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Autor(en): **McDuff, Dusa**

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Local homology of groups of volume-preserving diffeomorphisms, II

DUSA MCDUFF*

§1. Introduction

This is the second in a series of three papers on the local homology of groups of volume-preserving diffeomorphisms: see [9], [10]. It may be read independently of the other two papers since it uses none of their results or methods of proof. Here is a statement of the main theorem. (A slightly sharper version is stated in §2.) We will explain later how it is related to the results of [9], [10].

We consider a compact, oriented, smooth manifold W_1 with boundary ∂W_1. Let $W_0 \subset W_1$ be the complement of an open collar neighbourhood of ∂W_1. If ω is a volume form on Int W_1, we write $\mathrm{Diff}_\omega(W_i, \mathrm{rel}\,\partial)$ for the discrete group of all ω-preserving diffeomorphisms of W_i which are the identity near ∂W_i. Clearly $\mathrm{Diff}_\omega(W_0, \mathrm{rel}\,\partial) \subset \mathrm{Diff}_\omega(W_1, \mathrm{rel}\,\partial)$.

THEOREM 1. *The inclusion* $\mathrm{Diff}_\omega(W_0, \mathrm{rel}\,\partial) \subset \mathrm{Diff}_\omega(W_1, \mathrm{rel}\,\partial)$ *induces an isomorphism on (untwisted) integer homology.*

This theorem holds for any volume form on Int W_1. In particular, taking $\omega = dx_1 \wedge \cdots \wedge dx_n$ on $\mathbf{R}^n$, we see that the inclusion of the group of ω-preserving diffeomorphisms of $\mathbf{R}^n$ with support in the open unit disc into the group of compactly supported ω-preserving diffeomorphisms of $\mathbf{R}^n$ is a homology isomorphism.

Observe also that if we were considering the group $\mathrm{Diff}(W_i, \mathrm{rel}\,\partial)$ of all, not necessarily volume-preserving, diffeomorphisms with support in Int W_i, then the above result follows easily from the fact that $G_i = \mathrm{Diff}(W_i, \mathrm{rel}\,\partial)$ is the union of subgroups G_{ij}, where

$$G_{01} \subset G_{02} \subset \cdots \subset G_0 = G_{11} \subset G_{12} \subset \cdots \subset G_1,$$

and where, for any i, j, there is $g \in G_1$ which commutes with G_{0i} and conjugates

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G_{1j} to G_0. However these conjugating maps cannot preserve volume, and so one cannot argue in this way in the volume-preserving case.

The main application of Theorem 1 is to the study of the "local homology" of groups of volume-preserving diffeomorphisms. Recall from [7] that if $\mathcal{G}$ is a topological group whose underlying discrete group is G, then the homotopy fiber $\bar{B}\mathcal{G}$ of the natural map $BG \to B\mathcal{G}$ depends only on the algebraic and topological structure of a neighbourhood of the identity in $\mathcal{G}$. Therefore the homology of the space $\bar{B}\mathcal{G}$ is called the "local homology of $\mathcal{G}$ at the identity." When $\mathcal{G}$ is a Lie group, it follows from van Est's theorem that the "differentiable" part of its local cohomology is just the cohomology of the Lie algebra of $\mathcal{G}$. See [4]. If $\mathcal{D}iff_\omega(W_i, \mathrm{rel}\,\partial)$ denotes the group $\mathrm{Diff}_\omega(W_i, \mathrm{rel}\,\partial)$ in its usual C^∞-topology, it is not hard to see that the inclusion of $\mathcal{D}iff_\omega(W_0, \mathrm{rel}\,\partial)$ into $\mathcal{D}iff_\omega(W_1, \mathrm{rel}\,\partial)$ is a homotopy equivalence. For example, using [6] one can easily construct a family h_t of contractions of W_1 which preserve ω up to a constant and are such that $h_1(W_1) = W_0$. Then one can homotop $\mathcal{D}iff_\omega(W_1, \mathrm{rel}\,\partial)$ into $\mathcal{D}iff_\omega(W_0, \mathrm{rel}\,\partial)$ by conjugating by h_t. Thus Theorem 1 is equivalent to the statement:

THEOREM 1′. *The inclusion*

$$\bar{B}\,\mathcal{D}iff_\omega(W_0, \mathrm{rel}\,\partial) \hookrightarrow \bar{B}\,\mathcal{D}iff_\omega(W_1, \mathrm{rel}\,\partial)$$

induces an isomorphism on (*untwisted*) *integer homology.*

This is a very special case of a general theorem [10] which asserts that the local homology of groups such as $\mathcal{D}iff_\omega(W, \mathrm{rel}\,\partial)$ is a homotopy invariant of the pair consisting of W together with its tangent bundle. In fact, the local homology of $\mathcal{D}iff_\omega(W, \mathrm{rel}, \partial)$ is isomorphic to the homology of the space of sections of a bundle over W which is associated to the tangent bundle. In the ordinary, non-volume-preserving case, this theorem is due to Mather in dimension 1 and to Thurston in dimension >1. See [8]. In the volume-preserving case, Theorem 1′ is the crucial link which allows one to deduce the theorem for compact manifolds of finite volume from that for non-compact manifolds of infinite volume which is established in [9].

The proof of Theorem 1′ is surprisingly delicate. It is based on ideas of Thurston which he used in [11] to show that when $n = \dim W \geq 3$, the inclusion

$$\bar{B}\,\mathcal{D}iff^{\Phi}_{\omega 0}(V, \mathrm{rel}\,\partial) \hookrightarrow \bar{B}\,\mathcal{D}iff^{\Phi}_{\omega 0}(W, \mathrm{rel}\,\partial)$$

induces an isomorphism on H_1, where V is *any* compact n-dimensional submanifold of W, and where $\mathcal{D}iff^{\Phi}_{\omega 0}$ is the subgroup of $\mathcal{D}iff_\omega$ consisting of elements which are isotopic to the identity and have zero flux. This, in turn, is the key step in showing that $\mathcal{D}iff^{\Phi}_{\omega 0}(W, \mathrm{rel}\,\partial)$ is a simple group when $n \geqslant 3$. The case $n = 2$ coincides with the symplectic case and was considered by Banyaga in [1].

The proof of Theorem 1′ has two steps. First, one shows that the space $\bar{B}\,\mathcal{D}iff^{\Phi}_{\omega 0}(W_1, \mathrm{rel}\,\partial)$ deformation retracts onto a subspace which is made up from diffeomorphisms of "small" support in W_1. An elegant proof of this deformation lemma in the non-volume-preserving case is given by Mather in [8] §15, following ideas of Thurston. However that proof does not work either in the volume-preserving or the C^0 case. The present proof is much more complicated, but it does work in both these cases as well as in the symplectic case. See Remark 4.15 below. In fact, it is just a generalization to higher dimensions of Thurston and Banyaga's proof of a similar result for the 2-skeleton. Second, one shows that any cycle on this subspace made from diffeomorphisms of small support may have its support conjugated into W_0. The techniques used here, notably the construction of the maps h_κ in Lemma 3.6, do not appear to generalize immediately to the symplectic case.

§2. Basic definitions

First let us recall some facts about the flux homomorphism. We assume throughout that W_1 is a connected n-dimensional manifold with non-empty boundary. Then the volume form ω is exact and the flux is a continuous homomorphism Φ from the identity component $\mathcal{D}iff_{\omega 0}(W_1, \mathrm{rel}\,\partial)$ of $\mathcal{D}iff_{\omega}(W_1, \mathrm{rel}\,\partial)$ to $H^{n-1}_c(W_1; \mathbf{R}) \cong H^{n-1}(W_1, \partial W_1; \mathbf{R})$. It may be defined as follows. Given an $(n-1)$-cycle z in $(W_1, \partial W_1)$ then

$$\Phi(g)(z) = \int_c \omega,$$

where c is an n-chain with boundary $g_*(z) - z$. (This is independent of the choice of c because ω is exact.) We will write $\mathcal{D}iff^{\Phi}_{\omega 0}(W_1, \mathrm{rel}\,\partial)$ for the kernel of Φ. Thurston shows in [11] that $\mathrm{Diff}^{\Phi}_{\omega 0}(W_1, \mathrm{rel}\,\partial)$ is a perfect group when $n \geqslant 3$. In fact, he proves the slightly stronger result that $H_1(\bar{B}\,\mathcal{D}iff^{\Phi}_{\omega 0}(W_1, \mathrm{rel}\,\partial); \mathbf{Z}) = 0$. When $n = 2$, this is no longer true. There is a continuous surjective homomorphism

$$\rho : \mathcal{D}iff^{\Phi}_{\omega 0}(W_1, \mathrm{rel}\,\partial) \to \mathbf{R},$$

defined by Banyaga in [1] II.4.3, whose kernel we will denote by

$\mathcal{D}iff^{\Phi\rho}_{\omega 0}(W_1, \mathrm{rel}\,\partial)$. Banyaga shows in [1] that $H_1(\bar{B}\,\mathcal{D}iff^{\Phi\rho}_{\omega 0}(W_1, \mathrm{rel}\,\partial); \mathbf{Z}) = 0$. Let

$$\mathcal{G}_i = \begin{cases} \mathcal{D}iff^{\Phi}_{\omega 0}(W_i, \mathrm{rel}\,\partial) & \text{if} \quad n \geqslant 3, \text{ and} \\ \mathcal{D}iff^{\Phi\rho}_{\omega 0}(W_i, \mathrm{rel}\,\partial) & \text{if} \quad n = 2. \end{cases}$$

We will prove Theorem 1′ in the following sharpened form.

