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# Proper holomorphic mappings between circular domains

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## 1. Introduction

A now classical theorem of H. Cartan [3] states that if $f: \Omega_1 \to \Omega_2$ is a biholomorphic mapping between bounded circular domains in $\mathbf{C}^n$ which contain the origin, and if $f(0)=0$, then $f$ is a linear mapping. Cartan's theorem was later generalized by W. Kaup [4] to biholomorphic mappings between domains in a much wider class. In this note, we prove a generalization of Cartan's theorem which allows the mapping $f$ to be proper and non-biholomorphic. To be precise, we prove

THEOREM 1. *Suppose that $f: \Omega_1 \to \Omega_2$ is a proper holomorphic mapping between bounded circular domains in $\mathbf{C}^n$ which contain the origin, and suppose that $f^{-1}(0)=\{0\}$. Then the mapping $f$ is a polynomial mapping.*

The proof of this theorem uses only elementary properties of the Bergman projection associated to a bounded circular domain. Therefore, before we attempt to prove Theorem 1, it seems worthwhile to recall some basic definitions and to list the rudimental properties of the Bergman projection.

## 2. Basic definitions and facts

A circular domain contained in $\mathbf{C}^n$ is a connected open set such that if $z=(z_1, \ldots, z_n)$ is in the set, then for any real number $\theta$, the point $e^{i\theta}z = (e^{i\theta}z_1, \ldots, e^{i\theta}z_n)$ is also in the set.

The Bergman projection $P$ associated to a bounded domain $D$ contained in $\mathbf{C}^n$ is the orthogonal projection of $L^2(D)$ onto its closed subspace $H(D)$ consisting of $L^2$ holomorphic functions. Connected to the projection $P$ is the Bergman kernel function $K(w, z)$. This kernel is determined by the property that

$$P\phi(w) = \int_D K(w, z)\phi(z)\, dV_z$$

for all $\phi$ in $L^2(D)$. The kernel $K(w, z)$ is defined on $D \times D$, and is holomorphic in $w$ and antiholomorphic in $z$, and $K(w, z) = \overline{K(z, w)}$. Proofs of these elementary facts can be found in [2] and [5].

We shall require a lemma which is proved in [1].

LEMMA A. *Suppose that $f: \Omega_1 \to \Omega_2$ is a proper holomorphic mapping between bounded domains $\Omega_1$ and $\Omega_2$ contained in $\mathbf{C}^n$. Let $P_i$ denote the Bergman projection associated to $\Omega_i$, $i = 1, 2$, and let $u = \operatorname{Det}[f']$. The Bergman projections transform according to*

$$P_1(u \cdot (\phi \circ f)) = u \cdot ((P_2\phi) \circ f)$$

*for all $\phi$ in $L^2(\Omega_2)$.*

The classical Remmert proper mapping theorem states that the mapping $f$ in Lemma A is a branched cover of some finite order $m$. Since $|u|^2$ is equal to the real Jacobian determinant of $f$ viewed as a mapping of $\mathbf{R}^{2n}$, it follows from a simple change of variables that

$$\|u \cdot (\phi \circ f)\|_{L^2(\Omega_1)} = m^{1/2} \|\phi\|_{L^2(\Omega_2)}$$

for all $\phi$ in $L^2(\Omega_2)$. This fact will be used at a crucial step in the proof of Theorem 1.

It is well known ([2, 5]) that if the mapping $f$ in Lemma A is biholomorphic, then the Bergman kernel functions transform according to

$$u(w) K_2(f(w), f(z)) \overline{u(z)} = K_1(w, z)$$

where $K_i$ denotes the kernel function associated to $\Omega_i$, $i = 1, 2$.

Finally, before we proceed to prove Theorem 1, it is instructive to take a glance at the original proof of Cartan's theorem. Suppose $f: \Omega_1 \to \Omega_2$ is a biholomorphic mapping between bounded circular domains which contain the origin, and suppose $f(0) = 0$. For each real number $\theta$, the mapping $F_\theta$ defined by

$$F_\theta(z) = f^{-1}(e^{-i\theta} f(e^{i\theta} z)) \tag{2.1}$$

is an automorphism of $\Omega_1$ such that $F_\theta(0) = 0$ and such that the Jacobian matrix $F'_\theta(0)$ is equal to the identity matrix. Therefore, according to Cartan's lemma, the mapping $F_\theta$ is the identity. Now, writing equation (2.1) out in terms of power series reveals that $f$ must be a linear mapping. It is interesting that a very similar argument must be used at a key point in the proof of Theorem 1.

