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An algebro-geometric interpretation of the Bäcklund-transformation for the Korteweg-de Vries equation

FRITZ EHLERS and HORST KNÖRRER

1. Introduction

In 1882 G. Darboux has developed a method that associates to any one-dimensional Schrödinger-operator $Z=-d^2/dx^2+u(x)$ infinitely many other such operators the spectra of which are intimately related to the spectrum of Z itself (cf. [4] §408). This transformation is defined in the following way:
Let c be a constant, let $\psi(x)$ be a solution of the differential equation $Z\cdot\psi=c\psi$ with $\psi\neq 0$ and put $v:=\psi_x/\psi$. One easily verifies that $u=v^2+v_x+c$, and we put

$$\tilde{u}:=v^2-v_x+c=u-2v_x.$$

The differential operator

$$\tilde{Z}:=-\frac{d^2}{dx^2}+\tilde{u}$$

is called the Darboux–Bäcklund-transform of Z. It depends on the choice of c and ψ, a change of ψ by a constant factor does not effect the transformation.

It is well known that Korteweg–de Vries-equation

$$u_t=3uu_x-\tfrac{1}{2}u_{xxx}$$

is equivalent to the Lax-equation

$$\frac{\partial Z}{\partial t}=-[K,Z]$$

with

$$Z:=-\frac{d^2}{dx^2}+u,\qquad K:=2\frac{d^3}{dx^3}-3u\frac{d}{dx}-\frac{3}{2}u_x.$$

(For a discussion of the properties of the KdV-equation cf. [10]).

The transformation of Z described above is compatible with this Lax-equation, so it yields a transformation that produces new solutions of the KdV-equation out of given ones. Transformations of this type are called Bäcklund-transformations after A. Bäcklund who developed such a transformation for the sine-Gordon-equation in the last century (cf. [8]). For the KdV-equation the transformation described above has been worked out by R. Miura [13].

Successive application of the Bäcklund-transformation with diminishing values of $c \in \mathbf{R}$, $c<0$, produces out of the solution $u \equiv 0$ of the KdV-equation the multi-soliton-solutions of this equation.

In 1976 I. M. Krichever developed a method for constructing one-dimensional Schrödinger operators starting with certain line bundles on hyperelliptic curves (cf. [12], [14]; parts of this construction will be described in chapter 2). Since Krichever's construction is purely formal we work with differential operators with coefficients in the ring $\mathbf{C}[[x]]$ of complex formal power series in x (i.e. with elements of $\mathbf{C}[[x]][d/dx]$). The object of this paper is to describe how the Darboux–Bäcklund-transformation, applied to such a Schrödinger operator, changes the hyperelliptic curve and the line bundle on it. The precise result will be given in Chapter 3; its essential content is the following:

Let Z be a Schrödinger-operator constructed by Krichever's method from a line bundle on the hyperelliptic curve $y^2+F(z)=0$ (F a polynomial of odd degree), and let $\tilde{Z}$ be a Bäcklund-transform of Z as described above. If ψ was chosen generically[1] in the c-eigenspace of the linear map $Z: C[[x]] \to C[[x]]$ then $\tilde{Z}$ corresponds to a line bundle on the curve $y^2+(z-c)^2F(z)=0$, whereas for some special choices of ψ the operator $\tilde{Z}$ can be obtained from a line bundle on the curve $y^2+F(z)=0$ or $y^2+(z-c)^{-2}F(z)=0$.

Our work on this problem was motivated by discussions in a seminar on geometrical methods in mathematical physics, run by members and guests of the SFB 40 in Bonn in 1979 and 1980. We would like to thank all its participants, especially M. Adler, for many helpful and interesting suggestions.