THEOREM 2.1. *The inclusion $\bar{B}\mathcal{G}_0 \hookrightarrow \bar{B}\mathcal{G}_1$ induces an isomorphism on (untwisted) integer homology.*

Clearly, it suffices to consider the case when vol W_1 is finite. Therefore we will assume from now on that this is so. Also, it will be convenient to reformulate Theorem 2.1 slightly. Choose a point $x_1 \in \partial W_1$, and put $W_2 = W_1 -$(open disc *nbhd* of x_1). So W_2 has corners: see Fig. 1. Let $\mathcal{G}_2$ be $\mathcal{D}iff^{\Phi}_{\omega 0}(W_2, \mathrm{rel}\,\partial)$ if $n \geqslant 3$ and $\mathcal{D}iff^{\Phi\rho}_{\omega 0}(W_2, \mathrm{rel}\,\partial)$ when $n = 2$. If vol $W_0 =$ vol W_2, the discrete groups G_0 and G_2 are direct limits of subgroups which are conjugate in G_1. It follows that the inclusion $BG_0 \hookrightarrow BG_1$ induces an isomorphism on integer homology if and only if the inclusion $BG_2 \hookrightarrow BG_1$ does. Since the inclusions $\mathcal{G}_0 \hookrightarrow \mathcal{G}_1$ and $\mathcal{G}_2 \hookrightarrow \mathcal{G}_1$ are homotopy equivalences, a similar statement is true on the level of $\bar{B}\mathcal{G}$. Therefore it will suffice to show that $H_*(\bar{B}\mathcal{G}_1, \bar{B}\mathcal{G}_2; \mathbf{Z}) = 0$ for any W_2. Clearly, this is an immediate consequence of the following lemma.

LEMMA 2.2. *If* $N > 2d^2 + 2$, *then* $H_d(\bar{B}\mathcal{G}_1, \bar{B}\mathcal{G}_2; \mathbf{Z}) = 0$ *whenever* vol$(W_1 - W_2) < 1/N$ vol W_1.

Before beginning the proof we must make some definitions.

Let Sing $\mathcal{G}$ denote the singular complex of the topological group $\mathcal{G}$. The discrete group G acts freely on Sing $\mathcal{G}$ by multiplication on the right, and hence acts freely on the realization $|\mathrm{Sing}\,\mathcal{G}|$ of Sing $\mathcal{G}$. The quotient space $|\mathrm{Sing}\,\mathcal{G}|/G$ fits

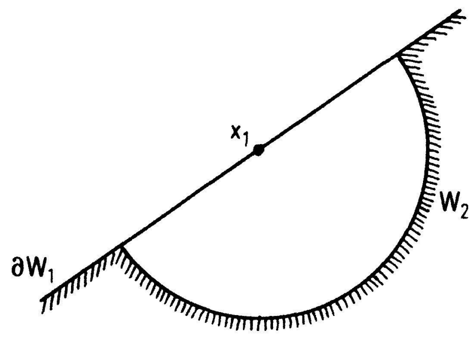

Fig. 1

into a fibration sequence

$$\mathscr{G} \simeq |\mathrm{Sing}\,\mathscr{G}| \to |\mathrm{Sing}\,\mathscr{G}|/G \to BG$$

and therefore is weakly equivalent to $\bar{B}\mathscr{G}$. In this paper, following [8], we will use $|\mathrm{Sing}\,\mathscr{G}|/G$ as our model for $\bar{B}\mathscr{G}$. Notice that $|\mathrm{Sing}\,\mathscr{G}|/G$ is the realization of the simplicial set $S = \mathrm{sing}\,\mathscr{G}/G$. If Δ^p denotes the standard p-simplex with vertices $(v_0, \ldots, v_p)$, a p-simplex σ of S is just a based continuous map

$$\theta_\sigma : (\Delta^p, v_0) \to (\mathscr{G}, \mathrm{id})$$

where id is the identity element of $\mathscr{G}$. Further, if Δ^q is a face of Δ^p with first vertex v_i, the corresponding face of σ is given by the map

$$\theta_\alpha \cdot \theta_\sigma(v_i)^{-1} \mid \Delta^q.$$

In other words, one must renormalize θ_σ as well as restricting its domain.

It turns out to be useful to represent elements of $H_*(\bar{B}\mathscr{G}; \mathbf{Z})$ by maps of *cubical* complexes into $\bar{B}\mathscr{G}$. Thus let K be a finite cell complex which is obtained from a disjoint union of oriented d-dimensional cubes by making certain linear identifications of the faces. We will suppose that the vertices of K are ordered and that for each cube $\kappa \subset K$ with first vertex v_κ we are given a map $f_\kappa : (\kappa, v_\kappa) \to (\mathscr{G}, \mathrm{id})$. If these maps are compatible, in other words, if

$$f_\lambda(a) = f_\kappa(a) f_\kappa(v_\lambda)^{-1}, \qquad a \in \lambda,$$

whenever $\lambda \subset \kappa$, then they fit together in a unique way to give a map $f: K \to \bar{B}\mathscr{G}$. To see this, let T be the triangulation of K obtained by starring each cube at its barycenter, and order the vertices of T lexicographically. Then each p-simplex σ in T is taken by f to the simplex in $\bar{B}\mathscr{G}$ which corresponds to the singular simplex

$$(\Delta^p, v_0) \xrightarrow{\iota} (\sigma, v_\sigma) \xrightarrow{f_\kappa \cdot f_\kappa(v_\sigma)^{-1}} (\mathscr{G}, \mathrm{id}),$$

where ι is the natural identification and where κ is some cube containing σ. The compatibility conditions ensure that this is independent of the choice of κ. Thus the f_κ define a d-chain (K, f). Its boundary $\partial(K, f)$ is obtained by restricting f to the $(d-1)$-cubes of K, where these are taken with the appropriate cancellations and multiplicities.

Notice that we do not collapse degeneracies here. Since the degenerate cubes are factored out when one defines homology by means of the cubical complex, it is necessary to check that every element of $H_*(\bar{B}\mathscr{G})$ may be represented by a cubical cycle (K, f) as above. However this follows because the standard simplex Δ^n, with barycentric coordinates $\lambda_0, \ldots, \lambda_n$, has a canonical subdivision into projectively embedded n-cubes $C_0, \ldots, C_n$. In fact, let C_p be the set of all points in Δ^n for which $\lambda_p = \max\{\lambda_0, \ldots, \lambda_n\}$. Then C_p is homeomorphic to the standard n-cube, with linear coordinates $0 \leq \lambda_i/\lambda_p \leq 1$ for $i \neq p$. Therefore one may obtain a suitable cubical representative of a homology class by subdividing a simplicial cycle.

An alternative way to describe the chain (K, f) is to define maps $f_C : C \to \mathscr{G}$, where C runs over a family of subcomplexes of K which cover K. These maps must satisfy the compatibility conditions

$$f_{C'}(a)f_{C'}(a_0)^{-1} = f_C(a)f_C(a_0)^{-1}, \qquad a \in C \cap C',$$

where a_0 is any fixed element of $C \cap C'$. For example, if K^* is a subdivision of K into little cubes, the f_κ, $\kappa \subset K$, define a chain (K^*, f^*). If we identify K and K^* as topological spaces, the maps f and f^* are not equal. However they are clearly homotopic.

If K' is a subcomplex of K, we will write (K', f) for the chain obtained by restricting f to K'.

EXAMPLE 2.3. Let K be the unit square $\kappa = \{(a, b) : 0 \leq a, b \leq 1\}$ with vertices ordered as $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$, and define $f_\kappa(a, b) = h(a)g(b)$ where $h(0) = g(0) = \mathrm{id}$. Then (K, f) is a 2-chain in $\bar{B}\mathscr{G}$. Its boundary is the union of the two 1-chains $b \mapsto g(b)$ and $b \mapsto h(1)g(b)h(1)^{-1}$, since the chains corresponding to $b = 0, 1$ cancel. Thus (K, f) is a 2-cycle if $h(1)$ commutes with the $g(b)$.

Now consider the case $\mathscr{G} = \mathscr{G}_1$, and let (K, f) be a d-chain as above. The *support* $\operatorname{supp} f_\kappa$ *of a cube* κ in K is defined to be the closure in W_1 of the set $\{x \in W_1 : f_\kappa(a)(x) \neq x \text{ for some } a \in \kappa\}$. Clearly

$$\operatorname{supp} f_\lambda \subseteq \operatorname{supp} f_\kappa \quad \text{whenever} \quad \lambda \subseteq \kappa.$$

We define $\operatorname{supp}(K, f)$ to be the union of $\operatorname{supp} f_\kappa$ over all $\kappa \in K$. Observe that (K, f) is a relative cycle in $(\bar{B}\mathscr{G}_1, \bar{B}\mathscr{G}_2)$ if and only if $\operatorname{supp} \partial(K, f) \subset \operatorname{Int} W_2$. Thus, in Example 2.3, (K, f) is a relative 2-cycle if $g(b)$ and $h(1)g(b)h(1)^{-1}$ have supports in $\operatorname{Int} W_2$ for all b.