## 3. Proof of Theorem 1

Theorem 1 is a relatively simple consequence of two basic lemmas which we now list.

LEMMA B. *Suppose that $K(w, z)$ is the Bergman kernel function associated to a bounded circular domain $\Omega$. Suppose that $w$ and $z$ are points in $\Omega$ and that $U$ is any connected neighborhood of the unit circle in $\mathbf{C}$ such that $tw$ and $\bar{t}z$ are in $\Omega$ for each $t$ in $U$. Then $K(tw, z) = K(w, \bar{t}z)$ for all $t$ in $U$.*

Let $B_R$ denote the ball in $\mathbf{C}^n$ of radius $R$ centered at the origin.

LEMMA C. *Suppose that $\Omega$ is a bounded circular domain in $\mathbf{C}^n$ which contains the unit ball $B_1$. Let $P$ denote the Bergman projection associated to $\Omega$. For each multi-index $\alpha$, there is a function $\phi_\alpha$ in $C_0^\infty(B_1)$ such that $P\phi_\alpha = z^\alpha$. Furthermore, $\phi_\alpha$ can be chosen so that if $\phi_{\alpha,\varepsilon}$ is defined via $\phi_{\alpha,\varepsilon}(z) = \varepsilon^{-2n-|\alpha|}\phi_\alpha(z/\varepsilon)$, then $P\phi_{\alpha,\varepsilon} = z^\alpha$ if $0 < \varepsilon < 1$.*

We shall now prove Theorem 1, assuming the truth of the lemmas. We may suppose, without loss of generality, that $\Omega_1$ and $\Omega_2$ both contain $\bar{B}_1$, the closure of the unit ball.

Let $K(w, z)$ denote the Bergman kernel function associated to $\Omega_1$. Lemma B has as an important consequence the fact that for $z$ close to the origin, the function $K(w, z)$ extends to be holomorphic in $w$ on a large neighborhood of $\bar{\Omega}_1$. Indeed, if $R$ is a large positive number, then $K(w, z)$ extends holomorphically as a function of $w$ to $B_R$ whenever $z$ is in $B_{1/R}$. This follows from the formula

$$K(w, z) = K\left(\frac{w}{R}, Rz\right)$$

which holds for $(w, z)$ in $B_1 \times B_{1/R}$, and which extends to hold for $(w, z)$ in $B_R \times B_{1/R}$ by analytic continuation.

Now notice that if $\phi_{\alpha,\varepsilon}$ is the function of Lemma C associated to $\Omega_2$, and if $u = \operatorname{Det}[f']$, then Lemma A yields that

$$u \cdot f^\alpha = u \cdot (z^\alpha \circ f) = P_1(u \cdot (\phi_{\alpha,\varepsilon} \circ f)) \tag{3.1}$$

where $P_1$ denotes the Bergman projection associated to $\Omega_1$. We may rewrite (3.1) in integral form:

$$u(w)f(w)^{\alpha} = \int_{\Omega_1} K(w, z)u(z)\phi_{\alpha,\varepsilon}(f(z))\, dV_z. \tag{3.2}$$

Equation (3.2) contains the core of the proof of Theorem 1.

We shall now prove that $u \cdot f^{\alpha}$ is a polynomial for each $\alpha$, including $\alpha = (0, 0, \ldots, 0)$, by showing that the functions $u \cdot f^{\alpha}$ are entire functions which satisfy an estimate of the form $|u(w)f(w)^{\alpha}| \le C\,|w|^q$. Then, since $u$ is a polynomial and $u \cdot f^{\alpha}$ is a polynomial for each $\alpha$, and since the ring of polynomials is a unique factorization domain, we will conclude that $f$ must be a polynomial mapping.

