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Bases for quadratic differentials*

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Let G be a non-elementary finitely generated Kleinian group and q be an integer bigger than one. It is well known (see, for example, Bers [7]) that every cusp form of weight $-2q$ is the Poincaré series of a rational function with poles only on the limit set of G. There are at least three interesting problems all related, but somewhat independent, associated with the spaces of cusp forms for the group G and the Poincaré series operator.

(I) Find necessary and sufficient conditions for the Poincaré series of a rational function to vanish identically.

(II) Construct bases for the spaces of cusp forms that vary holomorphically with moduli.

(III) Construct (finite dimensional) spaces of rational functions that vary holomorphically with moduli so that the Poincaré series operator establishes an isomorphism between these spaces and the spaces of cusp forms for the Kleinian groups.

The first problem is probably the hardest. There are many formal reasons that force a Poincaré series to vanish (see, for example, [19]). A computational algorithm for determining whether or not a Poincaré series vanishes is more difficult to obtain. Deep and interesting work of Hejhal [11], [12], [13] has resulted in one solution to problem (I) for Schottky and Fuchsian groups.

Bers [5] has obtained bases for cusp forms (supported on a single component) for quasi-fuchsian groups. Earle (private communication) has observed that a set of global coordinates for Teichmüller space always leads to bases for quadratic differentials (the case $q=2$) that vary holomorphically with moduli. These are partial solutions to problem (II).

In this paper we are mainly concerned with problem (III). In the first part of the paper, we give an algorithm for determining spaces of rational functions that solve problem (III) for geometrically finite function groups and for $q=2$. These spaces are constructed via the stratifications we have introduced in [18], and

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exploit techniques involving the tangent bundle to the deformation spaces of a Kleinian group.

As a by-product we exhibit large families of rational functions with non-vanishing Poincaré series, and we obtain global holomorphic trivializations of the cotangent bundle of the space of deformations of a finitely generated function group.

Similar (but by no means identical) bases for spaces of quadratic differentials were obtained previously by Bers [8] and Hejhal [11] for Schottky groups. Wolpert [27], [28] has shown that lengths of certain finite sets of geodesics are local coordinates on Teichmüller space. As a consequence he obtains bases for spaces of quadratic differentials for Fuchsian groups. The work of Earle [9] on coordinates for Teichmüller space can also be used to obtain such bases for certain Kleinian groups. Similar constructions enter the unpublished work of Bers and of Earle and Marden on coordinates for the spaces of Riemann surfaces with nodes.

The second part of our paper is an investigation of the simplest geometrically finite Kleinian groups which are not function groups; these are the groups with two components, neither invariant. Special cases of such groups appear in [9]. We show that every such group is a quasiconformal deformation of a $\mathbf{Z}_2$-extension of a Fuchsian group, and we stratify all such groups.

§1. Deformation spaces and fiber spaces

1.1. Let G be a finitely generated non-elementary Kleinian group (that is, G, is a discrete group of Möbius transformations which operates discontinuously at some point of the extended complex plane $\hat{\mathbf{C}}$). As usual, we denote the limit set of G by $\Lambda=\Lambda(G)$ and the region of discontinuity of G by $\Omega=\Omega(G)$. Let Δ be a G-invariant union of (connected) components of Ω.

The Banach space $L^\infty(G,\Delta)$ is the space of L^∞ functions with support in Δ, which satisfy

$$(\mu\circ g)\bar{g}'=\mu g' \quad \text{for all} \quad g\in G. \tag{1}$$

In general, a bounded function on $\hat{\mathbf{C}}$ satisfying (1) is called a *Beltrami differential* (for G).

In the special case that $\Delta=\Omega$, we set $L^\infty(G)=L^\infty(G,\Omega)$.

A point in the open unit ball $M(G,\Delta)\subset L^\infty(G,\Delta)$ is called a *Beltrami coefficient* (for G).

Now let x_1, x_2, x_3 be three distinct points of $\hat{\mathbf{C}}$, and let w be a quasiconformal homeomorphism of $\hat{\mathbf{C}}$. We say that w is *normalized at* (x_1, x_2, x_3) if $w(x_i)=x_i$, $i=1,2,3$. Every quasiconformal homeomorphism w has a well defined Beltrami coefficient $\mu = w_{\bar{z}}/w_z$ (we say that w is *μ-conformal*), and conversely, given a Beltrami coefficient μ; there is a unique μ-conformal homeomorphism w^μ normalized at (x_1, x_2, x_3) (see Ahlfors–Bers [2]).

A quasiconformal homeomorphism w is said to be *compatible* with G provided that for all $g \in G$, $w \circ g \circ w^{-1}$ is a Möbius transformation. In this case, w induces an isomorphism θ_w defined by

$$\theta_w(g) = w \circ g \circ w^{-1}, \qquad g \in G,$$

of G onto another Kleinian group.

For any normalization, w is compatible with G if and only if $w = a \circ w^\mu$ for some Beltrami coefficient μ for the group G and some Möbius transformation a.

If w and w^* are compatible with G and induce the same isomorphism up to conjugation (that is, there is a Möbius transformation a so that $\theta_w = \theta_{a \circ w^*}$) then we say that w and w^* are *equivalent*.

The *deformation space* $T(G, \Delta)$ is defined to be the set of equivalence classes of compatible quasiconformal homeomorphisms which are conformal off Δ. Equivalently, it is $M(G, \Delta)$ factored by the relation: $\mu \sim \mu^*$ if w^μ is equivalent to w^{μ^*}. The map

$$\Phi : M(G, \Delta) \to T(G, \Delta)$$

endows $T(G, \Delta)$ with a topology and complex structure.

It is well known that $T(G, \Delta)$ is a complex manifold (see [6], [15], [21]).

For the case (the primary one that concerns us here) that $\Delta = \Omega$, we set

$$d = d(G) = \dim T(G, \Omega).$$

A finitely generated Kleinian group is called *stratifiable* if there are $d+3$ distinct points $x_1, x_2, x_3, y_1, \ldots, y_d \in \hat{\mathbf{C}}$ so that if we use (x_1, x_2, x_3) to normalize each w^μ, then the mapping

$$\Psi(w^\mu) = (w^\mu(y_1), w^\mu(y_2), \ldots, w^\mu(y_d))$$

defines a biholomorphic embedding of $T(G)$ onto an open subset of $\hat{\mathbf{C}}^d$.

The set $x_1, x_2, x_3, y_1, \ldots, y_d$ is called a *stratification* of G; we also say that $x_1, \ldots, y_d$ *stratify* G.

The starting point in this paper is the main result of our preceding paper.

THEOREM (Kra-Maskit) [18]). *Every non-elementary geometrically finite function group G is stratifiable.*

To establish the above theorem, we use Maskit's description of geometrically finite function groups to obtain generators for G. Appropriate fixed points of these generators appear in the stratification for G (see also §6). We have essentially produced an algorithm for obtaining a stratification of G, once one knows how G is built up from simpler groups.

1.2. Given the Kleinian group G, a point $x \in \hat{\mathbf{C}}$ is called *sturdy* if the following holds. Whenever w, w^* are compatible with G and $\theta_w = \theta_{w^*}$, then $w(x) = w^*(x)$. We remark that limit points are always sturdy; points of Ω which are not fixed points are never sturdy; fixed points of elliptic elements of order >2 are always sturdy; fixed points of elliptic elements of order 2 are sometimes sturdy and sometimes not. We note that if G has exactly two components, then elliptic fixed points of order 2 are sturdy.

1.3. Let G be a non-elementary Kleinian group, and let x_1, x_2, x_3 be distinct sturdy points for G. Let Δ be an invariant union of components of G.

For each $\mu \in M(G, \Delta)$, we set

$$G^\mu = w^\mu G (w^\mu)^{-1},$$
$$\Delta^\mu = w^\mu(\Delta),$$

where w^μ is normalized at (x_1, x_2, x_3). Also, for each $g \in G$, we set

$$g^\mu = w^\mu \circ g \circ (w^\mu)^{-1},$$

and we denote the isomorphism $g \mapsto g^\mu$ by θ_μ instead of θ_{w^μ}.

1.4. We need the *Bers fiber space*

$$F(G, \Delta) = \{(\Phi(\mu), z) \mid \mu \in M(G, \Delta), z \in \Delta^\mu\}.$$

It is easy to see that $F(G, \Delta)$, is a complex manifold of dimension $\dim T(G, \Delta) + 1$.