We aim to show that any relative d-cycle (K, f) in $(\bar{B}\mathscr{G}_1, \bar{B}\mathscr{G}_2)$ is null-homologous. To keep control on the boundary, it is convenient to consider cycles for which there is no need to make cancellations when passing to their boundaries. Therefore, we say that a complex K is *reduced* if it may be obtained from a union of disjoint d-cubes by identifying pairs of oppositely oriented $(d-1)$-dimensional faces. Its boundary ∂K is then the subcomplex of K which is spanned by the $(d-1)$-cubes which lie in only one d-cube. Further, we define a *reduced relative d-cycle* to be a d-chain (K, f) where K is reduced and where $\operatorname{supp} f_\kappa \subset \operatorname{Int} W_2$ for all $\kappa \subset \partial K$. Thus $\operatorname{supp}(\partial K, f) \subset \operatorname{Int} W_2$, so that we can consider the boundary of (K, f) to be $(\partial K, f)$. For example, suppose that in (2.3) above the edges $a=0, 1$ have support in $\operatorname{Int} W_2$ while $b=0, 1$ do not. Then (K, f) is a relative cycle, but it is not reduced since one must cancel the edges $b=0, 1$ to obtain $\partial(K, f)$. However it may be reduced by identifying the edge $b=0$ with the edge $b=1$ and then subdividing κ into two embedded cubes by the line $b=\text{const}$. Note also that every homology class in $H_*(\bar{B}\mathscr{G}_1, \bar{B}\mathscr{G}_2)$ may be represented by a reduced relative d-cycle (K, f). This holds because every class is represented by a relative simplicial cycle, which may be reduced by changing the identifications of its $(d-1)$-dimensional faces and then subdividing. One then takes K to be a cubical subdivision of this simplicial cycle.

Now let $\mathscr{V} = \{V_i : i \in A\}$ be any open cover of W_1. The chain (K, f) will be said to be *supported by* $\mathscr{V}$ if there is a function α which assigns to every cube $\kappa \subset K$ a set $\alpha(\kappa) \subseteq A$ in such a way that

(2.4) (i) $|\alpha(\kappa)| \leq \dim \kappa$,
(ii) $\operatorname{supp} f_\kappa \subseteq \bigcup \{V_i : i \in \kappa(\kappa)\}$, and
(iii) $\alpha(\lambda) \subseteq \alpha(\kappa)$ if $\lambda \subseteq \kappa$.

Further, let $\mathscr{V}'$ be a subfamily $\{V_i : i \in A'\}$ of $\mathscr{V}$ and put $W' = \bigcup \{V_i : i \in A'\}$. If K' is a subcomplex of K, then we will say that the triple (K, K', f) is *supported by* $(\mathscr{V}, \mathscr{V}')$ if there is a function α which in addition to the above three conditions satisfies

(iv) $\alpha(\kappa) \subseteq A'$ for all $\kappa \subset K'$.

This condition clearly implies that $\operatorname{supp}(K', f) \subset W'$. A reduced relative d-cycle (K, f) will be said to be supported by $(\mathscr{V}, \mathscr{V}')$ if the triple $(K, \partial K, f)$ is so supported.

In §4 we will prove:

LEMMA 2.5 (Deformation Lemma). *Let (K, f) by any chain in $\bar{B}\mathscr{G}_1$ and let K' be a subcomplex of K such that* $\operatorname{supp}(K', f) \subset W'$. *Then there is a chain*

$(K \times I, F)$ *such that:*

(i) $(K \times 0, F) = (K, f)$,
(ii) $\operatorname{supp}(K' \times I, F) \subseteq W'$, *and*
(iii) *the triple* $(K \times 1, K' \times 1, F)$ *has a subdivision which is supported by* $(\mathcal{V}, \mathcal{V}')$,

COROLLARY 2.6. *Suppose that* (K, f) *is a reduced relative* d*-cycle such that* $\operatorname{supp}(\partial K, f) \subset W' \subset \operatorname{Int} W_2$. *Then* (K, f) *is homologous in* $(\bar{B}\mathcal{G}_1, \bar{B}\mathcal{G}_2)$ *to a reduced relative cycle which is supported by* $(\mathcal{V}, \mathcal{V}')$.

Notice that if a relative cycle (K, f) is supported by $(\mathcal{V}, \mathcal{V}')$ then the support of each cube in the cycle is small. However the support of the cycle (K, f) might still be almost the whole of $\operatorname{Int} W_1$. In the next section we describe a d-fold conjugation process which at each step takes a little more of the support of (K, f) into W_2.

§3. The conjugation lemma

Throughout this section we assume that (K, f) is a reduced relative d-cycle. Our first task is to construct a suitable cover $\mathcal{V}$.

(3.1) *The cover* $\mathcal{V}$

Let $N > 2d^2 + 2$. We will assume that $\operatorname{vol}(W_1 - W_2) < 1/N \operatorname{vol} W_1$ as in Lemma 2.2. The cover $\mathcal{V}$ will consist of sets $V_1, \ldots, V_N$ as in Fig. 2. Thus we require:

(i) $\bar{V}_i \cap \bar{V}_j = \varnothing$ if $|i - j| > 1$;
(ii) the sets $\bar{V}_2, \ldots, \bar{V}_{N-1}$ and $\bar{V}_1 \cap \bar{V}_2, \ldots, \bar{V}_{N-1} \cap \bar{V}_N$ are diffeomorphic to $D^{n-1} \times I$, where D^{n-1} is the closed $(n-1)$-disc;
(iii) each set $V_{(1,i)} = V_1 \cup \cdots \cup V_i$, $i < N$, is diffeomorphic to an open disc neighbourhood of x_1;
(iv) if $\mathring{V}_i = V_i - (\bar{V}_{i-1} \cup \bar{V}_{i+1})$, then

$$\operatorname{vol} \mathring{V}_1 < \operatorname{vol} \mathring{V}_i \quad \text{for} \quad 2 \leq i < N.$$

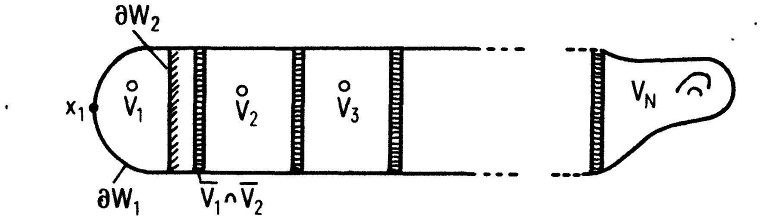

Fig. 2

We will assume the sets $V_i \cap V_{i+1}$ and V_N have very small volume so that the $\mathring{V}_i$, $1 \leq i < N$, fill out almost all of W_1.

If $\alpha \subset \{1, \ldots, N\}$ we write V_α for $\bigcup \{V_i : i \in \alpha\}$. Similarly $\mathring{V}_\alpha = \bigcup \{\mathring{V}_i : i \in \alpha\}$. Recall also the notation $V_{(1,i)} = V_1 \cup \cdots \cup V_i$ used in (ii) above.

Since supp $(\partial K, f)$ is a compact subset of Int W_2, we may choose V_1 and V_2 so that

(v) $\operatorname{supp}(\partial K, f) \subseteq W_1 - \bar{V}_1 \subseteq W_1 - \mathring{V}_1 \subseteq \operatorname{Int} W_2$.

This is compatible with (iv) above because $\operatorname{vol}(W_1 - W_2) < 1/N \operatorname{vol} W_1$. If $\mathcal{V}' = \{V_i : 2 \leq i \leq N\}$, the cycle (K, f) will then satisfy the conditions of Corollary 2.6 with respect to $(\mathcal{V}, \mathcal{V}')$, and so will be homologous to a cycle which is supported by $(\mathcal{V}, \mathcal{V}')$. Therefore, it will suffice to prove:

LEMMA 3.2 (Conjugation Lemma). *Suppose that (K, f) is a reduced relative d-cycle in $(\bar{B}\mathcal{G}_1, \bar{B}\mathcal{G}_2)$ which is supported by the cover $(\mathcal{V}, \mathcal{V}')$ of* (3.1) *above. Then (K, f) is null-homologous.*

Proof when $d = 1$. Because $\bar{B}\mathcal{G}_1$ has only one vertex, every 1-chain (K, f) is an (absolute) 1-cycle. In particular, since (K, f) is supported by $\mathcal{V}$, it is a sum of 1-cycles each supported by some V_i. Therefore we just have to show that any 1-cycle with support in V_1 is homologous to a cycle with support in Int W_2. Since $N > 2$, $\operatorname{vol} V_1 < \operatorname{vol}(V_1 \cap V_2) \cup \mathring{V}_2 < \operatorname{vol} W_2$. Therefore, given any compact subset S of $V_1 \cap \operatorname{Int} W_1$, there is a path h_t, $0 \leq t \leq 1$, in $\mathcal{G}_1$ with $h_0 = \mathrm{id}$ and such that $h_1(S) \subset \operatorname{Int} W_2$. (See Remark (3.3) below.) Thus the proof may be completed by constructing a 2-chain as in Example 2.3. □

Remark 3.3. In general, the only obstructions to constructing a volume-preserving isotopy which moves sets around in a prescribed way are the obvious ones involving volume. See [6]. One can ensure that h_t has zero flux by making its support lie in a contractible subset of W_1. When $n = 2$, one can also ensure that h_t lies in the kernel of ρ by replacing it by $h_t k_t$, where $\rho(k_t) + \rho(h_t) = 0$ and where k_t has support in a tiny disc which is disjoint from supp h_t. Then we will have $h_t \in \mathcal{G}_1$ in all cases.