First, notice that if $\varepsilon > 0$ is taken to be very small, the formula (3.2) implies that $u \cdot f^{\alpha}$ extends to be holomorphic in a large neighborhood of $\bar{\Omega}_1$. Indeed, since $f^{-1}(0) = \{0\}$, the nullstellensatz implies that there are holomorphic functions $a_{ij}(z)$ and positive integers $k_i$ such that $z_i^{k_i} = \sum_{j=1}^{n} a_{ij}(z)f_j(z)$ near $z = 0$. Hence, there are positive constants $m$ and $c$ such that $f$ satisfies an estimate of the form $|z|^m \le c\,|f(z)|$ for all $z$ in $\bar{\Omega}_1$. Therefore,

$$\operatorname{Supp}(\phi_{\alpha,\varepsilon} \circ f) \subset \{z : |f(z)| \le \varepsilon\} \subset \{z : |z| \le (c\varepsilon)^{1/m}\}.$$

Hence, if $R$ is a large positive number and if $\varepsilon$ is chosen so that $(c\varepsilon)^{1/m} < 1/R$, formula (3.2) in conjunction with the fact that $K(w, z)$ extends to $B_R \times B_{1/R}$ reveals that $u \cdot f^{\alpha}$ extends to be holomorphic on $B_R$. Therefore, we conclude that $u \cdot f^{\alpha}$ is an entire function.

We must now show that $|u(w)f(w)^{\alpha}| < C\,|w|^q$. Fix a point $w$ in $\mathbf{C}^n$. Pick $\varepsilon$ so that $(c\varepsilon)^{1/m} = |w|^{-1}$, i.e., let $\varepsilon = c^{-1}\,|w|^{-m}$. Note that $\operatorname{supp}(\phi_{\alpha,\varepsilon} \circ f) \subset B_{|w|^{-1}}$ and that

$$\begin{aligned}
\|u \cdot (\phi_{\alpha,\varepsilon} \circ f)\|_{L^2(\Omega_1)} &= (\text{constant})\, \|\phi_{\alpha,\varepsilon}\|_{L^2(\Omega_2)} \\
&= (\text{const.})\, \varepsilon^{-n-|\alpha|}\, \|\phi_{\alpha}\|_{L^2(B_1)} \\
&= (\text{const.})\, |w|^{m(n+|\alpha|)}.
\end{aligned}$$

We now use formula (3.2) and Lemma B to obtain that

$$\begin{aligned}
|u(w)f(w)^{\alpha}| &= \left| \int_{\Omega_1} K\!\left(\frac{w}{|w|}, |w|\, z\right) u \cdot (\phi_{\alpha,\varepsilon} \circ f)\, dV_z \right| \\
&\le (\text{const.}) \left( \operatorname*{Sup}_{\bar{B}_1 \times \bar{B}_1} |K(\zeta, \xi)| \right) \|u \cdot (\phi_{\alpha,\varepsilon} \circ f)\|_{L^2(\Omega_1)} \\
&\le (\text{const.})\, |w|^{m(n+|\alpha|)}
\end{aligned}$$

where the constant is independent of $w$. This completes the proof of Theorem 1.

## 4. Proofs of the lemmas

Lemma B is well known. We shall present a proof for the sake of completeness.

*Proof of Lemma B.* If $\theta$ is a real number, the mapping $\Phi$ defined via $\Phi(z)=e^{i\theta}z$ is an automorphism of the domain $\Omega$. Therefore, the Bergman kernel function $K(w, z)$ satisfies the identity

$$\operatorname{Det}[\Phi'(w)]K(\Phi(w), \Phi(z)) \operatorname{Det}[\overline{\Phi'(z)}]=K(w, z).$$

If we replace $z$ by $e^{-i\theta}z$ in this formula we obtain

$$K(e^{i\theta}w, z)=K(w, e^{-i\theta}z).$$

Now $K(tw, z)$ and $K(w, \bar{t}z)$ are holomorphic functions of $t$ on $U$ which agree on the unit circle. Hence, $K(tw, z)=K(w, \bar{t}z)$ for all $t$ in $U$.

