips, R. S., Livšic, M. S., Lumer, G., Meister, E., Putnam, C. R., Waelbroeck, L., Widom, H., Wolff, M., Zweifel, P. F./Greenberg, W. – Memorial papers: Köthe, G., Toeplitz, U., Dieudonné, J. – Programme of the Conference

FRICKER F., **Einführung in die Gitterpunktlehre**, Birkhäuser Verlag Basel–Boston–Stuttgart, 1982, 215 S., SFr. 86.–

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DIEDERICH, K., LIEB, I., **Konvexität in der komplexen Analysis** (DMV Seminar Band 2), Birkhäuser Verlag Basel–Boston–Stuttgart 1981, 150 S., SFr. 22.–, DM 24.–

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