2. Krichever's construction, as described by Mumford

Krichever's construction provides a relation between certain line bundles on hyperelliptic curves (see the end of this chapter) and certain Schrödinger

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[1] The notion "generically" has the meaning used in algebraic geometry. As the eigenspaces of Z are two-dimensional it means that ψ has to avoid some finite number of one-dimensional subspaces.

operators. These Schrödinger operators have the property that they commute with some other differential operator Y of odd degree.[2] Conversely, any Schrödinger operator with this property can be obtained from a line bundle on some hyperelliptic curve. In this chapter we will describe how the hyperelliptic curve and the line bundle on it can be reconstructed from the operator Z. In the exposition of these results we follow the approach of Mumford [14], §2.

Let Y be a differential operator of odd degree commuting with Z, and assume further that its degree r is minimal with respect to this property. It follows from the commutativity that its leading coefficient is constant and thus we may assume that it equals one. Y and Z generate the ring $R \subset \mathbf{C}[[x]][d/dx]$ of all differential operators commuting with Z. As in [14], p. 133 one sees that there are polynomials $F_1(z)$, $F_2(z) \in \mathbf{C}[z]$ such that $Y^2+2F_1(Z)\cdot Y+F_2(Z)=0$, $\deg F_1 \leq (r-1)/2$, $\deg F_2 = r$. Replacing Y by $Y+F_1(Z)$ one obtains $Y^2+F(Z)=0$ where $F(z):=F_2(z)-F_1(z)^2$ has leading coefficient one. It can be shown that

$$\mathbf{C}[z, y]/(y^2+F(z)) \rightarrow R, \qquad z \mapsto Z, \qquad y \mapsto Y$$

is an isomorphism.

Let C_0 be the affine curve $\{(x, y) \in C^2 \mid y^2+F(X)=0\}$. This is the spectrum Spec R of the commutative ring R in the sense of algebraic geometry. There is a unique compact curve $C \supset C_0$, such that the points of $C-C_0$ are nonsingular on C. Since $\deg F$ is odd, $C-C_0$ consists only of one point, P.

In some neighbourhood of P, $\zeta := y \cdot z^{-(r+1)/2}$ is a local parameter of C. If $D \in R$ is considered as a rational function on C then its pole order at P is its order as a differential operator.

In order to construct the line bundle on C we notice that for every $z \in \mathbf{C}$ the operator Y maps the two-dimensional space $\ker(Z-z\cdot \mathrm{id})$ to itself. Its eigenvalues on this space are the values $y \in \mathbf{C}$ with $y^2+F(z)=0$. So for each point $Q \in C_0$ the "eigenspace of R corresponding to Q"

$$E_Q := \ker(Z-z\cdot \mathrm{id}) \cap \ker(Y-y\cdot \mathrm{id})$$

has dimension ≥ 1.

Remark. $\dim E_Q = 1$ for all $Q \in C_0$.

Proof. Let $Q=(z, y)$ be a point of C_0. If E_0 were not one-dimensional then

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[2] The existence of such an operator Y implies that u is a so called finite gap potential.

the two-dimensional space $\ker(Z - z \cdot \mathrm{id})$ would be contained in $\ker(Y - y \cdot \mathrm{id})$. This implies that $(Y - y \cdot \mathrm{id}) = Y'.\ (Z - z \cdot \mathrm{id})$ with some operator Y' of degree $r-2$ (cf. [9], 5.4). Since $Y - y \cdot \mathrm{id}$ and $Z - z \cdot \mathrm{id}$ commute with Z the same is true for Y', in contradiction to the assumption that the degree of Y was minimal.

The spaces E_Q can be glued to a line bundle on the affine curve C_0. In order to describe its extension to C we use the following construction indicated by Mumford [14]:

Consider $M := \mathbf{C}[d/dx]$ as a right R-module with multiplication

$$X \cdot D = (X \circ D)_0, \qquad X \in M, \qquad D \in R, \qquad \left(\sum a_i(x) \frac{d^i}{dx^i}\right)_0 := \sum a_i(0) \frac{d^i}{dx^i}$$

M is finitely generated, so it is the space of sections of some coherent sheaf $\mathscr{L}_0$ on C_0, unique up to isomorphism. $\mathscr{L}_0$ extends to a sheaf $\mathscr{L}$ on C in such a way that the space of sections of $\mathscr{L} \otimes \mathcal{O}(nP)$ corresponds to the space of differential operators of order at most n.

