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Comment. Math. Helvetici 57 (1982) 11–21 0010–2571/82/000111–11$01.50+0.20/0

A Stiefel complex for the orthogonal group of a field

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In this paper we show that the poset of orthogonal frames in $(F^n, n\langle 1\rangle)$ with at most k elements is $(k-1)$-spherical if n is sufficiently large. Here $n\langle 1\rangle$ is the identity form, and F may be any field with finite pythagoras number, e.g., a local or global field, finite field or real-closed field. We then use this poset to show that for n large with respect to m, the inclusion $O_n \to O_{n+1}$ induces an isomorphism $H_m(O_n) \to H_m(O_{n+1})$, where homology is taken with integral coefficients.

§0. Introduction

It is often useful, in studying the homology of a group, to have a "combinatorial representation" of the group, i.e., a simplicial complex with a natural group action. If this complex has little or no homology, the spectral sequence arising from the group action will relate the homology of stabilizers of simplices with the homology of the group in a relatively uncomplicated way. This fact has been used, for example, to compute the cohomology of special linear groups [1], [3] and to prove homology stability theorems for the basic groups in algebraic K-theory [4], [6], [7].

In this paper we discuss a simplicial complex which can be used to study the orthogonal group of a quadratic form. This is the "Stiefel complex," i.e., the geometric realization of a partially ordered set of orthonormal frames in the underlying vector space of the form. The first part of the paper proves connectedness theorems for the complex associated to the identity form $n\langle 1\rangle$ and some of its subforms. The proof easily generalizes to general forms over a field of characteristic not equal to two, but for large Witt index the degree of connectivity goes down. As an application, we then use these complexes to prove a homology stability theorem for the orthogonal group O_n.

* Supported by National Science Foundation Grant MCS 77-18723 (A03).

§1. Stiefel complexes

In this section we construct some simplicial complexes (Stiefel complexes) associated to a quadratic module over a ring, and discuss their homotopy properties in special cases.

Let R be a commutative ring with unit, and let V be a quadratic R-module, i.e., a free R-module equipped with a bilinear symmetric form q.

DEFINITION. An *orthonormal k-frame* $[v_1, \ldots, v_k]$ in V is an (unordered) set of k elements $v_1, \ldots, v_k$ of V with $q(v_i, v_j) = \delta_{ij}$.

The set of orthonormal k-frames in V is partially ordered by inclusion: $[v_1, \ldots, v_k] < [u_1, \ldots, u_l]$ if $\{v_1, \ldots, v_k\} \subset \{u_1, \ldots, u_l\}$.

DEFINITION. The *realization* of a partially ordered set X, denoted $|X|$, is the simplicial complex whose i-simplices are totally ordered chains of $i+1$ elements of X; the simplices are glued together via the natural identifications.

An exposition of notations, definitions and basic techniques pertaining to partially ordered sets (posets) and their realizations may be found in [5]. We will use the notions of link, suspension and join (denoted lk, susp and $*$ respectively) from simplicial complexes, as well as the following facts:

LEMMA 1.1. *If X and Y are two subposets of a poset Z, and $x < y$ for all $x \in X$, $y \in Y$, then $|X \cup Y| = |X| * |Y|$.*

LEMMA 1.2. *If $f: X \to X$ is an inclusion preserving (or inclusion reversing) map from a poset X to itself, then $|X|$ is homotopy equivalent to $|\mathrm{im}\,(f)|$.*

We can now define and study Stiefel complexes.

DEFINITION. The k-th *Stiefel complex* of a quadratic R-module V (denoted $X_k(V)$) is the realization of the poset of orthonormal frames in V with at most k elements.

We first consider the case where R is the ring of integers in a totally real number field K, V is a free R-module with basis $\{e_1, \ldots, e_n\}$ and the matrix of the quadratic form in this basis is the identity matrix I_n.