The case $d = 1$ is so simple that one does not need the special properties of the cover $\mathcal{V}$. These will be useful later on, but first we must homotop (K, f) to a cycle (K_1, F) which is easier to manipulate.

(3.4) *The cycle (K_1, F)*

Let K^* be the first barycentric subdivision of K. It is an ordered cubical complex with one q-cube $D(\lambda, \kappa)$ for each pair of cubes λ, κ in K with $\lambda \subset \kappa$,

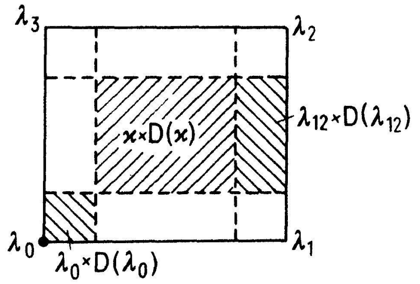

Fig. 3

where $q = \dim \kappa - \dim \lambda$. One can easily check that it is reduced. Let $D(\lambda)$ be the subcomplex $\bigcup \{D(\lambda, \kappa): \kappa \supseteq \lambda\}$ of K^*. Now consider the subcomplex

$$K_1 = \bigcup_{\lambda \subset \kappa} \lambda \times D(\lambda, \kappa) = \bigcup_{\lambda} \lambda \times D(\lambda)$$

of $K \times K^*$. This is an ordered d-dimensional cubical complex whose d-cubes have the form $\lambda \times D(\lambda, \kappa)$ where $\dim \kappa = d$. See Fig. 3.

We will write $\mu(\lambda)$ for the subcomplex $\lambda \times D(\lambda)$ of K_1. If $\lambda \subset \kappa$ then $\mu(\lambda) \cap \mu(\kappa) = \lambda \times D(\kappa)$. It is not hard to check that K_1 is homeomorphic to K and so may be considered as a subdivision of K. Thus it is reduced and has boundary $\partial(K_1) = (\partial K)_1$.

There is a natural map $\pi: K_1 \to K$ which projects each $\mu(\lambda) = \lambda \times D(\lambda)$ onto the first factor λ. The maps $F_{\mu(\lambda)} = f_\lambda \circ \pi$ are clearly compatible. They fit together to form a reduced relative d-cycle (K_1, F) which is homologous to (K, f).

(3.5) *The conjugating map h*

We wish to define a map $h: K_1 \times I \to \mathscr{G}_1$ which will conjugate the support of (K_1, F) into Int W_2. Let Z be a set of the form $W_1 -$(open collar *nbhd* of ∂W_1) which contains supp (K, f), and let $T \subset V_{(1,d+1)} = V_1 \cup \cdots \cup V_{d+1}$ be a thin tube which intersects the sets $\mathring{V}_1, \ldots, \mathring{V}_{d+1}$ in turn and which lies outside Z. See Fig. 4.

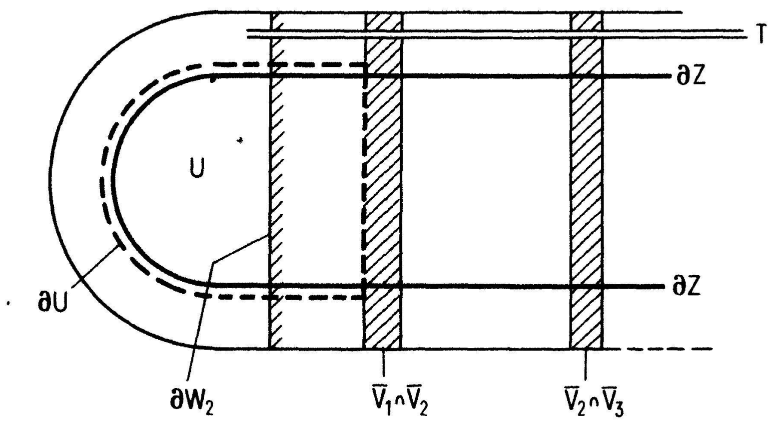

Fig. 4

Further, Let U be a neighbourhood of $Z \cap \mathring{V}_1$ in $\mathring{V}_1$ whose closure does not meet ∂W_1 or T. Then vol $U <$ vol $\mathring{V}_i$, $2 \leq i < N$, by (3.1)(iv). Therefore, one can find a path m_t, $0 \leq t \leq 1$, in $\mathscr{G}_1$ with support in $\mathring{V}_1 \cup T \cup \mathring{V}_2$ such that $m_0 = \mathrm{id}$ and $m_1(U) \subset \mathrm{Int}\, W_2$. See Remark (3.3). Observe that if supp $g \subset Z - \mathring{V}_2$ then supp $(m_1 g m_1^{-1}) = m_1(\text{supp } g) \subset \mathrm{Int}\, W_2$.

Now recall from (2.4) that because (K, f) is supported by $\mathcal{V}$ there is a function α : (cubes of K) $\to$ (subsets of $\{1, \ldots, N\}$) such that

$$|\alpha(\kappa)| \leq \dim \kappa; \qquad \alpha(\lambda) \subseteq \alpha(\kappa) \quad \text{if} \quad \lambda \subseteq \kappa;$$

and

$$\text{supp } f_\kappa \subset V_{\alpha(\kappa)} = \bigcup \{V_i : i \in \alpha(k)\}.$$

Choose a number $\beta(\kappa)$ in $\{1, \ldots, d+1\} - \alpha(\kappa)$ for each d-cube κ. In general, set $\beta(\lambda) = \bigcup \{\beta(\kappa) : \lambda \subset \kappa, \dim \kappa = d\}$. Then supp f_λ is disjoint from $\mathring{V}_{\beta(\lambda)}$ for all λ.

We are now ready to define the conjugating map h.

LEMMA 3.6. *There is a map $h : K_1 \times I \to \mathscr{G}_1$ such that*

(i) $h(\partial K_1 \times I \cup K_1 \times 0) = \mathrm{id}$;

(ii) *for each cube κ in K the restriction of h to $\kappa \times D(\kappa) \times I$ is a composite*

$$\kappa \times D(\kappa) \times I \xrightarrow{\text{proj.}} D(\kappa) \times I \xrightarrow{h_\kappa} \mathscr{G}_1;$$

(iii) *for each κ and $b \in D(\kappa)$*

$$\text{supp } h_\kappa(b, t) \subset \mathring{V}_1 \cup T \cup \mathring{V}_{\beta(\kappa)},$$

where $\beta(\kappa)$ is as above;

(iv) *for each $\kappa \subset K$, $\kappa \not\subset \partial K$ and $b \in D(\kappa)$*

$$h_\kappa(b, 1)(U) \subseteq \mathrm{Int}\, W_2 \cap V_{(1,d+1)}.$$

Proof. We define the h_κ first for cubes of dimension d, then for those of dimension $d-1$ and so on. If $\dim \kappa = d$, then $|\beta(\kappa)| = 1$ and $D(\kappa)$ is a single point, b_κ say. Therefore, we may choose the path $h_\kappa(b_\kappa, t)$ to be a suitable conjugate of the path m_t defined in (3.5) above. In general, if $\dim \kappa = q$, and h_λ has been defined for all λ of dimension $>q$, then h_κ is determined on $\partial D(\kappa) \times I$. Since

$\beta(\kappa)\subset\beta(\lambda)$ whenever $\lambda\subset\kappa$, the support of $h_\kappa \mid \partial D(\kappa)\times I$ satisfies (ii) and (iv). It is not hard to see that one can extend h_κ to the whole of $D(\kappa)\times I$. If $\kappa\subset\partial K$, one must take care to satisfy (i) also. Further details will be left to the reader. $\square$

(3.7) *The basic conjugation process*

We define a $(d+1)$-chain $(K_1\times I, H)$ as follows:

if $\mu=\kappa\times D(\kappa)\times I\subset K_1\times I$,

then

$$H_\mu(a, b, t)=h_\kappa(b, t)f_\kappa(a).$$

Observe that $H_\mu(a_0, b_0, 0)=\mathrm{id}$ by (3.6)(i) if (a_0, b_0) is the first vertex of $\kappa\times D(\kappa)$. Hence H_μ is properly normalized. Since h is globally defined on $K_1\times I$ and the f_κ are compatible, the H_μ are also compatible, and so fit together to form the chain $(K_1\times I, H)$. The boundary of $(K_1\times I, H)$ has three parts:

$$(K_1\times 0, H),\qquad (K_1\times 1, H)\quad\text{and}\quad(\partial K_1\times I, H).$$

By Lemma 3.6(i) we have $(K_1\times 0, H)=(K_1, F)$. Also $\operatorname{supp}(\partial K_1\times I, H)\subset\operatorname{Int} W_2$. Hence (K, f) is homologous to $(K_1\times 1, H)$.