We denote the projection on the first factor by

$$\pi : F(G, \Delta) \to T(G, \Delta).$$

The group G operates on $F(G, \Delta)$, so as to preserve the fiber of this projection: if $g \in G$,

$$g(\Phi(\mu), z) = (\Phi(\mu), g^{\mu}(z)), \qquad \mu \in M(G, \Delta), \qquad z \in w^{\mu}(\Delta).$$

The quotient space $V(G, \Delta) = F(G, \Delta)/G$ is also a complex manifold of the same dimension, but not necessarily connected ($V(G, \Delta)$ has the same number of components as Δ/G).

If $\Delta = \Omega$, we use the notation:

$$T(G) = T(G, \Omega), \qquad F(G) = F(G, \Omega), \qquad V(G) = V(G, \Omega).$$

§2. Quadratic differentials

2.1. Let G be a non-elementary stratifiable Kleinian group, and let y, x_1, x_2, x_3 be four distinct sturdy points for G. We normalize each w^{μ} at (x_1, x_2, x_3).

We define the function $\varphi_y : F(G) \to \hat{\mathbf{C}}$ as follows (here $\mu \in M(G)$, $z \in w^{\mu}(\Omega)$):

$$\begin{aligned} \varphi_y(\Phi(\mu), z) &= \frac{w^{\mu}(y) - x_1}{(z - x_1)(z - x_2)(z - x_3)(z - w^{\mu}(y))}, \quad \text{if} \quad x_1, x_2, x_3, w^{\mu}(y) \neq \infty; \\ \varphi_y(\Phi(\mu), z) &= \frac{-1}{(z - x_1)(z - x_2)(z - x_3)}, \quad \text{if} \quad w^{\mu}(y) = \infty; \\ &= \frac{w^{\mu}(y) - x_1}{(z - x_1)(z - x_2)(z - w^{\mu}(y))}, \quad \text{if} \quad x_3 = \infty; \\ &= \frac{w^{\mu}(y) - x_1}{(z - x_1)(z - x_3)(z - w^{\mu}(y))}, \quad \text{if} \quad x_2 = \infty; \\ &= \frac{1}{(z - x_2)(z - x_3)(z - w^{\mu}(y))}, \quad \text{if} \quad x_1 = \infty. \end{aligned} \tag{2}$$

Using the well known fact [2] that w^{μ} is holomorphic in μ, one sees at once that φ_y is a holomorphic map $F(G) \to \hat{\mathbf{C}}$.

It is also quite easy to see that φ_y is integrable over each fiber of $F(G)$; that is,

$$\iint_{\Omega^{\mu}} |\varphi_y(\Phi(\mu), z) \, dz \wedge d\bar{z}| < \infty$$

(in fact, the rational function $\varphi_y(\Phi(\mu), \cdot)$ is integrable over all of $\hat{\mathbf{C}}$).

We now define the *Poincaré series operator* Θ by the formula

$$(\Theta\varphi_y)(\Phi(\mu), z) = \sum_{g\in G} \varphi_y(\Phi(\mu), g^\mu(z))\left(\frac{d}{dz}\, g^\mu(z)\right)^2. \tag{3}$$

It is a routine computation to see that this series converges uniformly and absolutely on compact subsets of $F(G)$.

For each fixed μ, the series (3) is a classical Poincaré series on Ω^μ and $\Theta\varphi_y(\Phi(\mu), z)$ is an integrable quadratic differential in z on Ω^μ; that is,

$$(\Theta\varphi_y)(\Phi(\mu), g^\mu(z)) \cdot \left(\frac{d}{dz}\, g^\mu(z)\right)^2 = (\Theta\varphi_y)(\Phi(\mu), z), \tag{4}$$

for all $g \in G$; also

$$\iint_{\Omega^\mu/G^\mu} |(\Theta\varphi_y)(\Phi(\mu), z)\, dz \wedge d\bar{z}| \leq \iint_{\Omega^\mu} |\varphi_y(\Phi(\mu), z)\, dz \wedge d\bar{z}| < \infty. \tag{5}$$

We remark that $\Theta\varphi_y(\Phi(\mu), z)$ is holomorphic even if y or some x_i is an elliptic fixed point in $\Omega(G)$. To see this, we fix μ and renormalize so that $w^\mu(y) = 0$ (or $x_i = 0$), and the maximal elliptic subgroup of G^μ with fixed point at 0 is $\{z \mapsto e^{2\pi i p/q} z,\ p = 0, \ldots, q-1\}$. We let H be the corresponding subgroup of G and we write the sum in (3) as first a sum over H, then a sum over G/H. It suffices to show that the sum over H is regular at 0. We rewrite (2) in its partial fraction decomposition, and note that the coefficient of $1/z$ for $\varphi_y(\Phi(\mu), z)$ is $1/x_2x_3$ (assuming $x_2 \neq \infty \neq x_3$). Then the sum over H of the singular terms reduces to

$$\frac{1}{x_2x_3} \sum_{p=0}^{q-1} e^{2\pi i p/q} = 0.$$

For each $\mu \in M(G)$, we let $Q(G^\mu)$ be the space of holomorphic integrable quadratic differentials for G^μ (that is, the space of holomorphic functions on $\Omega(G^\mu)$ satisfying (4) and (5)). We denote the dimension of this space by d (of course, d is independent of μ); it is well known that $d = \dim T(G)$.

Our main result is the following:

THEOREM 1. *Let G be a stratifiable Kleinian group with stratification $x_1, x_2, x_3, y_1, \ldots, y_d$. Then for each $\mu \in M(G)$, the d functions*

$$\Theta\varphi_{y_1}(\Phi(\mu), z), \ldots, \Theta\varphi_{y_d}(\Phi(\mu), z),$$

defined on $\Omega(G^\mu)$, form a basis (over $\mathbf{C}$) for $Q(G^\mu)$.

As remarked before, for fixed μ the restriction of Θ to $\Omega(G^\mu)$ is the classical Poincaré Θ operator. In this situation, we will write $\varphi_y(z)$ for $\varphi_y(\Phi(\mu), z)$, etc.

2.2. We now fix μ, and consider the space R^μ of rational functions $f(z)$, where the poles of f are all simple and occur only at some of the points: $\{x_1, x_2, x_3, w^\mu(y_1), \ldots, w^\mu(y_d)\}$; we require further that $f(z) = O(|z|^{-4})$ near ∞, if ∞ is not one of these points, and we require $f(z) = O(|z|^{-3})$ if ∞ is one of these points. One easily sees that the vector space R^μ has dimension d.

Our theorem asserts that the Poincaré series operator establishes an isomorphism between R^μ and $Q(G^\mu)$.

Since $\{x_1, \ldots, y_d\}$ is a stratification for G, the space R^μ depends, not on μ, but on $\Phi(\mu) \in T(G)$. Then R^μ and $Q(G^\mu)$ are the fibers of trivial d-dimensional vector bundles over $T(G)$.

2.3. We again return to the case that Δ is an invariant union of components of G. We let $Q(G, \Delta)$ be the subspace of $Q(G)$ consisting of those quadratic differentials supported on Δ.

For $\varphi \in Q(G, \Delta)$ and $\mu \in L^\infty(G, \Delta)$, we introduce the pairing

$$\langle \varphi, \mu \rangle = \tfrac{1}{2} \iint_{\Delta/G} \varphi(z)\mu(z)\,|dz \wedge d\bar{z}|,$$

and we set

$$Q(G, \Delta)^\perp = \{\mu \in L^\infty(G, \Delta) \mid \langle \varphi, \mu \rangle = 0 \text{ for all } \varphi \in Q(G, \Delta)\}.$$

The pairing gives us a canonical isomorphism between the dual space $Q(G, \Delta)^*$ of $Q(G, \Delta)$ and $L^\infty(G, \Delta)/Q(G, \Delta)^\perp$ (see [14, Chapter III]).

It is well known (see for example [16] and the literature quoted there) that the tangent space to $T(G, \Delta)$ at $\Phi(\mu)$ is canonically isomorphic to $Q(G^\mu, \Delta^\mu)^*$, and (using the pairing) the cotangent space is (canonically isomorphic to) $Q(G^\mu, \Delta^\mu)$.