LEMMA 1.3. *For R and V as above, the only elements in V of length 1 are $\{\pm e_i\}$.*

Proof. Let $v = r_1e_1 + \cdots + r_ne_n$ be an element of V with $v\cdot v = \sum_{i=1}^n r_i^2 = 1$. The norm from K to $\mathbf{Q}$ is a multiplicative homomorphism taking $R-\{0\}$ to $\mathbf{Z}-\{0\}$; thus for each i we have either $r_i = 0$ or $N(r_i^2) = \prod_\sigma \sigma r_i^2 \geq 1$, where the product runs over all distinct embeddings of K into $\mathbf{R}$ which fix $\mathbf{Q}$. But for each such embedding σ,

$$\sum_{i=1}^n \sigma r_i^2 = \sigma\left(\sum_{i=1}^n r_i^2\right) = 1,$$

so $\sigma r_i^2 \leq 1$ for each i. Thus, for each i, either $r_i = 0$ or $\sigma r_i^2 = 1$ for all σ, which implies that $r_i = \pm 1$. Since $\sum_{i=1}^n r_i^2 = 1$, we must have $v = \pm e_j$ for some j. ■

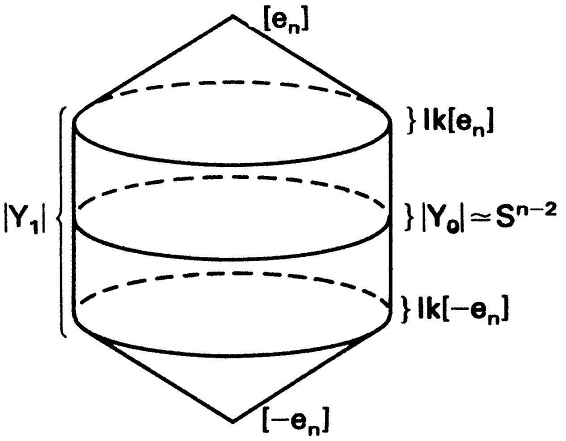
PROPOSITION 1.4. *Let R and $V = R^n$ be as above. Then the Stiefel complex $X_n(V)$ is homotopy equivalent to the $(n-1)$-sphere S^{n-1}.*

Proof. The proof proceeds by induction on n. For $n = 1$, Lemma 1.3 says that the only orthonormal frames are $[e_1]$ and $[-e_1]$; thus $X_1(V)$ consists of two points, i.e. $X_1(V) \simeq S^0$.

If $n > 1$, consider the subposet $Y_0 = \{$orthonormal frames in R^{n-1} (=span of $\{e_1, \ldots, e_{n-1}\})\}$. Then by induction, $|Y_0| \simeq S^{n-2}$.

Let $Y_1 = Y_0 \cup \{$orthonormal frames which strictly contain e_n or $-e_n\}$. Then the map $Y_1 \to Y_0$ which is the identity on Y_0 and sends $[\pm e_n, v_1, \ldots, v_k]$ to $[v_1, \ldots, v_k]$ gives a retraction $|Y_1| \simeq |Y_0|$ by Lemma 1.2.

By Lemma 1.3 again, the only orthonormal frames in V which are not in Y_1 are $[e_n]$ and $[-e_n]$. The inclusions $\mathrm{lk}\,[\pm e_n] \to Y_1$ induce homotopy equivalences, so $X_n(V) \simeq \mathrm{susp}\,|Y_1| \simeq S^{n-1}$:



■

Now let F be a field of characteristic $\neq 2$.

DEFINITION. The *pythagoras number* of F is the smallest integer $p = p(F)$ such that every sum of squares in F can be written as a sum of p squares. If there is no such number, we say $p(F) = \infty$.

EXAMPLES. (See [2]). If F is real-closed or pythagorean, $p(F)=1$. If F is a global field or a local field with finite residue field, $p(F)\leq 4$. If F is a function field of transcendence degree n over a real-closed field, then $p(F)\leq 2^n$. If $F=\mathbf{R}(x_1, x_2, \ldots)$, then $p(F)=\infty$. If F is not formally real, then $p(F)<\infty$.

NOTATION. We will use $\langle d_1, \ldots, d_n\rangle$ to denote the diagonal quadratic form on F^n with diagonal entries $d_1, \ldots, d_n$. If $\langle d_1, \ldots, d_n\rangle$ and $\langle e_1, \ldots, e_n\rangle$ are isometric, we write $\langle d_1, \ldots, d_n\rangle\cong\langle e_1, \ldots, e_n\rangle$.