Mumford shows that the pairing

$$\begin{array}{l} M \times E_Q \to \mathbf{C} \\ (X, f) \mapsto (Xf)(0) \end{array} \quad (Q \in C_0)$$

induces an isomorphism

$$\mathrm{Hom}_{\mathbf{C}}(\mathfrak{L}_Q / m_Q \mathfrak{L}_Q, \mathbf{C}) \cong E_Q$$

where m_Q denotes the maximal ideal of $Q \in C_0$.

Since E_Q is one-dimensional for all $Q \in C_0$ this implies that $\mathscr{L}$ is locally free of rank 1, hence $\mathscr{L}$ is the sheaf of sections of a line bundle L on C. (Obviously $L|_{C_0}$ is the dual of the line bundle obtained by glueing the E_Q's.) By construction we have that $\dim H^0(C, L(nP)) = \dim M_{n-1} = n$ for all $n \in \mathbf{N}$, by Riemann–Roch this is equivalent to $\dim H^0(C, L) = \dim H^1(C, L) = 0$. Hence L is a line bundle of degree $g-1$, where $g = (r-1)/2$ is the arithmetic genus of C.

We have thus obtained the following algebraic-geometric data:

(i) a hyperelliptic curve C in normal form $y^2 + F(z) = 0$[3], where F is a polynomial of odd degree with leading coefficient one.

(ii) a line bundle L on C with $\dim H^0(C, L) = \dim H^1(C, L) = 0$.

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[3] i.e. a hyperelliptic curve C together with two rational functions y, z on C that generate the function field of C and that fulfil the identity $y^2 + F(z) = 0$.

As we mentioned in the beginning one can recover the operator Z from these data. Furthermore it can be shown that the time-variation of the operator

$$Z_t = -\frac{d^2}{dx^2} + u(x, t)$$

according to the KdV-equation induces a linear variation of L in the Jacobian of line bundles of degree $g-1$ on C.

3. The Bäcklund-transformation in the algebraic-geometric picture

Let $Z = -(d^2/dx^2) + u(x)$ be a formal Schrödinger operator, and let R be the ring of all differential operators commuting with Z.

ASSUMPTION. R contains a differential operator of odd order.

As in the introduction let $\tilde{Z} = -(d^2/dx^2) + \tilde{u}(x)$ be Darboux-Bäcklund-transform of Z with parameters $c \in \mathbf{C}$, $\psi \in \ker(Z - c \cdot \mathrm{id})$, $\psi(0) \neq 0$. Denote by $\tilde{R}$ the ring of differential operators commuting with $\tilde{Z}$.

ASSERTION. $\tilde{R}$ also contains a differential operator of odd order.

As in Chapter 2 we associate to R resp. $\tilde{R}$ hyperelliptic curves C and $\tilde{C}$ with normal forms $y^2 + F(z) = 0$ resp. $y^2 + \tilde{F}(z) = 0$ and line bundles L and $\tilde{L}$ on C. resp. $\tilde{C}$. Let $\mathscr{L}$ resp. $\tilde{\mathscr{L}}$ be the sheaf of sections of L resp. $\tilde{L}$, and let P resp. $\tilde{P}$ be the point at infinity of C resp. $\tilde{C}$. We call $E := \bigcup_{Q \in C-P} E_Q$ the set of common eigenfunctions of R.

THEOREM.

(I) *If* $\psi \notin E$ *then* $\tilde{C}$ *is the hyperelliptic curve in the normal form* $y^2 + (z-c)^2 \cdot F(z) = 0$ *(i.e.* $\tilde{F}(z) = (z-c)^2 \cdot F(z)$*). If* $\pi : C \to \tilde{C}$ *denotes the map induced by* $(z, y) \mapsto (z, (z-c)y)$ *then* $\tilde{\mathscr{L}}$ *is naturally isomorphic to the subsheaf of* $\pi_* \mathscr{L} \otimes \mathcal{O}(\tilde{P})$ *of sections* s *with* $\langle s_Q, \psi \rangle = 0$.