Let us write $\bar H$ for the restriction of H to $K_1\times 1=K_1$. Then on the subcomplex $\mu(\kappa)=\kappa\times D(\kappa)$ of K_1 the map $\bar H$, when normalized at the vertex (a_0, b_0), takes the form:

$$\begin{aligned}\bar H_{\mu(\kappa)}(a, b)&=h_\kappa(b, 1)f_\kappa(a)h_\kappa(b_0, 1)^{-1}\\&=l_\kappa(b)g_\kappa(a),\end{aligned}$$

where $l_\kappa(b)=h_\kappa(b, 1)h_\kappa(b_0, 1)^{-1}$ and $g_\kappa(a)=h_\kappa(b_0, 1)f_\kappa(a)h_\kappa(b_0, 1)^{-1}$. Observe that

$$\operatorname{supp} l_\kappa(b)\subset V_{(1,d+1)}$$

by (3.6)(iii). Also, because $\operatorname{supp} f_\kappa\subset Z\cap V_{\alpha(\kappa)}$, and because $T\cup\mathring{V}_{\beta(\kappa)}$ is disjoint from $Z\cap V_{\alpha(\kappa)}$, conditions (iii) and (iv) of (3.6) imply that $\operatorname{supp} g_\kappa\subset\operatorname{Int} W_2$ when $\kappa\not\subset\partial K$. This holds also when $\kappa\subset\partial K$ because $h_\kappa(b, t)$ commutes with $f_\kappa(a)$ by (3.1)(v). One may also take $(a_0, b_0)\in\partial K_1$ so that $h_\kappa(b_0, 1)=\mathrm{id}$ by (3.6)(ii). Note further that

$$\operatorname{supp} l_\kappa(b)g_\kappa(a)l_\kappa(b)^{-1}\subset\operatorname{Int} W_2\cap(V_{(1,d+1)}\cup V_{\alpha(\kappa)})$$

for all $b \in D(\kappa)$. In particular, if $\dim \kappa = d$, then $\operatorname{supp} \bar{H}_{\mu(\kappa)} \subset \operatorname{Int} W_2$, so that the cubes $\mu(\kappa)$, $\dim \kappa = d$, contribute nothing to $(K_1, \bar{H})$.

Proof of Lemma 3.2

We prove this by induction, using the following inductive hypothesis:

$IH(k)$: There is a function

$\alpha:(\text{cubes in } K) \to (\text{subsets of } \{1, 2kd+2, \ldots, N\})$

such that

(i) $|\alpha(\kappa)| \leq d - k$,
(ii) $\alpha(\mu) \subset \alpha(\kappa)$ if $\mu \subset \kappa$; and
(iii) $\operatorname{supp} f_\kappa \subset V_{\alpha(\kappa)} \cup (\operatorname{Int} W_2 \cap V_{(1,2kd+1)})$.

Any reduced relative d-cycle which is supported by $(\mathcal{V}, \mathcal{V}')$ satisfies $IH(0)$. Also, if (K, f) satisfies $IH(d)$ then each $\alpha(\kappa)$ must be empty. Therefore (iii) implies that $\operatorname{supp}(K, f) \subseteq \operatorname{Int} W_2$. Hence it remains to prove:

LEMMA 3.8. *Any reduced cycle for which $IH(k)$ holds for some $k < d$ is homologous to a reduced cycle for which $IH(k+1)$ holds.*

We begin with the following result.

LEMMA 3.9. *Suppose that (K, f) satisfies $IH(k)$ for some $k < d$. Then (K, f) is homologous to a reduced cycle $(\bar{K}, \bar{f})$ which has a function α satisfying the conditions of $IH(k)$ as well as:*

(iv) $1 \in \alpha(\kappa)$ *if and only if* $\kappa \not\subset \partial \bar{K}$.

Proof. Because (K, f) is a relative cycle, $\operatorname{supp} f_\kappa \subset \operatorname{Int} W_2$ for all $\kappa \subset \partial K$. therefore if we define

$$\alpha_1(\kappa) = \alpha(\kappa) \quad \text{for} \quad \kappa \not\subset \partial K,$$

and

$$\alpha_1(\kappa) = \alpha(\kappa) - \{1\} \quad \text{for} \quad \kappa \subset \partial K,$$

the function α_1 satisfies $IH(k)$. Hence we may suppose that $1 \notin \alpha(\kappa)$ for all $\kappa \subset \partial K$.

Now consider the cycle (K_1, F) constructed in (3.4). Define α_1 for K_1 by setting

$$\alpha_1(\tau) = \bigcap_{\tau\subset\mu(\kappa)} \alpha(\kappa), \quad \text{for all cubes } \tau \text{ in } K_1,$$

where $\mu(\kappa) = \kappa \times D(\kappa) \subset K_1$. It is not hard to check that this satisfies $IH(k)$. Let $C = \{\kappa \in K : 1 \notin \alpha(\kappa)\}$ and put

$$\bar{K}_1 = K_1 - \operatorname{Int}\left(\bigcup_{\kappa\in C} \mu(\kappa)\right).$$

Since $\operatorname{supp} F_{\mu(\kappa)} = \operatorname{supp} f_\kappa \subset \operatorname{Int} W_2$ for all $\kappa \in C$, $(\bar{K}_1, F)$ is a (reduced) relative cycle homologous to (K, f). We claim that the function α_1 when restricted to $\bar{K}_1$ satisfies (iv) above. For, by construction, $\partial K \subset C$. It follows that $\tau \subset \partial\bar{K}_1$ if and only if the set $\{\kappa : \tau \subset \mu(\kappa)\}$ intersects C (but is not entirely contained in C.). Also, if $\tau \not\subset \partial\bar{K}_1$ then the set $\{\kappa : \tau \subset \mu(\kappa)\}$ is disjoint from C. Condition (iv) now follows easily. □

Proof of Lemma 3.8

We will suppose as we may that the function α on (K, f) satisfies condition (3.9)(iv). Choose a function

$$\beta : (\text{cubes in } K) \to (\text{subsets of } \{2kd+2, \ldots, 2kd+d+1\})$$

so that

$$\beta(\kappa) \cap \alpha(\kappa) = \varnothing, \qquad \beta(\kappa \cap \lambda) = \beta(\kappa) \cup \beta(\lambda),$$

and

$$|\beta(\kappa)| = 1 \quad \text{if} \quad |\alpha(\kappa)| = d - k.$$

We now want to define a map $h : K_1 \times I \to \mathscr{G}_1$ as in Lemma 3.6 with respect to this β. We will choose Z and U as before, and will take T to be a tube in $V_{(1,2kd+d+1)}$ which is disjoint from Z. Since (K, f) is a relative cycle there is a compact subset A of $(\operatorname{Int} W_2) \cap V_1$ such that $\operatorname{supp}(\partial K, f) \subset A \cup V_{(2,N)}$. Then we require h to satisfy conditions (i) and (ii) of Lemma 3.6 as well as:

(iii)′ for each $\kappa \subset K$

$$\operatorname{supp} h_\kappa \subset (\mathring{V}_1 - A) \cup T \cup \mathring{V}_{\beta(\kappa)} \subset V_{(1,2kd+d+1)},$$

(iv)′ for each $\kappa \subset K$, $\kappa \not\subset \partial K$ and $b \in D(\kappa)$

$$h_\kappa(b, 1)(U) \subset \operatorname{Int} W_2 \cap V_{(1,2kd+d+1)}.$$

Now observe that if $|\alpha(\kappa)| = d-k$ then $\alpha(\kappa) = \alpha(\lambda)$ for all $\lambda \supset \kappa$. Also, if S is a subset $\{1, 2kd+2, \ldots, N\}$ with $|S| = d-k$, the subcomplex

$$K_1(S) = \bigcup \{\mu(\kappa) : \alpha(\kappa) = S\}$$

does not intersect $K_1(S')$ unless $S = S'$. Therefore we may define β and h so that:

(v) on each set $K_1(S) \times I$, h depends only on $t \in I$,
and then extend h to the rest of $K_1 \times I$.