PROPOSITION 1.5. *Let F be a field with $p(F)=p<\infty$. Let $V=F^n$ with the identity form $n\langle 1\rangle$, and $W^{n-l}\subset V$ a codimension l nondegenerate subspace. Then if $n>pl$, W contains a unit vector.*

Proof. Let $\langle d_1, \ldots, d_{n-l}\rangle$ be a diagonalization of the restriction of the identity form to W. Since W is nondegenerate, we can extend this diagonalization to all of $V: n\langle 1\rangle\cong\langle d_1, \ldots, d_{n-l}, x_1, \ldots, x_l\rangle$.

Since x_i is a sum of at most p squares, we have $p\langle 1\rangle\cong\langle x_i, y_{i1}, \ldots, y_{ip-1}\rangle$ for some y_{ij}. Hence if $n>pl$,

$$\begin{aligned}\langle x_1, \ldots, x_l, y_{11}, \ldots, y_{l,p-1}, 1, \ldots, 1\rangle \\ \cong n\langle 1\rangle \\ \cong\langle x_1, \ldots, x_l, d_1, \ldots, d_{n-l}\rangle.\end{aligned}$$

By Witt cancellation, this gives

$$\langle y_{11}, \ldots, y_{l,p-l}, 1, \ldots, 1\rangle\cong\langle d_1, \ldots, d_{n-l}\rangle,$$

so $\langle d_1, \ldots, d_{n-l}\rangle$ represents 1; i.e., W contains a unit vector. ∎

COROLLARY 1.6. *Let F and V be as above, and let $E=[e_1, \ldots, e_m]$ and $F=[f_1, \ldots, f_l]$ be two orthonormal frames with $l\leq m$. Then $E^\perp\cap F^\perp$ contains a unit vector if $n>2pl+m$. If F is formally real, $E^\perp\cap F^\perp$ contains a unit vector if $n>pl+m$.*

Proof. If F is formally real, $E^\perp\cap F^\perp$ is a nondegenerate subspace of $E\cong(F^{n-m}, (n-m)\langle 1\rangle)$, so the result follows immediately from the proposition.

If F is not formally real, the largest possible dimension of a totally isotropic subspace of $E^\perp\cap F^\perp$ is l. Therefore, there is a nondegenerate subspace of dimension at least $n-m-2l$ in E, and the result follows again from the proposition. ■

D. Shapiro has pointed out to me that a field which has the property stated in the corollary must have finite pythagoras number:

PROPOSITION (Shapiro). *Let F be a field. Suppose there is a number n such that given any two unit vectors $e, f \in (F^n, n\langle 1\rangle)$, $e^\perp \cap f^\perp$ contains a unit vector. Then $p(F) \leq n-2$.*

Proof. Let c be a sum of squares in F. We must show it can be written as a sum of $n-2$ squares. It suffices to assume c is the sum of $n-1$ squares.

We can write $c = x^2 - y^2$, with $x, y \in \dot{F}$. Replacing c by c/x^2, we may assume $c = 1 - a^2$, with $a \in \dot{F}$.

Let $W = (F^{n-1}, (n-1)\langle 1\rangle)$, and let $w \in W$ be a vector with $w \cdot w = c = 1 - a^2$. Define $v = W \perp Fe$, *with* $e \cdot e = 1$. Then $V \cong (F^n, n\langle 1\rangle)$. Set $f = ae + w$; then $f \cdot f = 1$.

By hypothesis, $e^\perp \cap w^\perp \cap f^\perp$ contains a unit vector v. Now diagonalize the form on V, using e, v and w as the first three basis vectors; you get $n\langle 1\rangle \cong \langle 1, 1, c, d_1, \ldots, d_{n-3}\rangle$. By Witt cancellation, $(n-2)\langle 1\rangle \cong \langle c, d_1, \ldots, d_{n-3}\rangle$, so c is the sum of $n-2$ squares. ■

We will now use Corollary 1.6 to prove connectivity results for certain Stiefel complexes.