2.4. We now proceed to the proof of our theorem. For ease of computation we assume first that $x_3 = \infty$. It was shown by Ahlfors and Bers [2] that for fixed $\mu \in M(G, \Delta)$, for $t^{-1} > \|\mu\|_\infty$, and for fixed $z \neq x_1, x_2, x_3$, the function $t \mapsto w^{t\mu}(z)$ is a holomorphic function of t. They further showed that for $x_3 = \infty$,

$$\frac{d}{dt} w^{t\mu}(z)\Big|_{t=0} = \frac{1}{2\pi i}(z - x_1)(z - x_2) \iint_\Omega \frac{\mu(\zeta)\, d\zeta \wedge d\bar{\zeta}}{(\zeta - z)(\zeta - x_1)(\zeta - x_2)}. \tag{6}$$

We fix $\mu \in M(G)$ and we choose $\nu_1, \ldots, \nu_d \in L^\infty(G^\mu)$ so that $\nu_1, \ldots, \nu_d$ forms a basis for $L^\infty(G^\mu)/Q(G^\mu)^\perp$.

We return to our map $\Psi: T(G) \to \hat{\mathbf{C}}^d$, since $x_3 = \infty$, $\Psi: T(G) \to \mathbf{C}^d$, and we need to study the differential $D\Psi$ at the point $\Phi(\mu)$. We consider a point $\Psi(\Phi(\mu)) \in \mathbf{C}^d$, and we write $\Psi_1, \ldots, \Psi_d$ as natural parameters on $\mathbf{C}^d$. Regarding $L^\infty(G)/Q(G^\mu)^\perp$ as the tangent space to $T(G)$, an arbitrary tangent vector to $T(G)$ at $\Phi(\mu)$ can be written as $\sum_{i=1}^d z_i \nu_i$ with $z_i \in \mathbf{C}$. Then formula (6) yields

$$\frac{\partial}{\partial z_k}(\Psi_j(\Phi(\mu)))\left(\sum z_i \nu_i\right)\Big|_{z=0}$$

$$= \frac{\partial}{\partial z} w^{z\nu_k}(w^\mu(y_j))|_{z=0}$$

$$= \frac{1}{2\pi i}(w^\mu(y_j) - x_1)(w^\mu(y_j) - x_2) \iint_{\Omega^\mu} \frac{\nu_k(\zeta)\, d\zeta \wedge d\bar{\zeta}}{(\zeta - x_1)(\zeta - x_2)(\zeta - w^\mu(y_j))}$$

$$= \frac{1}{2\pi i}(w^\mu(y_j) - x_2) \iint_{\Omega^\mu} \nu_k(\zeta)\varphi_{y_j}(\Phi(\mu), \zeta)\, d\zeta \wedge d\bar{\zeta}$$

$$= \frac{1}{2\pi i}(w^\mu(y_j) - x_2)\langle \nu_k(\cdot), \Theta\varphi_{y_j}(\Phi(\mu), \cdot)\rangle.$$

We showed in [18] that Ψ is a holomorphic injection (onto an open set); hence $D\Psi$ is a linear isomorphism. Since the ν_k span the tangent space to $T(G)$ at $\Phi(\mu)$, the functions $\Theta\varphi_{y_j}(\Phi(\mu), \cdot)$ must span the cotangent space. We have shown that for every μ, the functions $\Theta\varphi_{y_j}(\Phi(\mu), \cdot)$, $j = 1, \ldots, d$, are linearly independent.

In order to complete the proof of our theorem, it remains only to remove the restriction $x_3 = \infty$. We do this by choosing a Möbius transformation a with $a(x_1) = x_1$, $a(x_2) = x_2$, and $a(\infty) = x_3$. An easy computation shows that there is a constant c, depending only on a, so that the quadratic differentials $\Theta\varphi_{y_1}, \ldots, \Theta\varphi_{y_d}$ for G, and $\Theta\varphi_{a(y_1)}, \ldots, \Theta\varphi_{a(y_d)}$ for aGa^{-1} are related by

$$\Theta\varphi_{a(y_i)}(\Phi(\nu), a(z))(a'(z))^2 = c\Theta\varphi_{y_i}(\Phi(\mu), z),$$

where $\mu(z)a'(z) = \nu(az)\overline{a'(z)}$, $z \in \Omega^\mu$.

2.5. For any finitely generated Kleinian group, $T(G)$ is a domain in $\mathbf{C}^n$ (see [18]), hence the cotangent space is trivial. We have shown the following.

COROLLARY 1. *Let G be a stratifiable Kleinian group, with* $\dim T(G) = d$. *Then there are d holomorphic functions $\varphi_1, \ldots, \varphi_d$ on $F(G)$, so that $\Theta\varphi_1, \ldots, \Theta\varphi_d$ exhibits a global holomorphic trivialization of the cotangent bundle of $T(G)$.*

2.6. We now turn to the case where we have selected an invariant union of components Δ. Since $T(G, \Delta)$ is a submanifold of $T(G)$, the cotangent space of $T(G, \Delta)$ is a subspace of the cotangent space of $T(G)$. This yields the following.

COROLLARY 2. *Let G be a stratifiable Kleinian group and let Δ be an invariant union of components. Let $r = \dim T(G, \Delta)$. Then there are r functions $\psi_1, \ldots, \psi_r$ meromorphic on $F(G, \Delta)$, so that for any $\mu \in M(G, \Delta)$, the functions $\Theta\psi_1, \ldots, \Theta\psi_r$ form a basis for $Q(G^\mu, \Delta^\mu)$ and vanish identically on $\Omega^\mu \setminus \Delta^\mu$.*

§3. Fuchsian groups

3.1. Let Γ be a finitely generated Fuchsian group of the first kind operating on the upper half plane U (and on the lower half plane U^*). We assume that $d = \dim T(\Gamma, U) > 0$. We showed in [18] that Γ can be stratified by real points, $x_1, x_2, x_3, y_1, \ldots, y_{2d}$, and that the Fuchsian groups in $T(\Gamma)$ are precisely those points for which the $2d$ coordinates $w^\mu(y_i)$ are all real.

One easily sees that if Γ^μ is Fuchsian, and $w^\mu(y_j)$ are all real, then for each j,

$$\Theta\varphi_{y_j}(\Phi(\mu), z) = \overline{\Theta\varphi_{y_j}(\Phi(\mu), \bar{z})}.$$

We restate our main result in this case as follows.

THEOREM 2. *Let Γ be a finitely generated Fuchsian group of the first kind, with* $\dim T(\Gamma, U) = d$. *Let $x_1, x_2, x_3, y_1, \ldots, y_{2d}$ be a real stratification for Γ. Let $\mu \in M(\Gamma)$ be such that Γ^μ is Fuchsian. Then the functions $\Theta\varphi_{y_1}, \ldots, \Theta\varphi_{y_{2d}}$* (a) *commute with complex conjugation,* (b) *form a basis over* $\mathbf{C}$ *for $Q(\Gamma^\mu)$, and* (c) *form a basis over* $\mathbf{R}$ *for $Q(\Gamma^\mu, U)$.*

We remark that conclusion (a) is false if Γ^μ is not Fuchsian; conclusion (b) is true even if Γ^μ is quasifuchsian. Nothing is known about conclusion (c) if Γ^μ is not Fuchsian.

3.2. Our main theorem asserts that certain sets of Poincaré series form a basis for quadratic differentials; in particular, these Poincaré series do not identically vanish.

CORROLLORY 3. *Let Γ be a finitely generated Fuchsian group of the first kind, and let x_1, x_2, x_3, y be part of a stratification for Γ. Then the Poincaré series $\Theta\varphi_y(0, \cdot)$ does not vanish identically in U or in U^*.*

We showed in [18], that every elliptic fixed point can be made part of a stratification of a Fuchsian group. Hence we obtain:

COROLLORY 4. *Let y be an elliptic fixed point of the Fuchsian group of the first kind Γ. Then there are real points x_1, x_2, x_3 so that the Poincaré series $\Theta\varphi_y(0, \cdot)$ does not vanish identically in both U and U^*.*

One often can choose both fixed points of an elliptic element as part of a stratification set.

COROLLORY 5. *Let x_1, x_2, x_3, y be part of a stratification set for the Fuchsian group of the first kind Γ, where $\{x_1, x_2, x_3, y\}$ is invariant under complex conjugation. Then the Poincaré series $\Theta\varphi_y(0, \cdot)$ does not vanish identically in either U or U^*.*

Remarks. (1) Let Γ be a torsion free Fuchsian group operating on U such that U/Γ is a compact surface of genus $g \geq 2$. In [27], Wolpert constructs $6g-6$ Poincaré series that form a basis for $Q(\Gamma, U)$ over $\mathbf{R}$. Further, by using the Petersson scalar product and a geometric interpretation of the Poincaré series he constructs, he is able to select $3g-3$ series that form a basis over $\mathbf{C}$.