(ii) *If* $\psi \in E_Q$ *for some smooth point* $Q \in C$ *then the curves* C *and* $\tilde{C}$ *coincide (i.e.* $F = \tilde{F}$*) and* $\tilde{L} \cong L \otimes \mathcal{O}(P) \otimes \mathcal{O}(-Q)$.

(iii) *If* $\psi \in E_Q$ *for a singular point* Q *of* C *then* $\tilde{C}$ *is the hyperelliptic curve with the normal form* $y^2 + (z-c)^{-2} F(z) = 0$*. If* $\tau : \tilde{C} \to C$ *denotes the map induced by* $(z, y) \mapsto (z, (z-c)y)$ *then* $\tilde{L} \cong \tau^* L \otimes \mathcal{O}(-\tilde{P})$.

Remarks.

1. For almost all choices of $\psi \in \ker(Z-c\cdot \mathrm{id})$, ψ is not contained in E. In this case $\tilde{L}$ is a line bundle on $\tilde{C}$ with $\pi^*(\tilde{L})\otimes \mathcal{O}(-P)\cong L$. By [15], V. 12 the Jacobian[4] of $\tilde{C}$ is a central extension of the Jacobian of C:

$$e\to G\to \mathrm{Jac}(\tilde{C})\to \mathrm{Jac}(C)\to 0, \qquad \tilde{L}\to \pi^*(\tilde{L})\otimes \mathcal{O}(-P)$$

with $G=\mathbf{C}^*$ or $G=\mathbf{C}$, e the neutral element of G. Given a line bundle L on C with $H^0(L)=H^1(L)=0$ we have an isomorphism between G and the set of elements $[\psi]\in \mathbf{P}(\ker(Z-c\cdot \mathrm{id}))$ such that $\psi\notin E$. The condition $\psi(0)\neq 0$ corresponds to the condition $H^0(\tilde{L})=H^1(\tilde{L})=0$.

2. If $\psi\in E_Q$ for a singular point $Q\in C$ (which then has the coordinates $(c, 0)$) the Bäcklund transformation from Z to $\tilde{Z}$ is the inverse of a Bäcklund transformation from $\tilde{Z}$ to Z as described in part (i) of the theorem and in the remark above.

3. The "transference" in $\mathrm{Jac}(C)$ caused by the Bäcklund transformation with $\psi\in E_Q$ for a smooth point $Q\in C$ has been considered in greater generality by Burchnall and Chaundy [3], §VI, VII.

4. J. Drach noticed in 1912 that the Darboux–Backfund transformation does not increase the effective genus of the associated hyperelliptic curve (see [6]).[5]

5. In [1] it is shown that iterated application of the Bäcklund transformation with $c=0$ to the solution $u\equiv 0$ of the KdV-equation leads to the rational solutions discovered by Airault-McKean-Moser [2]. So the theorem shows that these solutions belong to curves of the form $y^2+z^{2n+1}=0$.

Similarly one can construct the multi-soliton-solutions by repeated application of the Bäcklund-transformation with diminishing values of c (cf. [5]). This gives a simple proof that these solutions belong to curves of the form $y^2+(z-c_1)^2\cdots(z-c_n)^2z=0$ (see also [11]).

6. For a real valued Schwartz function $u(x)$ the effect of the Bäcklund-transformation on the spectral data has been described in [7].

4. Proofs

Replacing u by $u-c$ we may assume that $c=0$.

4.1. The interpretation of the Bäcklund-transformation given in [1], [5] shows

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[4] By abuse of notation we denote by $\mathrm{Jac}(C)$ the moduli space of line bundles of degree $g-1$ on C. As base point we take the point at infinity.

[5] This reference was shown to us by D. V. and G. V. Chudnovsky during the 1980 Arbeitstagung of the SFB 40 in Bonn.

that $\tilde{Z}$ is obtained from Z by conjugation in the ring PsD [[x]] of pseudo-differential operators. (For the definition of PsD [[x]] see [14], p. 140.)