THEOREM 1.7. *Let F be a field with pythagoras number $p = p(F) < \infty$. Let $[e_1, \ldots, e_m]$ and $[f_1, \ldots, f_l]$ be two orthonormal frames in $(F^n, n\langle 1\rangle)$, with $l \leq m$, and let $V = [e_1, \ldots, e_m] \cap [f_1, \ldots, f_l] \subset F^n$. Then for $n > 2p(l+k-1) + (m+k-1)$ (or, if F is formally real, for $n > p(l+k-1) + (m+k-1)$), $X_k(V)$ is homotopy equivalent to a wedge of $(k-1)$-spheres.*

COROLLARY 1.8. *Let $V = (F^n, n\langle 1\rangle)$. Then for $n > (2p+1)(k-1)$ (or $n > (p+1)(k-1)$ for F formally real), $X_k(V) \cong \bigvee S^{k-1}$.*

Proof of Theorem 1.7. The proof proceeds by induction on k. For $k = 1$, Corollary 1.6 says that $X_1(V)$ is non-empty and hence contains at least two 1-frames; therefore, $X_1(V) \cong \bigvee S^0$.

Now assume $k \geq 2$. Choose a unit vector g in V and let $H = g^\perp \cap V = [f_1, \ldots, f_l]^\perp \cap [e_1, \ldots, e_m, g]^\perp$. We check that $n > 2p(l+k-2) + (m+k-1)$, so by induction, $X_{k-1}(H) \cong \bigvee S^{k-2}$.

Let

$$Y_0'' = \{[\pm g]\} \cup \{[h_1, \ldots, h_r] \in X_{k-1}(H)\}$$
$$\cup \{[\pm g, h_1, \ldots, h_r] : [h_1, \ldots, h_r] \in X_{k-1}(H)\}.$$

Then, as in the proof of Proposition 1.4, we have

$$|Y_0''| \simeq \operatorname{susp}(X_{k-1}(H)) \simeq \bigvee S^{k-1}.$$

Let $Y_0' = Y_0'' \cup \{[h_1, \ldots, h_r, \pm g, v_1, \ldots, v_s] = 0 < r+1+s \leq k$ and $[h_1, \ldots, h_r, \pm g] \in Y_0''\}$.

Then the map $Y_0' \to Y_0''$ which is the identity on Y_0'' and sends $[h_1, \ldots, h_r, \pm g, v_1, \ldots, v_s]$ to $[h_1, \ldots, h_r, \pm g]$ induces a homotopy equivalence $|Y_0'| \simeq |Y_0''|$.

Let $Y_0 = Y_0' \cup \{k\text{-frames } [h_1, \ldots, h_k] \text{ in } H\}$. If $[h_1, \ldots, h_k] \in Y_0 - Y_0'$, we have $\operatorname{lk}[h_1, \ldots, h_k] \cap |Y_0'| = \operatorname{lk}[h_1, \ldots, h_k] = |\{\text{proper subframes of } [h_1, \ldots, h_k]\}|$. The set of proper subsets of a finite set with k elements can be identified with the barycentric subdivision of the boundary of a $(k-1)$-simplex; thus $\operatorname{lk}[h_1, \ldots, h_k] \cap |Y_0'| \simeq S^{k-2}$, which implies that $|Y_0|$ is the wedge product

$$|Y_0'| \quad \bigvee_{[h_1, \ldots, h_k] \in Y_0 - Y_0'} \operatorname{susp}(\operatorname{lk}[h_1, \ldots, h_k])$$
$$\simeq \left(\bigvee S^{k-1}\right) \bigvee_{[h_1, \ldots, h_k] \in Y_0 - Y_0'} S^{k-1}$$
$$\simeq \bigvee S^{k-1}.$$

For $1 \leq i \leq k$, define $Y_i = Y_{i-1} \cup \{[v_1, \ldots, v_i] : v_r \cdot g \neq 0 \text{ for some } 1 \leq r \leq i\}$. Then given $[v_1, \ldots, v_i] \in Y_i - Y_{i-1}$, we have

$$\operatorname{lk}[v_1, \ldots, v_i] \cap |Y_{i-1}|$$
$$= |\{\text{subframes of } [v_1, \ldots, v_i]\} \cup \{[v_1, \ldots, v_i, x_1, \ldots, x_j] : j \geq 1,$$
$$i + j \leq k \text{ and } x_r \in H \text{ for some } 1 \leq r \leq j\}|$$
$$\simeq S^{i-2} * |\{[v_1, \ldots, v_i, x_1, \ldots, x_j] : j \geq 1, i+j \leq k, \text{ and } x_r \in H \text{ for all } r\}|$$
$$\simeq S^{i-2} * X_{k-i}(H \cap [V_1, \ldots, v_i]^{\perp}).$$