(2) In [13], Hejhal studies Poincaré series of rational functions that are invariant under conjugation. He produces finite spanning sets for $Q(\Gamma)$ (and hence also $Q(\Gamma, U)$) that are not necessarily linearly independent. His methods are not limited to quadratic differentials, but also work for q-differentials, $q > 2$.

§4. Other applications

4.1. If G is a stratifiable function group with invariant component Δ, where $T(G) = T(G, \Delta)$ (for example G might be a Schottky group, or G might be such that all the components other than Δ are thrice punctured spheres), then a stratification gives us global coordinates for $T(G)$, and Theorem 1 gives us a global holomorphic trivialization of the cotangent space. In the special case that Δ is simply-connected, we get global coordinates for the Teichmüller space (every Teichmüller space can be so realized; see Maskit [24]).

4.2. We also remark that if G is as in the preceding section, we can achieve the same result with functions which are not necessarily rational. To this end, we let Γ be the Fuchsian model of G; that is, there is a holomorphic projection $h: U \to \Delta$, where for each $\gamma \in \Gamma$, there is a $g \in G$ so that $h \circ \gamma = g \circ h$. This projection map induces a holomorphic covering $T(\Gamma, U) \to T(G, \Delta)$, (see [6], [15], [21]) and this map can be further extended to a fiber preserving holomorphic covering $H: F(\Gamma, U) \to F(G, \Delta)$ (see, for example, [10]).

The map H induces a map H_* from functions on $w^\mu(\Delta)$ to functions on $w^\sigma(U)$ (where μ and σ are appropriately related). If we transform functions as quadratic differentials (that is, $H_*\varphi(z) = \varphi(h(z))h'(z)^2$), then, denoting the Poincaré series operator for G by Θ_G, an easy computation shows that $\Theta_G \circ H_* = H_* \circ \Theta_\Gamma$. We choose $\varphi_1, \ldots, \varphi_d$ so that $\Theta_\Gamma\varphi_1, \ldots, \Theta_\Gamma\varphi_d$ form a basis for $Q(\Gamma, U)$. Then $\Theta_G H_*\varphi_1, \ldots, \Theta_G H_*\varphi_d$ form a basis for $Q(G)$, and conversely a basis for $Q(\Gamma, U)$ can be obtained from a basis for $Q(G)$.

§5. Extensions of Fuchsian groups of the second kind

5.1. In this and the next section we give stratifications for all $\mathbf{Z}_2$-extensions of quasifuchsian groups of the first kind. In this section we focus on certain groups which are $\mathbf{Z}_2$-extensions of Fuchsian groups of the second kind.

Let Γ be a finitely generated, non-elementary Fuchsian group of the second kind. There are standard generators for Γ of the form $A_1, B_1, \ldots, A_g, B_g, E_1, \ldots, E_n, F_1, \ldots, F_m$, where the A_i, B_i and F_j are hyperbolic and the E_j are elliptic or parabolic. The signature of Γ is $(g, n, m; \nu_1, \ldots, \nu_n)$ where ν_i is the order of E_i if E_i is elliptic, and $\nu_i = \infty$ if E_i is parabolic. Since Γ is of the second kind, $m > 0$. The defining relations for Γ are

$$\prod_{j=1}^{g} [A_j, B_j] \circ \prod_{k=1}^{n} E_k \circ \prod_{l=1}^{m} F_l = 1,$$

$$E_k^{\nu_k} = 1, \quad \text{if} \quad \nu_k < \infty, \quad k = 1, \ldots, n,$$

where as usual

$$[A_j, B_j] = A_j \circ B_j \circ A_j^{-1} \circ B_j^{-1}.$$

One easily sees that viewing Γ as a Kleinian group, $\Omega(\Gamma)/\Gamma$ is a surface of genus $2g + m - 1$ with $2n$ distinguished points on it. Hence $\dim T(\Gamma) = 3(2g + m) - 6 + 2n$.

5.2. We want to adjoin square roots f_p of the elements F_p, where f_p interchanges upper and lower half planes. We do this as follows.

We choose some $F_p (1 \le p \le m)$ and we let I_p denote the axis of F_p in U. We denote reflection in the real axis by j, and set

$$C_p = I_p \cup j(I_p) \cup \{\text{fixed points of } F_p\}.$$

The circle C_p bounds two discs; one of them, call it D_p is precisely invariant under F_p in Γ (that is, if $A \in \Gamma$ is not a power of F_p then $A(D_p) \cap D_p = \varnothing$).

Since F_p is hyperbolic, there are two distinct Möbius transformations whose square is F_p. One square root is hyperbolic and it preserves D_p; call the other f_p.

We let $\Gamma_0 = \Gamma$, and for $p = 1, \ldots, m$, we define Γ_p to be generated by Γ_{p-1} and f_p. The construction of Γ_p from Γ_{p-1} and f_p requires a version of Combination Theorem II (see, for example, [22]) which is unfortunately not in print. We briefly outline the proof here.

We let ω_p be a fundamental domain for Γ_p (acting on $\Omega(\Gamma_p)$), where $\omega_p \cap D_{p+1}$ is a fundamental domain for the action of $\langle F_{p+1} \rangle$ on D_{p+1}, $p = 0, 1, \ldots, m-1$. We denote the complement of D_p by $\hat{D}_p$.

LEMMA. *For $p = 0, \ldots, m-1$, the group Γ_{p+1} is discrete; $\hat{D}_{p+1} \cap \omega_p$ is a fundamental domain for Γ_{p+1}; the relations in Γ_{p+1} are the relations of Γ_p together with $f_{p+1}^2 = F_{p+1}$.*

Proof. Any element of Γ_{p+1} can be written as $A = A_{k+1} \circ f_{p+1}^{\pm 1} \circ A_k \circ \cdots \circ f_{p+1}^{\pm 1} \circ A_1$, where A_1, A_{k+1} might be trivial, but otherwise no A_i is a power of F_{p+1} (there is also the easy case $A = f_{p+1}^{\pm 1} \circ F_{p+1}^l$). Let z be a point of $\hat{D}_{p+1} \cap \omega_p$. Then $A_1(z) \notin \omega_p$ (if $A_1 \ne 1$) and $A_i(z) \in \hat{D}_{p+1}$; $f_{p+1}^{\pm 1} \circ A_1(z) \notin \hat{D}_{p+1}$, $A_2 \circ f_{p+1}^{\pm 1} \circ A_1(z) \in \hat{D}_{p+1} \setminus \omega_p$, etc.

We have shown that no two distinct points of $\omega_p \cap \hat{D}_{p+1}$ are equivalent under any word of the above form; hence Γ_{p+1} is discrete, and the relations are as stated. The remainder of the proof that $\hat{D}_{p+1} \cap \omega_p$ is a fundamental domain is standard (see for example [22]).

We will later need a similar version of Combination Theorem II, where instead of $f_p^2 = F_p \in G_p$, we have $f_p^2 = 1$. The proof is almost identical and is left to the reader.

Looking at the identifications of the sides of the fundamental domains, we see that for $p < m$, $\Omega(\Gamma_p)$ is connected and the surfaces $\Omega(\Gamma_p)/\Gamma_p$ are all surfaces of the same conformal type $(2g+m-1, 2n)$.

However $\Omega(\Gamma_m)$ has two components U and U^*, but $\Omega(\Gamma_m)/\Gamma_m$ is still just one surface of the same conformal type $(2g+m-1, 2n)$. In particular the spaces $T(\Gamma_p)$ all have the same dimension.

5.3. THEOREM 3. *The Kleinian groups Γ_p are all stratifiable.*

Proof. For $p<m$, these are geometrically finite function groups and so there is nothing to prove [18].

We now assume that $p=m$, and notice that the two components of Γ_m are simply connected. Hence $T(\Gamma_m)$ is simply connected and so for every element γ of Γ_m, $\operatorname{tr} w\circ\gamma\circ w^{-1}$, the trace of $w\circ\gamma\circ w^{-1}$ is well defined on $T(\Gamma_m)$ once we have chosen $\operatorname{tr}\gamma$.

There are several cases to consider.

Case I. $g>0$.

For $i=1,\ldots,m$, let $z_{i,1}$ and $z_{i,2}$ be the attractive and repulsive fixed points, respectively of F_i.