LEMMA 1. *$\tilde{Z} = A \circ Z \circ A^{-1}$ where A is the differential operator*

$$A := \psi \circ \frac{d}{dx} \circ \psi^{-1} = \frac{d}{dx} - \frac{\psi_x}{\psi} = \frac{d}{dx} - v.$$

Proof. Since $Z\psi = -\psi_{xx} + u\psi = 0$ we have $u = \psi_{xx}/\psi$ and hence $Z = A^* \circ A$ where $A^* = -(d/dx) - v = -\psi^{-1} \circ d/dx \circ \psi$ denotes the adjoint of A. Then one easily verifies that $\tilde{Z} = A \circ A^*$, which proves the lemma.

The ring R is generated by Z and some operator Y with $Y^2 + F(Z) = 0$. $\psi \in E$ if and only if ψ is an eigenvector of Y.

LEMMA 2. *Let $\kappa: D \to A \circ D \circ A^{-1}$ be the conjugation by A in* PsD[[x]].
(i) *If $\psi \notin E$, let $R' := \mathbf{C}[Z, Y \circ Z] \subset R$. Then $\kappa(R) \cap \mathbf{C}[[x]][d/dx] = \kappa(R')$.*
(ii) *$\kappa(R) \subset \mathbf{C}[[x]][d/dx]$ if and only if $\psi \in E$. In this case, call $R' = R$.*

Proof. Let $X \in \mathbf{C}[[x]][d/dx]$ be any formal differential operator. Then

$$X \circ \psi = X\psi + X' \circ \frac{d}{dx}$$

where X' is some differential operator. It follows that

$$A \circ X \circ A^{-1} = \psi \circ \frac{d}{dx} \circ \psi^{-1} \circ X\psi \circ \left(\frac{d}{dx}\right)^{-1} \circ \psi^{-1} + \psi \circ \frac{d}{dx} \circ \psi^{-1} \circ X' \circ \psi^{-1}.$$

Thus $A \circ X \circ A^{-1} \in \mathbf{C}[[x]][d/dx]$ if and only if $\psi^{-1} \cdot X\psi$ is a constant function, i.e. if ψ is an an eigenfunction of X. This proves (ii) and the "$\supset$"-part of (i). Since $\dim_{\mathbf{C}} R/R' \leq 1$ the lemma is proved.

The assertion in Chapter 3 follows from Lemma 1 and 2. We now want to compare $\kappa(R') = \kappa(R) \cap \mathbf{C}[[x]][d/dx]$ with the ring $\tilde{R}$ of all operators commuting with $\tilde{Z}$. Obviously $\kappa(R) \cap \mathbf{C}[[x]][d/dx] \subset \tilde{R}$, and we have

LEMMA 3. *If $\kappa(R) \cap \mathbf{C}[[x]][d/dx] \neq \tilde{R}$ then $\psi \in E_Q$ for some singular point $Q \in C$.*

Proof. We put $Y' := \kappa(Y \circ Z)$ if $\psi \notin E$ and $Y' := \kappa(Y)$ if $\psi \in E$. In the first case $Y'^2 + Z^2 \circ F(\tilde{Z}) = 0$, in the second case $Y'^2 + F(\tilde{Z}) = 0$. In both cases $\kappa(R) \cap \mathbf{C}[[x]][d/dx] = \mathbf{C}[Y', \tilde{Z}]$. Since $\tilde{R} \supset C[Y', \tilde{Z}]$, $\tilde{R} = \mathbf{C}[\tilde{Y}, \tilde{Z}]$ with some differential operator $\tilde{Y}$ fulfilling an equation of the form $\tilde{Y}^2 + \tilde{F}(\tilde{Z}) = 0$ with a polynomial $\tilde{F}$ of odd degree. Then $Y' = g(\tilde{Z}) \circ \tilde{Y}$ with $g(z) \in C[z]$.