We now check our induction hypothesis for $X_{k-i}(H \cap [v_1, \ldots, v_i]) : \|H \cap [v_1, \ldots, v_i]^{\perp} = [f_1, \ldots, f_l, g]^{\perp} \cap [e_1, \ldots, e_m, v_1, \ldots, v_i]^{\perp}$, and we have $n > 2p(l+$

$1+k-i-1)+(m+i+k-i-1)$, so the hypothesis is satisfied. Thus

$$\begin{aligned}\operatorname{lk}[v_1,\ldots,v_i]\cap|Y_{i-1}| &\simeq S^{i-2} * \bigvee S^{k-i-1}\\ &\simeq \bigvee S^{k-2}, \quad \text{so} \quad |Y_i| \simeq \bigvee S^{k-1}.\end{aligned}$$

Since $|Y_k| = X_k(V)$, this proves the theorem. ∎

An inspection of the proof shows that the essential problem is to show the existence of a unit vector in a given subspace. For many fields this can be done more efficiently than was done above. Suppose there is a number m_F such that every non-degenerate form $d_1, \ldots, d_{m_F}$ with d_i a sum of squares, represents 1. (This is the case for pythagorean and real-closed fields ($m_F = 1$), finite fields ($m_F = 2$), global and local fields ($m_F \le 4$). It is not the case for $\mathbf{C}(x_1, x_2, \ldots)$, though this field has pythagoras number 2 (see [2])). If we use the number m_F to ensure the existence of unit vectors in the proof of Theorem 1.7, we obtain the following theorem.

THEOREM 1.9. *Let F be a field, m_F as above, and $V = (F^n, n\langle 1\rangle)$. If F is formally real and $n \ge 2k + m_F - 2$, then $X_k(V) \simeq \bigvee S^{k-1}$. If F is not formally real and $n \ge 3k + m_F - 3$, then $X_k \simeq \bigvee S^{k-1}$.*

EXAMPLES. Let $F = \mathbf{F}_3$, the field with three elements, then $X_4(\mathbf{F}_3^4)$ is the disjoint union of three 3-spheres, containing the 1-frames $[(1,0,0,0)]$, $[(1,1,1,1)]$ and $[(-1,1,1,1)]$ respectively. Thus $X_2(\mathbf{F}_3^4) \subset X_4(\mathbf{F}_3^4)$ is not connected, so $n \ge 3\cdot 2 + 2 - 3 = 5$ is necessary to get connectivity. However, $X_3(\mathbf{F}_3^7)$ is simply connected; also $X_2(\mathbf{F}_5^4)$ and $X_2(\mathbf{F}_7^4)$ are connected, showing that the bound in the theorem can often be improved.

Let $F = \mathbf{R}$. Then $X_2(\mathbf{R}^2)$ is the disjoint union of uncountably many circles, so is not connected. It can be shown, using an argument which essentially suspends this case; that $\pi_{n-2}(X_n(\mathbf{R}^n))$ is uncountable. A more complicated combinatorial argument shows $X_3(\mathbf{R}^4)$ is not simply connected, supporting the bound $n \ge 2k + 1 - 2$.

We will now produce a chain complex which gives the homology of X. We filter X by subcomplexes X_i = realization of $\{j\text{-frames}, j \le i\}$. Then $\phi \subset X_1 \subset \cdots \subset X_k = X$, and $X_i \simeq \bigvee S^{i-1}$. The spectral sequence of this filtration shows that the complex

$$\cdots \xrightarrow{\partial} H_{i-1}(X_i, X_{i-1}) \xrightarrow{\partial} H_{i-2}(H_{i-1}, H_{i-2}) \xrightarrow{\partial} \cdots$$

gives the homology of X; thus the sequence

$$\begin{gathered} 0 \to H_{k-1}(X) \to H_{k-1}(X_k, H_{k-1}) \to \cdots \to H_{i-1}(X_i, H_{i-1}) \\ \to \cdots \to H_0(X_1) \to \mathbf{Z} \to 0 \\ 0 \to C_{k+1} \to C_k \to \cdots \to C_i \to \cdots \to C_1 \to C_0 \to 0 \end{gathered}$$

is exact.