For $i=1,\ldots,g-1$, we let $x_{i,1},\ldots,x_{i,6}$, be respectively, the attractive fixed point of A_i, the repulsive fixed point of A_i, the attractive fixed point of B_i, the repulsive fixed point of B_i, $A_i(z_{1,1})$, and $B_i(z_{1,1})$.

We let C be the commutator $[A_g, B_g]$; we set $u_1=A_g$ (attractive fixed point of C), $u_2=A_g$ (repulsive fixed point of C), $u_3=B_g$ (attractive fixed point of C).

For $i=1,\ldots,n$, we let $y_{i,1}$, $y_{i,2}$ be the fixed points of E_i if E_i is elliptic; if E_i is parabolic, we let $y_{i,1}$ be the fixed point of E_i, and $y_{i,2}=E_i(z_{1,1})$.

Finally for $i=1,\ldots,m$, we set $z_{1,3}=f_i(u_1)$.

We note that we have defined $6g-6+3+2n+3m=\dim T(\Gamma_m)+3$ complex parameters.

Let w be some deformation in $T(\Gamma_m)$. We need to show that the parameters $w(x_{1,1}),\ldots,w(z_{m,3})$ determine the generators $w\circ A_1\circ w^{-1},\ldots,w\circ f_m\circ w^{-1}$ of $w\Gamma_m w^{-1}$. By changing the origin of the deformation space, this and all similar arguments in subsequent cases, are reduced to showing that the parameters $x_{1,1},\ldots,z_{m,3}$ determine the generators $A_1,\ldots,f_m$ of Γ_m.

Obviously $A_1, B_1,\ldots,A_{g-1}, B_{g-1}, E_1,\ldots,E_n, f_1,\ldots,f_m$ are all determined. Hence C is determined; the choice of $\operatorname{tr} C^{-1}$ together with u_1, u_2, u_3, determine A_g and B_g [18].

Case II. $g=0$, $n\geq 2$.

For $i=1,\ldots,m$, we define $z_{i,1}$ and $z_{i,2}$ as above. Then for $i=1,\ldots,n-2$, we define $y_{i,1}$ and $y_{i,2}$ as above, and we let y be a fixed point of E_{n-1}. Finally, for $i=1,\ldots,m$, we define $z_{i,3}=f_i(y)$.

The elements $E_1,\ldots,E_{n-2}, f_1,\ldots,f_m$ are all determined. Hence $E_{n-1}\circ E_n$ is determined. It was shown in [18] that $E_{n-1}\circ E_n$, together with y determine E_{n-1} and E_n.

Case III. $g=0$, $n=1$.

We remark that in this case $m \geq 2$, and we choose our parameters as follows. Let y be a fixed point of E_1. For $i = 1, \ldots, m-1$, let $z_{i,1}$, $z_{i,2}$, $z_{i,3}$ be, respectively, the attractive fixed point of F_i, the repulsive fixed point of F_i, and $f_i(y)$. Let z_m be the attractive fixed point of F_m.

We have defined $3m-1 = \dim T(\Gamma_m)+3$ parameters. We see at once that $f_1, \ldots, f_{m-1}$ are determined. We normalize Γ_m so that $z_m = \infty$, $y=0$; we write $F_1 \circ \cdots \circ F_{m-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad-bc=1$, $E_1 = \begin{pmatrix} K & 0 \\ p & K^{-1} \end{pmatrix}$, $f_m = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}$, $|\alpha|>1$. Since $T(\Gamma_m)$ is simply connected, we can choose K, a, b, c, d so that in $SL(2;\mathbf{C})$

$$\begin{pmatrix} K & 0 \\ p & K^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} \alpha^{-1} & -\beta \\ 0 & \alpha \end{pmatrix}.$$

This yields

(1) $Ka\alpha = \alpha^{-1}$,
(2) $Ka\beta + Kb\alpha^{-1} = -\beta$,
(3) $\alpha(pa + K^{-1}c) = 0$.

Since $a\alpha \neq 0$, we can solve (3) for p. Equation (1) yields α up to sign. We can solve (2) to obtain $\beta = -(Kb/1+aK)\alpha^{-1}$. Hence f_m is determined in $PSL(2;\mathbf{C})$ (note that $1+aK \neq 0$; since otherwise $b=0$ which is impossible).

Case IV. $g=0$, $n=0$.

In this case $m \geq 3$. Let y_1 be the repulsive fixed point of F_1. For $i = 2, \ldots, m-1$, we let $z_{i,1}$, $z_{i,2}$, $z_{i,3}$ be, respectively, the attractive fixed point of F_i, the repulsive fixed point of F_i, $f_i(y_1)$. Let y_2 be the repulsive fixed point of F_m, and let $y_3 = f_1(y_2)$.

We look at a deformation of Γ_m and we note that $f_2, \ldots, f_{m-1}$ are determined by the $z_{i,j}$. We normalize Γ_m so that $y_1=0$, $y_2=\infty$, $y_3=1$. Then we can write $F_2 \circ \cdots \circ F_{m-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad-bc=1$, $f_1 = \begin{pmatrix} K & 0 \\ K & K^{-1} \end{pmatrix}$, $|K|>1$, $f_m = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}$, $|\alpha|< 1$, where a, b, c, d are known, and the choice is made so that

$$\begin{pmatrix} K & 0 \\ K & K^{-1} \end{pmatrix}\begin{pmatrix} K & 0 \\ K & K^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} \alpha^{-1} & -\beta \\ 0 & \alpha \end{pmatrix}$$

(that is, we choose a, b, c, d, K, α, β for the original Γ_m so that this relation holds;

then we can regard a, b, c, d, K, α, β as functions on $T(\Gamma_m)$, where this relation continues to hold; since $T(\Gamma_m)$ is simply connected, the functions are well defined).

From routine calculation we obtain the following four equations:

(1) $K^2 a\alpha = \alpha^{-1}$,

(2) $K^2 a\beta + K^2 b\alpha^{-1} = -\beta$,

(3) $(K^2+1)a\alpha + K^{-2}c\alpha = 0$,

(4) $(K^2+1)(a\beta + b\alpha^{-1}) + K^{-2}(c\beta + d\alpha^{-1}) = \alpha$.

Since $\alpha \neq 0$, we can solve (3) for K^2 and obtain

$$K^2 = -\tfrac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{c}{a}}.$$

Since $|K| > 1$, we can never have $c/a = \frac{1}{4}$, hence on $T(\Gamma_m)$ there is a unique solution for K^2, and equally well for K.

We then solve (1) for α^2 and (2) for β in terms of α^{-1}; hence as above, we can solve for α and β.

§6. Global coordinates of Teichmüller spaces II: Earle slices

6.1. Let G be a finitely generated Kleinian group with exactly two components, neither of them invariant. We shall see (Lemma 6.3) that such a group must be a $\mathbf{Z}_2$-extension of a finitely generated quasifuchsian group of the first kind. It was remarked by Earle [9] that $T(G)$, the deformation space of G, is in this case the Teichmüller space of Δ_0/G_0, where

Δ = one of the components of $\Omega(G)$,

G_0 = stabilizer of Δ in G, and

$\Delta_0 = \{z \in \Delta;\ z$ is not a fixed point of an elliptic element of $G_0\}$.

In this section we shall prove the following.

THEOREM 4. *Let G be a finitely generated Kleinian group with two components and $\Omega(G)/G$ connected. Then the group G is stratifiable.*

6.2. The results of the next two sections, while apparently obvious, have, to the best of our knowledge, never appeared in print. The proofs seem to require deep results. For the convenience of the reader, we include complete details.

LEMMA. *Let G_0 be a finitely generated quasifuchsian group with components Δ and Δ^*. Then there exists a unique extremal quasireflection $J:\hat{\mathbf{C}}\to\hat{\mathbf{C}}$ which commutes with every element of G_0 and which maps Δ onto Δ^*.*

Proof. As usual let U be the upper half-plane and $f:\Delta\to U$, a Riemann map. Define the finitely generated Fuchsian group F_0 of the first kind by $F_0=fGf^{-1}$. Let U^* be the lower half-plane and let $j(z)=\bar{z}$. We let $f^*:U^*\to\Delta^*$ be the unique Teichmüller (extremal) mapping that induces the isomorphism

$$F_0\ni\gamma\mapsto f\circ\gamma\circ f^{-1}\in G_0.$$

Define

$$J(z)=\begin{cases} f^*\circ j\circ f(z), & z\in\Delta\cup\Lambda(G),\\ f^{-1}\circ j\circ(f^*)^{-1}(z), & z\in\Delta^*\cup\Lambda(G).\end{cases}$$

It is easy to see that J commutes with every element of G_0. Since J preserves the fixed points of elements of G_0, J is the identity on $\Lambda(G)$. Hence J is a global quasireflection. Further J has minimal maximal dilatation among all quasireflections commuting with G_0, and J is the unique quasireflection with these properties.