Consider the cycle $(K_1, \bar{H})$ constructed from h as in (3.7). For each $\kappa \in K$ we have

$$\bar{H}_{\mu(\kappa)}(a, b) = l_\kappa(b) g_\kappa(a)$$

where $l_\kappa(b) = h_\kappa(b, 1) h_\kappa(b_0, 1)^{-1}$ and where

$$\operatorname{supp} g_\kappa(a) \subset \operatorname{Int} W_2 \cap (V_{(1,2kd+d+1)} \cup V_{\alpha(\kappa)}). \tag{*}$$

(When $\kappa \subset \partial K$, this holds by (iii)′ above.) In particular, $l_\kappa(b) = \mathrm{id}$ for all κ such that $\mu(\kappa) \subset K_1(S)$, by (v), so that the chain $(K_1(S), \bar{H})$ has support in Int W_2. Therefore, if

$$K_2 = K_1 - \bigcup_S \operatorname{Int} K_1(S),$$

then $(K_2, \bar{H})$ is a cycle homologous to $(K_1, \bar{H})$ and hence to (K, f).

This cycle nearly satisfies $IH(k+1)$. To see this, let

$$\mu(\partial K) = \bigcup \{\mu(\kappa) : \kappa \subset \partial K\}$$

and define α_1 on K_1 by

$$\begin{aligned} \alpha_1(\tau) &= \{1\} \cup \Big(\{2(k+1)d+2, \ldots, N\} \cap \Big(\bigcap_{\tau \subset \mu(\kappa)} \alpha(\kappa)\Big)\Big) \quad \text{if} \\ &\quad \tau \subset \mu(\partial K) \text{ and } \tau \not\subset \partial K_1 \\ &= \{1, 2(k+1)d+2, \ldots, N\} \cap \Big(\bigcap_{\tau \subset \mu(\kappa)} \alpha(\kappa)\Big) \quad \text{otherwise.} \end{aligned}$$

It follows from condition (3.9)(iv) that $1 \in \alpha_1(\tau)$ for all $\tau \not\subset \partial K_1$. Now, (*) together with conditions (iii)′ and (iv)′ above imply that

$$\operatorname{supp} \bar{H}_{\mu(\kappa)} \subset V_{(1,2(k+1)d+1)} \cup V_{\alpha(\kappa)}.$$

Therefore, if $\tau \not\subset \partial K_1$,

$$\operatorname{supp} \bar{H}_{\mu(\kappa)} \subset V_{\alpha_1(\tau)} \cup (\operatorname{Int} W_2 \cap V_{(1,2(k+1)d+1)}).$$

On the other hand, if $\tau \subset \partial K_1$, then the function h is constant on τ by Lemma 3.6(i). Hence $l_\tau(b) = \mathrm{id}$, which implies that

$$\operatorname{supp} \bar{H}_\tau \subset \operatorname{Int} W_2 \cap (V_{(1,2(k+1)d+1)} \cup V_{\alpha(\kappa)})$$

for all κ with $\tau \subset \mu(\kappa)$. Thus the function α_1 satisfies condition (iii) of $IH(k+1)$ for all $\tau \subset K_1$. It clearly also satisfies (ii). As for (i), observe that if $\tau \not\subset \mu(\partial K)$ then $|\alpha_1(\tau)| = d-k$ only if the sets $\alpha(\kappa)$, $\tau \subset \mu(\kappa)$, are all the same and all have $d-k$ elements. But this implies that $\tau \subset \operatorname{Int} K_1(S)$ for some S, so that τ is not in K_2. Observe also that because $1 \in \alpha(\kappa)$ for all $\kappa \not\subset \partial K$, we must have $|\alpha(\kappa)| \leq d-k-1$ for $\kappa \subset \partial K$. Thus $\mu(\partial K) \subset K_2$ and $|\alpha_1(\tau)| \leq d-k$ for $\tau \subset \mu(\partial K)$. Therefore all we have to do now is cut out from K_2 the cubes in $\mu(\partial K)$ with $|\alpha_1(\tau)| = d-k$.

To do this, let R be a subset of $\{2kd+2, \ldots, N\}$ with $d-k-1$ elements and define

$$L(R) = \bigcup \{\mu(\kappa) : \kappa \subset \partial K, \alpha(\kappa) = R\} \subset \mu(\partial K),$$
$$L'(R) = \partial K_1 \cap L(R).$$

Since $|\alpha(\kappa)| \leq d-k-1$ on ∂K, the subcomplexes $L(R)$ are disjoint for distinct R. Let $K_3 = K_2 - \bigcup_R \operatorname{Int} L(R)$. Then the restriction of α_1 to K_3 satisfies the conditions of $IH(k+1)$ since we have removed all the cubes for which (i) fails. However, $(K_3, \bar{H})$ is no longer a relative cycle. Our aim now is to define chains $(M(R), G)$ for each R so that $(L(R), \bar{H}) + (M(R), G)$ is a relative boundary and so that $(K_3, \bar{H}) + \sum_R (M(R), G)$ is a relative cycle which satisfies $IH(k+1)$.

Note that $L(R) = L'(R) \times [0, 1]$. If we identify $L'(R) \times 0$ with $L'(R) \subset \partial K_1$, then it follows from (3.9)(iv) that $L'(R) \times 1 \subset K_1(S)$, where $S = \{1\} \cup R$. We chose h so that h is constant on each $K_1(S)$: see (v) above. Clearly, we may also assume that on each set $L'(R) \times [0, 1]$ the map h depends only on $s \in [0, 1]$. Then, for each $\tau \subset L'(R)$, we will have

$$\bar{H}_{\tau \times [0,1]}(a, s) = l_\tau(s) F_\tau(a), \quad \text{for} \quad (a, s) \in \tau \times [0, 1],$$

where $F_\tau = f_\kappa \circ \pi$ for some $\kappa \subset \partial K$ as in (3.4). Note also that the $l_\tau(s)$ commute with the $F_\tau(a)$ because of condition (iii)′ in the definition of h.

Now consider $\partial(L(R), \bar{H})$. The pieces $(L'(R)\times\{i\}, \bar{H})$, $i=0, 1$, have support in Int W_2. Therefore $\partial(L(R), \bar{H})$ is a sum of chains of the form $(\tau\times[0, 1], \bar{H})$, where τ is a $(d-2)$-cube in ∂K_1 with $\alpha(\tau)\subsetneqq R$. We will write $\partial L'(R)$ for this set of $(d-2)$-cubes. Since $|R|<d$, we may choose an integer $j\notin R$ such that $2kd+d+1<j\leqslant 2(k+1)d+1$. Let m_t, $0\leqslant t\leqslant 1$, be a path in $\mathcal{G}_1$ with support in $(\mathring{V}_1-A)\cup T'\cup \mathring{V}_j$ such that $m_1(U)\subset W_2$. Here T' is a tube in $V_{(1,2(k+1)d+1)}$ which does not meet supp F_τ for $\tau\subset L'(R)$. Therefore m_t commutes with the F_τ. Further, because supp $l_\tau \subset V_{(1,2kd+d+1)}$ by (iii)′, we may assume that $m_1 l_\tau(s) m_1^{-1}$ has support in Int W_2 for all s. Now define

$$G_\tau(t, a, s) = m_t l_\tau(s) F_\tau(a) \quad \text{for} \quad (t, a, s)\in I\times\tau\times[0, 1].$$

Because l_τ commutes with F_τ, the faces $(I\times\tau\times\{i\}, G_\tau)$, $i=0, 1$, of the chain $(I\times\tau\times[0, 1], G_\tau)$ cancel. Further $(\{1\}\times\tau\times[0, 1], G_\tau)$ has support in Int W_2, and $(\{0\}\times\tau\times[0, 1], G_\tau)=(\tau\times[0, 1], \bar{H})$. Therefore, if we put

$$(M(R), G) = \sum_{\tau\subset\partial L'(R)} (I\times\tau\times[0, 1], G_\tau),$$

then $(L(R), \bar{H})+(M(R), G)$ is a relative boundary. Thus the cycle $(K_3, \bar{H})+\sum_R (M(R), G)$ is homologous to $(K_1, \bar{H})$ and hence to (K, f). This cycle is not reduced since in the calculation of its boundary one must cancel the face $(I\times\tau\times\{1\}, G_\tau)$ with $(I\times\tau\times\{0\}, G_\tau)$ and must cancel $(\{0\}\times\tau\times[0, 1], G_\tau)$ with $(\tau\times[0, 1], \bar{H})$ in K_3. However, it is homologous to a reduced relative cycle $(K_4, \bar{G})$, where K_4 is formed from K_3 and the $M(R)$ by making the identifications which correspond to the above cancellations and then subdividing, and where $\bar{G}$ is induced by G and $\bar{H}$ in the obvious way.