§2. Proof of homology stability for O_n

THEOREM. *Let O_n be the orthogonal group of the standard identity form I_n over R, where R = ring of integers in a totally real number field, or R = field with finite pythagoras number. Then for n sufficiently large with respect to j, $H_j(O_{n+1}, O_n) = 0$.*

The proof follows what is by now a standard pattern (see, e.g., [7]). We outline it below.

Let E_* be a free $\mathbf{Z}[O_n]$-resolution of $\mathbf{Z}$, and C_* as in the end of §1, where $V = R^n$ with the standard basis and form. Then the double complex $E_* \otimes_{O_n} C_*$ gives a spectral sequence with

$$E^1_{p,q} = H_q(O_n; C_p) = 0.$$

We have (notation as in §1)

$$\begin{aligned} C_p &= H_{p-1}(X_p, X_{p-1}) \cong H_{p-1}(X_p/X_{p-1}) \cong H_{p-1}\Big(\bigvee_{\text{p-frames}} S^{p-1}\Big) \\ &= \bigoplus_{\text{p-frames}} H_{p-1}(S^{p-1}) = \bigoplus_{\text{p-frames}} \mathbf{Z}. \end{aligned}$$

LEMMA. *O_n acts transitively on the set of orthonormal p-frames, for any p.*

Proof. If $p = 1$, let v and e be any two vectors of length 1. Then either $v - e$ or $v + e$ is anisotropic, so reflection in the hyperplane perpendicular to this anisotropic vector is an orthogonal transformation taking v to e. By Witt cancellation, $v^\perp$ is isometric to $e^\perp$, and we proceed by induction on n. ■

Thus $\bigoplus_{\text{p-frames}} \mathbf{Z} = \mathbf{Z}[O_n] \bigoplus_{\mathbf{Z}[\text{stab}[e_1, \ldots, e_p]]} \mathbf{Z}$ where stab $[e_1, \ldots, e_p]$ is the stabilizer in

O_n of the frame $[e_1, \ldots, e_p]$. It is easy to check that

$$\operatorname{stab}[e_1, \ldots, e_p] = \left(\begin{array}{c|c} \Sigma_p & 0 \\ \hline 0 & O_{n-p} \end{array}\right),$$

where Σ_p is the symmetric group on p letters.

Our E^1_{pq} term may now be written

$$E^1_{p,q} = H_q\Big(O_n; \mathbf{Z}[O_n] \underset{\mathbf{Z}\left[\begin{array}{c|c} \Sigma_p & 0 \\ \hline 0 & O_{n-p} \end{array}\right]}{\otimes} \mathbf{Z}\Big)$$

$$= H_q\left(\left(\begin{array}{c|c} \Sigma_p & 0 \\ \hline 0 & O_{n-p} \end{array}\right); \mathbf{Z}\right).$$

We have the following picture of the spectral sequence:

$$\begin{array}{l|llll} j & H_j(O_n) \xleftarrow{d_1} H_j\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & O_{n-1} \end{array}\right) & & & \\ & & H_{j-1}\left(\begin{array}{c|c} \Sigma_2 & 0 \\ \hline 0 & O_{n-2} \end{array}\right) & & \\ & & & \cdot \quad \cdot \quad \cdot & \\ & & & & H_0\left(\begin{array}{c|c} \Sigma_{j+1} & 0 \\ \hline 0 & O_{n-j-1} \end{array}\right) \\ \hline \end{array}$$

The inclusion map $O_n \to O_{n+1}$ induces an inclusion of simplicial complexes $X_n \to X_{n+1}$ and thus a natural map of spectral sequences for O_n and for O_{n+1}. The mapping cone spectral sequence has

$$E^1_{p,q} = \begin{cases} H_q(O_{n+1}, O_n), & p = 0 \\ H_q\left(\left(\begin{array}{c|c} \Sigma_p & 0 \\ \hline 0 & O_{n-p+1} \end{array}\right)\left(\begin{array}{c|c} \Sigma_p & 0 \\ \hline 0 & O_{n-p} \end{array}\right); \mathbf{Z}\right), & p \geq 1 \end{cases}$$

$$\Rightarrow 0.$$

We prove the stability theorem by induction on j; i.e., we assume that for n sufficiently large with respect to $q < j$, $H_q(O_{n+1}, O_n) = 0$.