We shall call J the *extremal quasireflection* for G_0.

6.3. LEMMA. *Let G be an arbitrary finitely generated Kleinian group with two components, neither of them invariant. Then there exists a $\mathbf{Z}_2$-extension F of a finitely generated Fuchsian group of the first kind so that G is a quasiconformal deformation of F.*

Proof. Let Δ and Δ^* be the two components of G, and let G_0 be the stabilizer of Δ (therefore also of Δ^*). We conclude that G_0 is a finitely generated quasifuchsian group of the first kind (see, for example, [20] or [17]) and that G is a $\mathbf{Z}_2$-extension of G_0.

Let g be some element of $G\backslash G_0$. Let J be the extremal quasireflection for G_0. Since J is unique, $g\circ J\circ g^{-1}=J$; that is, g commutes with J. Hence $(g\circ J)^2=g^2\in G_0$; that is, $g\circ J$ acts as an orientation reversing (quasiconformal) involution on Δ/G_0. It is classical (see, for example, [4]) that there is a $\mathbf{Z}_2$-extension F' of a finitely generated Fuchsian group of the first kind F_0 by an orientation reversing conformal self-map of U which topologically uniformizes Δ/G', where G' is the group generated by G_0 and $g \circ J$. Thus there exists a quasiconformal map $f : \Delta \to U$ that conjugates G' to F'. For $z \in \Delta^*$, we define $f(z) = j \circ f \circ J(z)$, and observe that f extends to a global quasiconformal homeomorphism which conjugates G' onto F' [1]. Obviously f conjugates J to j and G_0 to F_0, and so $f \circ g \circ f^{-1} = (f \circ g \circ J \circ f^{-1}) \circ (f \circ J \circ f^{-1})$ is a fractional linear transformation interchanging U and U^*, where $(f \circ g \circ f^{-1})^2 \in F_0$. Finally, we let F be the group generated by F_0 and $f \circ g \circ f^{-1}$.

6.4. Using Lemma 6.3, we conclude that to prove Theorem 4, it suffices to assume that G is a $\mathbf{Z}_2$-extension of a finitely generated Fuchsian group G_0 of the first kind acting on U, and that the extra generator of G interchanges U and U^*. We shall assume that we are in a slightly more general situation. We are studying extensions G of non-elementary finitely generated Fuchsian groups of the first or second kind G_0 acting on U by an element g_0 that maps U onto U^*. We let $g \in G \setminus G_0$, and we form the group G' generated by G_0 and $j \circ g$. Then G' acts as a group of conformal and anti-conformal automorphisms of U, G' is isomorphic to G, and G' is independent of our choice of the element $g \in G \setminus G_0$. Furthermore $j \circ g$ induces an anti-conformal involution J on $S = U/G_0$.

It is classical that S/J is a surface, perhaps non-orientable, of some genus, with some number of boundary curves and some number of cross-caps. We give this a precise statement, and for the convenience of the reader, we include a proof.

There is a unique closed orientable surface $\bar{S}$ which conformally contains S; the difference $\bar{S} \setminus S$ is a finite set of parabolic punctures.

LEMMA. *There is a finite set of simple disjoint loops $w_1, \ldots, w_s$ on $\bar{S}$ with the following properties.*

(1) *The loops $w_1, \ldots, w_s$ divide $\bar{S}$ into two subsurfaces; J interchanges these two subsurfaces and keeps each w_i invariant.*
(2) *For each i, J either fixes every point of w_i, or has no fixed point on w_i.*
(3) *If J fixes every point of w_i, then w_i may pass through some elliptic ramification points or parabolic punctures. Off these punctures, w_i is a geodesic on $S_0 = S \setminus \{\text{ramification points}\}$.*
(4) *If J has no fixed point on w_i, then w_i is a smooth geodesic on S_0, (and doesn't pass through any elliptic or parabolic punctures).*
(5) *Every fixed point of J is a point of some w_i.*

Proof. We look at the set of fixed points in U of elements of G'; these consist of fixed points of elliptic elements of G_0 and fixed axes of orientation reversing elements of G'. A point of intersection of two or more of these axes is necessarily an elliptic fixed point. Looking at all these axes near an elliptic fixed point; one sees that they project onto a simple path on S. We conclude that the projection of the fixed axes is a set of simple disjoint paths.

We next show that if A is the line of fixed points of the element $g \in G'$; then either A is the axis of a hyperbolic element of G_0, or both endpoints of A are parabolic fixed points. Let x and y be the endpoints of A, and assume x is not a parabolic fixed point. Then [3] x is a point of approximation for G_0 and so there is a sequence g_n of distinct elements of G_0, with $g_n(x) \to x'$, $g_n(y) \to y' \neq x'$. Since G' is discrete, we must have $g_n(x) = x'$, $g_n(y) = y'$ for almost all n; that is, for fixed m and n sufficiently large $g_n \circ g_m^{-1}(x) = x$, $g_n \circ g_m^{-1}(y) = y$.

We have shown that the projection to $\bar{S}$ of the set of fixed points of reflections in G' is a set $w_1, \ldots, w_q$ of simple disjoint loops.

By looking at paths connecting these loops one easily sees that $w_1, \ldots, w_q$ divides S into at most two surfaces.

If $w_1, \ldots, w_q$ does not divide S then there is a homotopically non-trivial loop v on S, where v is disjoint from all w_i and the element of G_0 corresponding to v is hyperbolic (this follows easily from the fact that G_0 is non-elementary). Let h be some hyperbolic element of G_0 whose axis is disjoint from all reflection axes in G', and let r be some reflection in G'. Normalize G' so that $r(z) = -\bar{z}$, and $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$, a, b, c, d real; then $-bc > 0$, and $|a+d| > 2$. Observe that

$$(h \circ r)^2 = \begin{pmatrix} a^2 - bc & b(d-a) \\ c(a-d) & d^2 - bc \end{pmatrix}.$$

If this were the identity, then we would have $a = d$, $-bc > 0$ and $a^2 - bc = 1$, so that $|a+d| < 2$. We have shown that if $w_1, \ldots, w_q$ does not divide S, then S/J is non-orientable, or equivalently, that G' contains freely acting orientation reversing elements.

Let u be the shortest orientation-reversing loop on $(S \backslash (w_1 \cup \cdots \cup w_q))/J$. Then $u^2 = w_{q+1}$, is a simple loop on $S \backslash (w_1 \cup \cdots \cup w_q)$ which is invariant under J.

If $w_1, \ldots, w_{q+1}$, does not divide S, then we repeat the above argument; after a finite number of steps we arrive at the required $w_1, \ldots, w_s$.

6.5. In proving Theorem 4 under the simplifying assumption of §6.4, we first take up the case that none of the loops $w_1, \ldots, w_s$ is pointwise fixed by J.

This means that $j \circ g$ has no fixed points in U, for any choice of $g \in G \backslash G_0$.

Every element $g \in G \backslash G_0$ is either loxodromic or elliptic of order 2; one easily sees that g is elliptic if and only if $j \circ g$ has fixed points in U. Hence J has no fixed points if and only if every element of $G \backslash G_0$ is loxodromic.

We look at all lifts of all w_i; these divide U into regions; we choose one of these regions, call it R, and let Γ be the stability subgroup of R in G_0. One sees at once that R/Γ is one of the halves of S cut along $w_1, \ldots w_s$. Hence Γ is a finitely generated Fuchsian group of the second kind representing a surface of some genus, with some number of elliptic or parabolic punctures, and s holes. We observe that for each primitive $F \in \Gamma$ representing one of these holes, there is an element $f \in G$, with $f^2 = F$, and f interchanges upper and lower half planes (that is, $j \circ f$ preserves the axis of F but interchanges the two non-euclidean half planes bounded by it).