*Proof of Lemma C.* Let $K(w, z)$ denote the Bergman kernel function associated to $\Omega$. We shall use the shorthand notation,

$$K^{\bar{\alpha}}(w, z)=\frac{\partial^\alpha}{\partial\bar{z}^\alpha} K(w, z)$$

and

$$K^{\alpha}(w, z)=\frac{\partial^\alpha}{\partial w^\alpha} K(w, z)$$

for multi-indices $\alpha$. We shall also abreviate the operators $\partial^\alpha/\partial z^\alpha$ and $\partial^\alpha/\partial\bar{z}^\alpha$ by $\partial^\alpha$ and $\bar{\partial}^\alpha$, respectively.

Let $\theta$ be a radially symmetric function in $C_0^\infty(B_1)$ such that $\int\theta=1$. For small $\varepsilon>0$, let $\theta_\varepsilon(z)=\varepsilon^{-2n}\theta(z/\varepsilon)$. Since holomorphic functions assume their average values, it follows that if $h(w)$ is a function in $H(\Omega)$, then

$$\partial^\alpha h(0)=\int_\Omega \partial^\alpha h\overline{\theta_\varepsilon(z)}\, dV_z=\int_\Omega h(-1)^{|\alpha|}\overline{\bar{\partial}^\alpha\theta_\varepsilon}\, dV=\int_\Omega K^\alpha(0, z)h(z)\, dV_z.$$

Therefore, the Bergman projection of $(-1)^{|\alpha|}\bar{\partial}^\alpha\theta_\varepsilon$ is equal to $K^{\bar{\alpha}}(w,0)$ as a function of $w$.

Now suppose that $w$ and $z$ are in $B_1$. If we differentiate the formula $K(tw, z)=K(w, \bar{t}z)$ with respect to $\bar{z}$, we obtain that

$$K^{\bar{\alpha}}(tw, z)=t^{|\alpha|}K^{\bar{\alpha}}(w, \bar{t}z). \tag{4.1}$$

The formula (4.1) holds for all $t$ in the unit disc of $\mathbf{C}$. If we set $z=0$ in (4.1), we see that

$$K^{\bar{\alpha}}(tw, 0) = t^{|\alpha|}K^{\bar{\alpha}}(w, 0).$$

This implies that $K^{\bar{\alpha}}(w, 0)$ is a homogeneous polynomial of degree $|\alpha|$ in $w$.

We now claim that the set of homogeneous polynomials $H^N = \{K^{\bar{\alpha}}(w, 0) : |\alpha| = N\}$ forms a basis for the set of all homogeneous polynomials of degree $N$. Indeed, the functions in $H^N$ are linearly independent because if

$$\sum_{|\alpha|=N} c_\alpha K^{\bar{\alpha}}(w, 0) = 0$$

then $\sum_{|\alpha|=N} \bar{c}_\alpha \partial^\alpha h(0) = 0$ for every $h$ in $H(\Omega)$, which is absurd. Furthermore, the cardinality of $H^N$ is equal to the dimension of the vector space of all homogeneous polynomials of degree $N$. Hence, each monomial $z^\alpha$ can be written in the form

$$z^\alpha = \sum_{|\beta|=|\alpha|} c_\beta K^{\bar{\beta}}(z, 0).$$

Therefore,

$$z^\alpha = P\Big(\sum_{|\beta|=|\alpha|} c_\beta (-1)^{|\beta|}\bar{\partial}^\beta \theta_\varepsilon\Big).$$

If we set $\phi_\alpha = \sum_{|\beta|=|\alpha|} c_\beta (-1)^{|\beta|}\bar{\partial}^\beta \theta$, then the conditions of Lemma C are met.

*Remark.* Formula (3.2) can be used to prove the following generalization of a result of Kaup [4] on biholomorphic mappings between Reinhardt domains.

THEOREM 2. *Suppose $f:\Omega_1 \to \Omega_2$ is a proper holomorphic mapping between bounded circular domains in $\mathbf{C}^n$. Suppose further that $\Omega_2$ contains the origin and that the Bergman kernel function $K(w, z)$ associated to $\Omega_1$ is such that for each compact subset $G$ of $\Omega_1$, there is an open set $U = U(G)$ containing $\bar{\Omega}_1$ such that $K(w, z)$ extends to be holomorphic on $U$ as a function of $w$ for each $z$ in $G$. Then $f$ extends holomorphically to a neighborhood of $\bar{\Omega}_1$.*

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