PROPOSITION. *For $q < j$, $E^1_{p,q} = 0$.*

Proof. Consider the exact sequences

$$\begin{tikzcd}
1 \arrow[r] & O_{n-p+1} \arrow[r] & \left(\begin{array}{c|c} \Sigma_p & \\ \hline & O_{n-p+1} \end{array}\right) \arrow[r] & \Sigma_p \arrow[r] & 1 \\
1 \arrow[r] & O_{n-p} \arrow[u, hook] \arrow[r] & \left(\begin{array}{c|c} \Sigma_p & \\ \hline & O_{n-p} \end{array}\right) \arrow[u, hook] \arrow[r] & \Sigma_p \arrow[u] \arrow[r] & 1.
\end{tikzcd}$$

The relative Leray–Serre spectral sequence for this diagram has

$$E^2_{s,t} = H_s(\Sigma_p; H_t(O_{n-p+1}, O_{n-p}))$$
$$\Rightarrow H_{s+t}\left(\left(\begin{array}{c|c} \Sigma_p & \\ \hline & O_{n-p+1} \end{array}\right)\left(\begin{array}{c|c} \Sigma_p & \\ \hline & O_{n-p} \end{array}\right)\right).$$

By our induction hypothesis, $E^2_{s,t}=0$ for n large and $t<j$; therefore,

$$H_q\left(\left(\begin{array}{c|c} \Sigma_p & \\ \hline & O_{n-p+1} \end{array}\right)\left(\begin{array}{c|c} \Sigma_p & \\ \hline & O_{n-p} \end{array}\right)\right) = 0 \quad \text{for} \quad q=s+t<j. \quad \blacksquare$$

This proposition implies that

$$H_j(O_{n+1}, O_n) \xleftarrow{d^1} H_j\left(\left(\begin{array}{c|c} 1 & \\ \hline & O_n \end{array}\right)\left(\begin{array}{c|c} 1 & \\ \hline & O_{n-1} \end{array}\right)\right)$$

in onto (since the spectral sequence converges to 0). Then a diagram chase of the following diagram proves the theorem.

$$\begin{tikzcd}
 & H_j\left(\begin{array}{c|c} n & \\ \hline & 1 \end{array}\right)\left(\begin{array}{c|c|c} n-1 & & \\ \hline & 1 & \\ \hline & & 1 \end{array}\right) \arrow[r, "\partial"] & H_{j-1}\left(\begin{array}{c|c|c} n-1 & & \\ \hline & 1 & \\ \hline & & 1 \end{array}\right) \arrow[r, "i"] & H_{j-1}\left(\begin{array}{c|c} n & \\ \hline & 1 \end{array}\right) \arrow[ddl, equal] \\
H\left(\begin{array}{c|c} 1 & \\ \hline & n \end{array}\right) \arrow[r] \arrow[d] & H_j\left(\begin{array}{c|c} 1 & \\ \hline & n \end{array}\right)\left(\begin{array}{c|c|c} 1 & & \\ \hline & n-1 & \\ \hline & & 1 \end{array}\right) \arrow[u, "f"] \arrow[r, "\partial"] \arrow[d, "i+d_1"] & H_{j-1}\left(\begin{array}{c|c|c} 1 & & \\ \hline & n-1 & \\ \hline & & 1 \end{array}\right) \arrow[u, "f+g"] \arrow[d, "i"] & \\
H_j(n+1) \arrow[r] \arrow[d, equal, "h"] & H_j(n+1)\left(\begin{array}{c|c} n & \\ \hline & 1 \end{array}\right) \arrow[r, "\partial"] \arrow[d, "h"] & H_{j-1}\left(\begin{array}{c|c} n & \\ \hline & 1 \end{array}\right) & \\
H_j(n+1) \arrow[r] & H_j(n+1)\left(\begin{array}{c|c} 1 & \\ \hline & n \end{array}\right) & &
\end{tikzcd}$$

where the maps f are induced by conjugation by $\begin{pmatrix} 0 & 1 \\ I_n & 0 \end{pmatrix}$,

g are induced by conjugation by $\left(\begin{array}{c|c|c} 0 & 1 & 0 \\ \hline I_{n-1} & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$,

h are induced by conjugation by $\begin{pmatrix} 0 & I_n \\ 1 & 0 \end{pmatrix}$,

i are induced by inclusion. ■

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Received June 5/November 20, 1981