We claim that $(K_4, \bar{G})$ satisfies $IH(k+1)$. To see this, define the function α_2 on K_4 by

$$\begin{aligned} \alpha_2(\lambda) &= \alpha_1(\lambda) && \text{if} \quad \lambda\subset K_3 \\ &= \alpha_1(\tau)\cup\{1\} && \text{if} \quad \lambda\subset I\times\tau\times[0, 1]\subset M(R), \lambda\not\subset K_3. \end{aligned}$$

Since $|\alpha_1(\tau)|<d-k-1$ for $\tau\subset\partial L'(R)$, the function α_2 satisfies condition (i) of $IH(k+1)$. It is easy to check that the other conditions hold. This completes the proof of Lemma 3.8, and hence of Lemma 3.2. □

§4. The deformation lemma

This section is concerned with the proof of Lemma 2.5. In [11] Thurston gave a very brief outline of a proof in the case of a 1-chain on a manifold of dimension $\geqslant 3$. His method was later fully worked out in the symplectic case by Banyaga, both for 1-chains and 2-chains. The argument for $d>2$ is essentially the same: one just has to be very systematic, so that one can keep track of what is going on. We will begin by making some definitions and will describe the strategy of the proof in (4.3). Throughout we consider a triple (K, K', f) such that $\operatorname{supp}(K', f) \subset X \subset W'$, for some compact submanifold X of W'.

(4.1) *Coverings associated to a triangulation*

Put a Riemannian metric on W_1 and choose $\varepsilon>0$ so that ε-balls are geodesically convex and so that the ε-neighbourhood X_ε of X is contained in W'. Then choose a smooth convex triangulation $T=\{\Delta_i^k : i \in I_k, 0 \leqslant k \leqslant n\}$ of W_1 which restricts to a triangulation T' of X and is such that the ε-neighbourhood of any simplex in T, resp. T', is contained in a set of $\mathcal{V}$, resp. $\mathcal{V}'$. As in [1] ChIII.2, we associate to such a triangulation an open cover $\mathcal{U}=\{U_i^k : i \in I_k, 0 \leqslant k \leqslant n\}$ of W_1 with the following properties:

(a) each U_i^k is an ν-neighbourhood of a deformation retract of Δ_i^k for some $\eta<\varepsilon$,
(b) $\bar{U}_i^k \cap \bar{U}_j^l = \varnothing$ if either $k=l$ and $i \neq j$ or $k<l$ and Δ_i^k is not a face of Δ_j^l.
(c) For each k, the sets $U_i^j : 0 \leqslant j \leqslant k$, $i \in I_j$ cover the k-skeleton of T.

One should construct the U_i^k in order of increasing k. See Fig. 5. Note that $\mathcal{U}$ is a refinement of $\mathcal{V}$. Also $\mathcal{U}'=\{U_i^k : \Delta_i^k \in T'\}$ refines $\mathcal{V}'$.

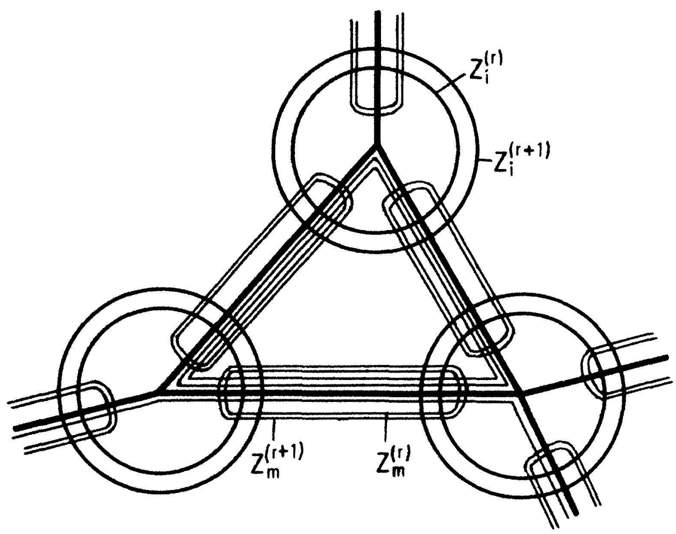

Fig. 5.

It will be convenient to renumber the sets U_i^k. Let $M_k = |I_0| + \cdots + |I_k|$ for each $0 \leq k \leq n$ and put $M = M_n$. We may assume that $I_k = \{1, \ldots, M_k - M_{k-1}\}$. Then U_i^k will be called U_m where $m = M_{k-1} + i$. In particular $U_m = U_m^0$ for $m \leq M_0$. (Our M_k slightly differ from Banyaga's N_k but the renumbering is the same as his.) We now choose a nested sequence of open covers $\mathscr{Z}^{(r)} = \{Z_m^{(r)} : 1 \leq m \leq M\}$, $-Q \leq r \leq Q + 2d + 1$, each of which is a slightly smaller version of $\mathscr{U}$, where $Q = 2Md$. Thus, for all r, s, m, l we have

$$Z_m^{(r)} \subset \bar{Z}_m^{(r)} \subset Z_m^{(r+1)} \subset \bar{Z}_m^{(Q+2d+1)} \subset U_m,$$
$$\bar{Z}_m^{(r)} \cap \bar{Z}_l^{(s)} \cong \bar{U}_m \cap \bar{U}_l.$$

A typical pair of such covers is shown in Fig. 5. We will often write $Z^{(r)}$ for a union of sets from $\mathscr{Z}^{(r)}$ and $Z^{(s)}$ for the corresponding union of sets from $\mathscr{Z}^{(s)}$. Further, we will write J' for the subset of $\{1, \ldots, M\}$ which corresponds to elements of T'. Thus $j \in J'$ if and only if $U_j = U_i^k$ where $\Delta_i^k \in T'$. We write $\mathscr{Z}'^{(r)}$ for $\{Z_j^{(r)} : j \in J'\}$. It is a refinement of $\mathscr{U}'$ and of $\mathscr{V}'$.

(4.2) *Neighbourhoods of the identity in* $\mathscr{G}_1$

Let $\mathcal{N}$ be any neighbourhood of the identity in $\mathscr{G}_1$. Then a k-simplex $\sigma \subset \bar{B}\mathscr{G}_1$ will be called *$\mathcal{N}$-small* if

$$\theta_\sigma(v)\theta_\sigma(w)^{-1} \in \mathcal{N}$$

for all v, w in the standard simplex Δ^k. Similarly, the chain (K, f) will be called $\mathcal{N}$-small if

$$f_\kappa(a)f_\kappa(b)^{-1} \in \mathcal{N}$$

for all a, b in κ and all cubes $\kappa \subset K$. Clearly, one may subdivide K to get a chain (K^*, f^*) which is $\mathcal{N}$-small and is homotopic to (K, f). Therefore, we may assume that our original triple (K, K', f) is $\mathcal{N}$-small for any given $\mathcal{N}$.

Let $\mathcal{M}$ be the set of all elements $g \in \mathscr{G}_1$ such that both g and g^{-1} take $Z_m^{(r)}$ into $Z_m^{(r+1)}$ for all m, r. Let $\mathcal{N}_i$, $0 \leq i \leq 2d+1$, be an increasing sequence of contractible C^1-neighbourhoods of the identity in $\mathscr{G}_1$ such that for each i we have:

(a) $\mathcal{N}_i = \mathcal{N}_i^{-1} \subset \mathcal{M}$; and
(b) for every union $Z^{(Q+i)}$ of sets from $\mathscr{Z}^{(Q+i)}$, every compact subset of the space $\{g \in \mathcal{N}_i\mathcal{N}_0 : \operatorname{supp} g \subset Z^{(Q+i)}\}$ contracts inside $\{g \in \mathcal{N}_{i+1} : \operatorname{supp} g \subset Z^{(Q+i+1)}\}$.

Finally $\mathcal{N}$ will be a very small neighbourhood of the identity which is contained in $\mathcal{N}_0$. Other conditions on $\mathcal{N}$ will be given later. We will assume from now on that (K, K', f) is $\mathcal{N}$-small.

(4.3) *The main construction*

For each p we identify the standard p-cube C^p with

$$\{x \in \mathbf{R}^p : 0 \leqslant x_1, \ldots, x_p \leqslant M\}$$

by a linear transformation. Then the hyperplanes $x_j \in \mathbf{Z}$ divide C^p into a collection of little cubes whose set of vertices is the integer lattice Λ in C^p. Since the hyperplanes $x_j \in \mathbf{Z}$ are preserved by the face inclusions $C^q \to C^p$, the cubical complex K has a corresponding subdivision K^*. Thus each p-cube κ in K is divided into M^p little cubes c in K^* whose vertices lie on the integer lattice Λ_κ. We will identify Λ_κ with Λ. In particular, the first vertex v_κ of κ is identified with $(0, \ldots, 0) \in \Lambda$.

Most of the effort involved in the proof of the deformation lemma is taken up in establishing the following result. It will be proved in (4.7)–(4.14) below.