First we show that $g(z) = az^m$ for some $a \in \mathbf{C}$, $n \in \mathbf{N}$. Otherwise Y' splits $Y' = (\tilde{Z} - b) \circ Y''$ with $b \neq 0$, $Y'' \in \tilde{R} - R$. But this implies that $b \cdot \kappa^{-1}(Y'') = \kappa^{-1}(\tilde{Z} \cdot Y'') - \kappa^{-1}(Y') = A^* \circ Y'' \circ A - \kappa^{-1}(Y')$ is a differential operator in R, thus $Y'' \in R'$.

Put $C_0' := \operatorname{Spec} R'$, $\tilde{C}_0 := \operatorname{Spec} \tilde{R}$. Since $\kappa : R[Z^{-1}] \xrightarrow{\cong} \kappa(R')[\tilde{Z}^{-1}] = \tilde{R}[\tilde{Z}^{-1}]$ the maps π, $\tilde{\pi}$ in the diagram below are isomorphisms outside of $\{(z, y) \in C_0' / z = 0\}$

$$
\begin{tikzcd}
C_0 \arrow[dr, "\pi"] & & \tilde{C}_0 \arrow[dl, "\tilde{\pi}"'] \\
R & C_0' & \tilde{R} \\
& R' \arrow[ul, hook] \arrow[ur, hook, "\kappa"'] &
\end{tikzcd}
$$

From $\dim R/R' \leq 1$ and the condition $\kappa(R') \neq \tilde{R}$ it follows that there exists a regular map $\pi_1 : \tilde{C}_0 \to C_0$ such that $\pi \circ \pi_1 = \tilde{\pi}$. Thus $\kappa(R)$ is contained in $\tilde{R}$ and by Lemma 2, $\psi \in E_Q$ for some point $Q = (0, y) \in C_0$. If Q were a regular point of C_0 then $\pi_1 : \tilde{C}_0 \to C_0$ were an isomorphism and $\kappa(R) = \tilde{R}$.

Remark. An application of Lemma 2 (i) shows that in this case Z is obtained from $\tilde{Z}$ by a Bäcklund-transformation with parameters $0, \psi^{-1}$.

4.2. Let us suppose that $\kappa(R) \cap \mathbf{C}[[x]][d/dx] = \tilde{R}$, i.e. $\tilde{R} = A \circ R' \circ A^{-1}$. If $\psi \notin E$, then $\tilde{C}$ is given by $y^2 + z^2 F(z) = 0$, if $\psi \in E$ then $\tilde{C} = C$. Let $\phi : \tilde{R} \to R$ be the map $D \mapsto A^{-1} \circ D \circ A$ and $\pi : C \to \tilde{C}$ the dual map. If $\psi \notin E$ then π is the map induced by $(z, y) \mapsto (z, zy)$; if $\psi \in E$ then it is the identity map of C.

In order to determine the line bundle on $\tilde{C}$ the space $M = \mathbf{C}[d/dx]$ will be considered as a module over various rings (cf. Chapter 2). The ring in question will be indicated by an index: M_R, $M_{\tilde{R}}$ etc. As in Chapter 2 we have the following natural identifications

$$
\Gamma(C - P, L) \cong M_R, \qquad \Gamma(\tilde{C} - \tilde{P}, \tilde{L}) \cong M_{\tilde{R}}, \qquad \Gamma(\tilde{C} - \tilde{P}, \pi_* \mathscr{L}) = M_{\phi(\tilde{R})}.
$$

(By $M_{\phi(\tilde{R})}$ we denote the $\tilde{R}$-module M where the scalar-multiplication is given by $X \cdot D := (X \circ \phi(D))_0$, $X \in M$, $D \in \tilde{R}$.)