We choose $F_1, \ldots, F_s$ to be non-conjugate primitive such elements and let $f_1, \ldots, f_s$ be their square roots. Then the group $\tilde{G}$ generated by Γ and $f_1, \ldots, f_s$ is a subgroup of G, has two components, neither invariant, and, as we observed in §5, $\Omega(\tilde{G})/\tilde{G}$ is the two halves of S with their boundaries glued together; that is, $\Omega(\tilde{G})/\tilde{G} = \Omega(G)/G$. We conclude that $\tilde{G} = G$ and hence, using §5.3, G is stratifiable.

6.6. We turn now to the case that J has fixed points on S. As we remarked earlier, this is equivalent to there being an involution $g \in G$, which interchanges the upper and lower half planes. We conjugate G so that $g(z) = -z$.

Then the involution J is induced by $j \circ g : z \mapsto -\bar{z}$; it has the positive imaginary axis as fixed point set. Let w be the fixed loop of J on S containing the projection of the positive imaginary axis, and let n be the number of elliptic and parabolic punctures on w. There are several cases to consider.

Case V. $n = 0$.

In this case w is a simple closed curve. Let $\mathscr{A}$ be the set of all translates in both U and U^* of $\{\operatorname{Re} z = 0, \operatorname{Im} z > 0\}$ under G. Let R be the subset of $\hat{\mathbf{C}}$ cut out by $\mathscr{A}$, where R is bounded by the imaginary axis and lies in the right half plane. Let G_1 be the stabilizer of R in G and let $G_{01} = G_1 \cap G_0$.

Exactly as in §5.2 (except that $g^2 = 1$), we can form the group $\tilde{G}$ generated by G_1 and g. We know that $\tilde{G} \subset G$, $\tilde{G}$ has two components, neither invariant, and $\Omega(\tilde{G})/\tilde{G}$ is homeomorphic to $\Omega(G_1)/G_1$. Since G_1 has one component and contains no degenerate subgroups, it is stratifiable. The fixed points of g are hyperbolic fixed points of G_1; hence $\tilde{G}$ is stratifiable. It remains to show that $\tilde{G} = G$.

If w does not divide $S = U/G_0$, then $R \cap U/G_{01}$, and $R \cap U^*/G_{01}$, are both equal to S cut along w. Then there is some loop w_1, disjoint from w, which is also invariant under J. Lifting J so that it keeps a lift $C \subset R \cap U$ of w_1 invariant, we get an orientation reversing element $g_1 \in G'$, which keeps $R \cap U$ invariant. Then $j \circ g_1$

maps $R \cap U$ onto $R \cap U^*$; that is, $j \circ g_1 \in G_1$. We conclude that $R/G_1 = R \cap U/G_{01}$ is S cut along w. Hence $\Omega(\tilde{G})/\tilde{G} = \Omega(G)/G$, so $\tilde{G} = G$.

If w divides S, then $R \cap U/G_{01}$ and $R \cap U^*/G_{01}$ are either equal or they are the two halves of s cut along w. Since $j(R \cap U) = R \cap U^*$, it must be the latter. The result now follows as above.

Case VI. $n = 1$.

We let P be the point of ramification or the puncture on w. Deforming w to lie on "either side" of P, we get two non-homotopic simple loops on S; we let w' and w'' be the geodesics on S in the corresponding homotopy classes (such geodesics exist except when S is a sphere with three elliptic or parabolic punctures in which case $\dim T(G) = 0$).

We note that $J(w') = w''$. The loops w' and w'' bound a subsurface $S_2 \subset S$, where $P \in S_2$. We let $S_1 = S \backslash \bar{S}_2$.

We let C' be a geodesic in U lying over w'. For the sake of definiteness we assume that C' is in the first quadrant. We extend C' to be a complete circle in $\hat{\mathbf{C}}$, and let $\mathscr{A} = \bigcup_{\gamma \in G} \gamma(C')$. As before $\mathscr{A}$ is a G'-invariant union of disjoint circles accumulating at all points of $\hat{\mathbf{R}}$. Let R_1 be the region in the first quadrant cut out by $\mathscr{A}$, bounded in part by C', where the projection of R_1 to U/G_0 does not contain the curve w.

As before we let G_1 be the stabilizer of $R_1 \cup jR_1$ in G, and G_{01} be the stabilizer of R_1 in G_0.

The surface S_1 may or may not be connected. If S_1 is not connected, then R_1/G_{01} is half of S_1 and $(R_1 \cup jR_1)/G_1 = S_1$. In this case $G_{01} = G_1$. If S_1 is connected then G_1 is a $\mathbf{Z}_2$-extension of G_{01} and $R_1/G_{01} = S_1 = (R_1 \cup jR_1)/G_1$. In either case $\Omega(G_1)/G_1$ is S_1 with a tube attaching the two boundary components; that is, $\Omega(G_1)/G_1$ is the surface S with the point P no longer a puncture or ramification point. We conclude that

$$d = d(G) = d(G_1) + 1.$$

We also remark that G_1 is a function group – it is either a finitely generated non-elementrary Fuchsian group of the second kind or a $\mathbf{Z}_2$-extension of such a group. In particular, every structure subgroup of G_1 is elementary. Hence G_1 is geometrically finite [23], and so it is stratifiable [18]. We let $x_1, \ldots, x_{d+2}$ be a stratification of G_1.

We now let R_2 be the region cut out by $\mathscr{A}$ on the other side of C'. Observe that the projection of R_2 to S contains the curve w. As before we let G_2 be the stabilizer of $R_2 \cup jR_2$ in G and G_{02} be the stabilizer of R_2 in G_0. Note that R_2

contains the positive imaginary axis, and that hence G_2 is G_{02} extended by $g: z \mapsto -z$. Thus $S_2 = R_2/G_{02} = (R_2 \cup jR_2)/G_2$. Observe that R_2 is invariant under the map $z \mapsto -\bar{z}$. As a matter of fact R_2/G_{02} is a sphere with two holes and one point of ramification order $\nu (2 \le \nu \le \infty)$, $JS_2 = S_2$, J has the reflection line w on S_2, this line passes through the ramification point P, and J interchanges the two holes. We conclude that G_{02} is a Fuchsian group of signature $(0, 1, 1; \nu)$. We also see that $\Omega(G_2)/G_2$ is a torus with one ramification point of order ν.

We let H be the stabilizer of C' in G. Then H is a hyperbolic cyclic group with generator h; also $H = G_1 \cap G_2$. It is quite easy to see that G_2 is generated by h and g, and that these satisfy the relations

$$g^2 = 1, (g \circ h^{-1} \circ g \circ h)^\nu = 1 \tag{7}$$

($\gamma^\infty = 1$ means that γ is parabolic). (One can see from this that there is a loop v on $\Omega(G_2)/G_2$, where v cuts w, w', w'' each exactly once, and v^2 lifts to a loop on $\Omega(G_2)$. This in fact proves that there can be no other relations in G_2 (see [23], [25]).)

We show finally how to extend the stratification of G_1 by adding one parameter to obtain a stratification of G. Our last parameter is

$$y = g(\text{attractive fixed point of } h).$$

Note that $h \in G_1$; hence h is determined by the stratification of G_1. We must show that the extra parameter determines the extra generator of G. We normalize so that

$$h = \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix}, \qquad 0 < |\tau| < 1,$$

$$y = g(0) = 1.$$

(We can assume that τ is determined since $T(G)$ is simply connected.) Hence (because $g^2 = 1$)

$$g = \begin{pmatrix} \alpha & -\alpha \\ \beta & -\alpha \end{pmatrix}, \qquad \alpha(\beta - \alpha) = 1.$$

The second of the relations (7) implies that

$$\operatorname{tr}(g \circ h^{-1} \circ g \circ h) = 2\alpha^2 - \alpha\beta(\tau^2 + \tau^{-2})$$

is constant on $T(G)$. Since h^2 is loxodromic, the last two equations determine $\alpha\beta$ and α^2 uniquely. Hence these two equations have solutions (α, β) and $(-\alpha, -\beta)$, and so we have determined g from our parameters. This completes Case VI.

Case VII. $n>1$.

We let P_1 and P_2 be two adjacent ramification points or parabolic punctures on some reflection arc w of the anti-conformal involution J on S. We denote the orders of these ramification points by ν_j, $2\leq\nu_j\leq\infty$, and we find a simple loop v on S with the following properties. The loop v divides S into two subsurfaces S_1 and S_2; both invariant under J. The subsurface S_2 has genus 0, contains the two points P_1 and P_2 and no other ramification points or punctures.

We may assume that $d(G)>0$ (as otherwise there is nothing to prove). Then there is a shortest geodesic in the homotopy class of v. We now replace v by the geodesic in its equivalence class and note that the statement that P_1 and P_2 are adjacent means that if $\nu_1<\infty$, $\nu_2<\infty$, then S_2 contains exactly two fixed points of J. We remark that if $\nu_1=2=\nu_2$, then the geodesic is no longer a loop, but a segment between the ramified points, and all our arguments require minor modifications, which we will ignore.

Exactly as before, we let C' be a lift of v, where C' intersects the positive imaginary axis; we complete it to a circle and let $\mathcal{A}=\bigcup_{\gamma\in G}\gamma(C')$. We let R_1 and R_2 be the regions in U cut out by $\mathcal{A}$ with the boundaries of R_1 and R_2 containing C' so that the projections of these regions are S_1 and S_2 respectively. Both R_1 and R_2 are $j\circ g$-invariant. As in the preceding cases, for $i=1, 2$ we let G_i be the stabilizer of $R_i\cup jR_i$ in G, and let G_{0i} be the stabilizer of R_i in G_0. We know that R_i is invariant under the reflection $j\circ g$, and thus $(R_i\cup jR_i)/G_i=R_i/G_{0i}$. Further R_i/G_{0i} are two parts of S cut along v. We also know that $j\circ g$ is a reflection that conjugates G_{01} into itself, and that the fixed line of $j\circ g$ cuts C'. If we let H_0 be the hyperbolic cyclic group stabilizing C', then we see that $gH_0g^{-1}=H_0$, and we conclude that the stabilizer H of C' in G_1 is a non-abelian $\mathbf{Z}_2$-extension of H_0; that is, as a Fuchsian group acting on the inside of C', H represents a disc with two ramification points each of order 2. We conclude that $\Omega(G_1)/G_1$ is homeomorphic to $\Omega(G)/G$ as a surface with ramification points, except that the points P_i $(i=1,2)$ no longer have ramification index ν_i; now they both have ramification index 2. Hence

$$d=d(G)=d(G_1).$$

Of course G_1 is a geometrically finite function group; hence stratifiable. We let $x_1,\ldots,x_{d+3}$ be a stratification for G_1.

We repeat the above analysis for G_2 and we conclude that $\Omega(G_2)/G_2$ is a sphere with four ramification points of indices ν_1, ν_2, 2, 2. We choose generators g_1, g_2 for G_{02}, where $g_i^{\nu_i}=1$ $(i=1,2)$, $g_2\circ g_1=h$, a generator for H_0, and $(g\circ j)\circ g_1\circ(g\circ j)=g_1^{-1}$. Then G_2 is generated by g_1, g_2, g. These satisfy the relations $g_1^{\nu_1}=g_2^{\nu_2}=g^2=(g\circ g_2\circ g_1)^2=(g\circ g_1)^2=1$. (We remark that from the theory of signatures of Kleinian groups [26], we know $\Omega(G_2)/G_2$, and so G_2 has a presentation of three elliptic or parabolic generators where a product of two of them is elliptic, parabolic or the identity, and a product of all three is elliptic or parabolic.)

We must show that the stratification of G_1 already stratifies G. The stratification of G_1 determines h and g. Again we change normalization so that (G_0 is no longer Fuchsian)

$$h=\begin{pmatrix}\tau & 0\\ 0 & \tau^{-1}\end{pmatrix},\qquad g=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

We write

$$g_1=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},\qquad \alpha\delta-\beta\gamma=1.$$

Now $\operatorname{tr} g_1$ and $\operatorname{tr} g_2$ are known constants. But

$$\operatorname{tr} g_2=\operatorname{tr} h\circ g_1^{-1}=\tau\delta+\tau^{-1}\alpha,$$

$$\operatorname{tr} g_1=\alpha+\delta.$$

Hence we can solve for α and δ. Since $(g\circ g_1)^2=1$, we also have $\gamma=\beta$. Finally to solve for γ we use $\gamma^2=\alpha\delta-1$. In general we have two solutions (note that $\alpha\delta\neq 1$). The connectivity and simply connectivity of $T(G)$ force the selection of square root.

This completes the proof of Theorem 4.

Remark. The cuts we made in the surface S had to be chosen with care. For example, had we chosen in the last case to cut S along a simple non-dividing loop v where J has exactly two fixed points on v, then we would have obtained a group G_1 with $d(G_1)>d(G)$.

REFERENCES

[1] Ahlfors, L. V., *Lectures on quasiconformal mappings*, Van Nostrand, New York, 1966.

[2] Ahlfors L. V. and Bers, L., *Riemann's mapping theorem for variable metrics*, Ann. of Math., 72 (1960), 385–404.

[3] BEARDON, A. F. and MASKIT, B. *Limit points of Kleinian groups and finite sided fundamental polyhedra*, Acta Math., *132* (1974), 1–12.
[4] BERS, L., *Uniformization by Beltrami equations*, Comm. Pure. Appl. Math., *14* (1961), 215–228.
[5] BERS, L., *A non-standard integral equation with applications to quasiconformal mappings*, Acta Math., *116* (1966), 113–134.
[6] BERS, L., *Spaces of Kleinian groups*, in "Several Complex Variables, Maryland 1970," Lecture Notes in Mathematics, *155* (1970), Springer, Berlin, 9–34.
[7] BERS, L., *Poincaré series for Kleinian groups*, Comm. Pure Apl. Math., *26* (1973), 667–672 and *27* (1974), 583.
[8] BERS, L., Automorphic forms for Schottky groups, Advances in Math., *16* (1975), 332–361.
[9] EARLE, C. J., *Some intrinsic coordinates on Teichmüller space*, Proc. Amer. Math. Soc., *83* (1981), 527–531.
[10] EARLE, C. J. and KRA, I., *On sections of some holomorphic families of closed Riemann surfaces*, Acta Math., *137* (1976), 49–79.
[11] HEJHAL, D. A., *Quelques remarques à propos des séries de Poincaré sur les groupes de Schottky*, C.R. Acad. Sci. Paris, *280* (1975), A341–A344.
[12] HEJHAL, D. A., *Sur les séries de Poincaré des groupes fuchsiens*, C.R. Acad. Sci. Paris, *284* (1977), A607–A610.
[13] HEJHAL, D. A., *Monodromy groups and Poincaré series*, Bull. Amer. Math. Soc., *84* (1978), 339–376.
[14] KRA, I., *Automorphic forms and Kleinian groups*, Benjamin, Reading, Massachusetts, 1972.
[15] KRA, I., *On spaces of Kleinian groups*, Comment. Math. Helv., *47* (1972), 53–69.
[16] KRA, I., *Canonical mappings between Teichmüller spaces*, Bull. Amer. Math. Soc., *4* (1981), 143–179.
[17] KRA, I. and MASKIT, B. *Involutions on Kleinian groups*, Bull. Amer. Math. Soc., *78* (1972), 801–805.
[18] KRA, I. and MASKIT, B., *The deformation space of a Kleinian group*, Amer. J. Math., *103* (1981), 1065–1102.
[19] LJAN, G. M., *The kernel of the Poincaré θ-operator*, Dokl. Akad. Nauk SSSR, *230* (1976), 269–270. English translation: Soviet Math. Dokl., *17* (1976), 1283–1285.
[20] MASKIT, B., *On boundaries of Teichmüller spaces and on Kleinian groups:* II, Ann. of Math., *91* (1970), 607–639.
[21] MASKIT, B., *Self-maps of Kleinian groups*, Amer. J. Math., *93* (1971), 840–856.
[22] MASKIT, B., *On Klein's combination theorem III, Advances in the theory of Riemann surfaces*, Ann. of Math. Studies, *66* (1971), 297–316.
[23] MASKIT B., *Decomposition of certain Kleinian groups*, Acta Math., *130* (1973), 243–263.
[24] MASKIT, B., *Moduli of marked Riemann surfaces*, Bull. Amer. Math. Soc., *80* (1974), 773–777.
[25] MASKIT, B., *On the classification of Kleinian groups: I – Koebe groups*, Acta Math., *135* (1975), 249–270.
[26] MASKIT, B., *On the classification of Kleinian groups II – signatures*, Acta Math., *138* (1977), 17–42.
[27] WOLPERT, S., *The Fenchel–Nielsen deformation*, Ann. of Math., *115* (1982), 501–528.
[28] WOLPERT, S., On *Poincaré series and the symplectic deformation geometry of Riemann surfaces*, to appear.

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