LEMMA 4. *The map* $\phi : M_{\tilde{R}} \to M_{\phi(\tilde{R})}$, $X \mapsto (X \circ A)_0$ *is* $\tilde{R}$*-linear.*

Proof. Let $D \in \tilde{R}, X \in M_{\tilde{R}}$. Then

$$\Phi(X \cdot D) = ((X \circ D)_0 \circ A)_0 = (X \circ D \circ A)_0 =$$
$$= (X \circ A \circ A^{-1} \circ D \circ A)_0 = ((X \circ A)_0 \circ (A^{-1} \circ D \circ A))_0 = \Phi(X) \cdot \phi(D).$$

(The above identities hold because $(X \circ X')_0 = (X_0 \circ X')_0$ for any two differential operators $X, X' \in \mathbf{C}[[x]][d/dx]$.)

LEMMA 5. $\Phi(M_{\tilde{R}}) = \{X \in M_{\phi(\tilde{R})} / (X\psi)(0) = 0\}$.

Proof. Since A has order 1 the image $\Phi(M_{\tilde{R}})$ has codimension 1 in $M_{\phi(\tilde{R})}$. On the other hand the non-trivial map $X \mapsto (X\psi)(0)$ vanishes on $\Phi(M_{\tilde{R}})$ since $A\psi = 0$.

Second proof: By an argument as in the proof of Lemma 2 one sees that $(X\psi)(0) = 0$ if and only if $(X \circ A^{-1})_0$ is a differential operator.

Let $\mathcal{F}$ be the sheaf of sections s of $\pi_* \mathcal{L}$ with $\langle s_Q, \psi \rangle = 0$. Obviously

$$\Phi(M_{\tilde{R}}) \subset \Gamma(\tilde{C} - \tilde{P}, \mathcal{F}) \subset M_{\phi(\tilde{R})} = \Gamma(\tilde{C} - \tilde{P}, \pi_* \mathcal{L}).$$

Since $\Phi(M_{\tilde{R}})$ has codimension 1 in $M_{\phi(\tilde{R})}$ and $\Gamma(\tilde{C} - \tilde{P}, \mathcal{F}) \neq \Gamma(\tilde{C} - \tilde{P}, \pi_* \mathfrak{L})$ it follows that $\Phi(M_{\tilde{R}}) = \Gamma(\tilde{C} - \tilde{P}, \mathcal{F})$, hence

$$\Gamma(\tilde{C} - \tilde{P}, \tilde{\mathcal{L}}) \cong \Gamma(\tilde{C} - \tilde{P}, \mathcal{F}).$$

Because A has order 1 Φ induces a sheaf map $\tilde{\mathcal{L}} \to \pi_* \mathcal{L}$ with a pole of order 1 at $\tilde{P}$. Hence $\tilde{\mathcal{L}} \cong \mathcal{F} \otimes \mathcal{O}(\tilde{P})$, whenever $\kappa(R) \cap \mathbf{C}[[x]][d/dx] = \tilde{R}$. This, together with Lemma 3, proves part (i) and (ii) of the theorem.

4.3. Now suppose that $\psi \in E_Q$ for a singular point $Q \in C$ (which then has the coordinates $(0, 0)$). If $\kappa(R) \cap \mathbf{C}[[x]][d/dx]$ were equal to $\tilde{R}$ then by the results of 4.2 $\tilde{\mathcal{L}}$ were isomorphic to $\mathcal{F} \otimes \mathcal{O}(\tilde{P})$. But this sheaf is not invertible at Q, whereas $\tilde{\mathfrak{L}}$ is invertible.

So $\kappa(R) \cap \mathbf{C}[[x]][d/dx] \subset \tilde{R}$. In the remark after Lemma 3 we noticed that $\tilde{C}$ has the normal form $y^2 + 1/z^2 F(z) = 0$ and that u is obtained by a Bäcklund transformation applied to $\tilde{u}$. By part (i) of the theorem this implies that $\tilde{L} = \tau^*(L) \otimes \mathcal{O}(-P)$, where $\tau : \tilde{C} \to C$ is the map $(z, y) \mapsto (z, yz)$. Thus also part (iii) of the theorem is proved.

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Mathematisches Institut
der Universität
Wegelerstr. 10
D 5300 Bonn 1
W.-Germany

Mathematisch Instituut
Rijksuniversiteit Leiden
Wassenaarseweg 80
2300 RA Leiden
The Netherlands

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