LEMMA 4.4. *There is a family of maps $\psi_\kappa : \Lambda_\kappa \to \mathcal{N}_0$, $\kappa \subset K$, with the following properties.*

(a) (compatibility) *If $\iota \in \Lambda_\kappa$ and $\kappa \subset \mu$ then*

$$\psi_\kappa(\iota) = \psi_\mu(\iota)\psi_\mu(v_\kappa)^{-1}.$$

(b) (agreement with f_κ) *For each κ and all vertices v of κ*

$$\psi_\kappa(v) = f_\kappa(v).$$

(c) *For each p-cube κ we have*

$$\operatorname{supp}\,(\psi_\kappa(j_1, \ldots, j_l, \ldots, j_p) \circ \psi_\kappa(j_1, \ldots, j_l - 1, \ldots, j_p)^{-1}) \subset Z_{j_l}^{(Q)},$$

(d) *If $\kappa \in K'$, then*

$$\psi_\kappa(j_1, \ldots; j_l, \ldots, j_p) = \psi_\kappa(j_1, \ldots, j_l - 1, \ldots, l_p) \quad \textit{unless} \quad j \in J'.$$

Condition (c) implies that for every little cube c in κ there are p integers

$j_1, \ldots, j_p$ such that

$$\text{(e)}\ \operatorname{supp} \psi_\kappa(\iota)\psi_\kappa(\iota')^{-1} \subseteq Z_{j_1}^{(Q)} \cup \cdots \cup Z_{j_p}^{(Q)},$$

where ι, ι' are any vertices of c. By (a), the diffeomorphism $\psi_\kappa(\iota)\psi_\kappa(\iota')^{-1}$ is independent of the choice of cube κ containing c. Therefore if ι_c is the first vertex of c we may put

$$\psi_c(\iota) = \psi_\kappa(\iota)\psi_\kappa(\iota_c)^{-1}$$

where $c \subset \kappa$. Clearly, this defines a compatible family of maps on the vertices of the subdivision K^* of K. Using (d) and (e), one can easily find a function α from the little cubes in K^* to the subsets of $\{1, \ldots, M\}$ which has the properties: $|\alpha(c)| \leq \dim c$; $\alpha(c) \subset \alpha(c')$ if $c \subset c'$; $\alpha(c) \in J'$ if $c \subset K'^*$;

$$\operatorname{supp} \psi_c(\iota) \subset Z_{\alpha(c)}^{(Q)} = \bigcup_{j \in \alpha(c)} Z_j^{(Q)}.$$

Therefore the triple (K^*, K'^*, ψ) is supported by $(\mathscr{Z}^{(Q)}, \mathscr{Z}'^{(Q)})$ and hence also by $(\mathscr{V}, \mathscr{V}')$. It remains to extend the ψ_c to the whole of K^* and to show that the resulting triple $(K^*, K'^*, \bar{\psi})$ is homotopic to (K, K', f). This will be the case because the ψ_c are close to the f_κ by (b).

(4.5) *Extending the ψ_c to $\bar{\psi}_c$*

We extend the ψ_c to a compatible family of maps $\bar{\psi}_c : c \to \mathcal{N}_p$, where $p = \dim c$, in such a way that

$$\operatorname{supp} \bar{\psi}_c \subset Z_{\alpha(c)}^{(Q+p)} \quad \text{for all} \quad c.$$

This may be done inductively over the skeleta of K^*. When $\dim c = 1$, the values of ψ_c at the two points of ∂c lie in $\mathcal{N}_0$ and one can extend ψ_c to the rest of c by (4.2)(b). Now suppose inductively that $\bar{\psi}_c$ has been defined for all c of dimension $<p$. If c has dimension p, then the compatibility conditions imply that $\bar{\psi}_c$ is already defined on ∂c. In fact, if c' is a face of c with first vertex $\iota_{c'}$, we must have

$$\bar{\psi}_c(a) = \bar{\psi}_{c'}(a)\psi_c(\iota_{c'}), \quad \text{for all} \quad a \in c'.$$

Hence $\bar{\psi}_c(a) \in \mathcal{N}_{p-1}\mathcal{N}_0$ for all $a \in \partial c$ by the inductive hypothesis. Also $\operatorname{supp} \bar{\psi}_c \mid \partial c \subset Z_{\alpha(c)}^{(Q+p-1)}$. Therefore one can extend $\bar{\psi}_c$ to the whole of c by condition (4.2)(b).

Thus we have a triple $(K^*, K'^*, \bar{\psi})$ which is supported by $(\mathscr{Z}^{(Q+d)}, \mathscr{Z}'^{(Q+d)})$ and hence also by $(\mathscr{V}, \mathscr{V}')$.

(4.6) *Construction of the homotopy* $(K\times I, F)$

Clearly there is a cycle $(K, \bar{\bar{\psi}})$ which gives $(K^*, \bar{\psi})$ upon the subdivision of K. In fact

$$\bar{\bar{\psi}}_\kappa(a)=\bar{\psi}_c(a)\psi_\kappa(\iota_c) \quad \text{for} \quad a\in c\subset\kappa.$$

Hence $\bar{\bar{\psi}}_\kappa(a)\in\mathscr{N}_{d+1}$ for all $a\in\kappa$ by (4.2)(b). Further, if $Z'^{(r)}=\bigcup\{Z_j'^{(r)}: j\in J'\}\subset W'$, then for each $\kappa\in K'$ we have

$$\operatorname{supp}\bar{\bar{\psi}}_\kappa\subset\bigcup_{c\subset\kappa} Z'^{(Q+d+1)}_{\alpha(c)}\subset Z'^{(Q+d+1)}\subset W'$$

By repeating the argument of (4.5) one can easily define maps $F_\kappa:\kappa\times I\to\mathscr{N}_{d+p+1}$, where $p=\dim\kappa$, so that the following conditions are satisfied:

(i) $F_\kappa(a,0)=f_\kappa(a)$ and $F_\kappa(a,1)=\bar{\bar{\psi}}_\kappa(a)$ for $a\in\kappa$;
(ii) $F_\kappa(v,t)=f_\kappa(v)$ for each vertex v of κ;
(iii) $F_\kappa(a,t)=F_\lambda(a,t)f_\kappa(v_\kappa)$ if $a\in\lambda\subset\kappa$;
(iv) for each p-cube κ in K'

$$\operatorname{supp}F_\kappa\subset Z'^{(Q+d+p+1)}\subset W'.$$

Note that conditions (i) and (ii) are consistent by (4.4)(b). Also (i) and (iv) are consistent because $\operatorname{supp}f_\kappa\subset X\subset Z'^{(Q)}$ for all $\kappa\in K'$. By (iii) these F_κ are compatible and so define a chain $(K\times I, F)$ which clearly has all the properties required by Lemma 2.5. Observe in particular that the restriction of F to $K\times 1$ is just $(K, \bar{\bar{\psi}})$. This holds because, by (ii), no renormalization is needed: compare (3.7).

This completes the proof of the deformation lemma, modulo the proof of Lemma 4.4.

(4.7) *Proof of Lemma* 4.4

We prove the lemma first for $d=1$ and 2. The general case will be proved by induction in (4.14). When $d=1$, the compatibility conditions are irrelevant and it suffices to define, for each 1-cube κ in K, elements $\psi_\kappa(i)$, $0\leq i\leq M$, of $\mathscr{N}_0$ such that $\psi_\kappa(0)=\mathrm{id}$, $\psi_\kappa(M)=f_\kappa(M)$ and

$$\operatorname{supp}(\psi_\kappa(j)\psi_\kappa(j-1)^{-1})\subset Z_j^{(Q)}.$$

Our construction will be based on a very careful choice of these $\psi_\kappa(j)$.

If Y is a (compact) submanifold of Int W_1 and if $g \in \mathscr{G}_1$ we will say that **supp** $g \subset Y$ if g has support in Int Y and if the flux of g with respect to Y is zero. The second condition means that the element $\Phi_Y(g)$ of $H_c^{n-1}(\text{Int } Y; R)$ is zero. Clearly **supp** $g \subset W_i$ for all $g \in \mathscr{G}_i$, $i = 1, 2$.

The following lemma is essentially due to Thurston [11]. A complete proof is given by Ismagilov in [5] 2.4, and in Banyaga [1] III.3.2 for the case $n = 2$. We will give a proof here in a form convenient for our purposes, using Ismagilov's method.

LEMMA 4.8 (Fragmentation Lemma). *Let s be any integer, $1 \le s \le Q$, and let $\mathcal{M}_1$ be any neighbourhood of the identity in $\mathscr{G}_1$. Then there is a neighbourhood of the identity $\mathcal{M}_0 \subset \mathcal{M}_1$ such that every $g \in \mathcal{M}_0$ may be decomposed into a sequence $g(0) = \text{id}, g(1), \ldots, g(M) = g$ which satisfies the following conditions for each j:*

(i) $g(j) \in \mathcal{M}_1$;
(ii) $\text{supp}\,(g(j)g(j-1)^{-1}) \subset Z_j^{(s)}$;
(iii) $\textbf{supp}\,(gg(j)^{-1}) \subset W_1 - \bigcup_{i \le j} Z_i^{(-s)}$.

Proof. We construct the $g(j)$ by induction on j. Since the construction involves M steps, it will be clear that there is a C^1-neighbourhood $\mathcal{M}_0$ such that the $g(j)$ may all be chosen in $\mathcal{M}_1$. Moreover, we will assume that $\mathcal{M}_0$ is so small that any diffeomorphism which we encounter, for example $gg(j-1)^{-1}$ below, is in the neighbourhood $\mathcal{N}_0$ defined in (4.2).

If $g(j-1)$ is already defined, then $g(j)$ must have the form $s(j)^{-1}g(j-1)$ where

$$s(j) = \begin{cases} g(j-1)g^{-1} & \text{on } \bigcup\limits_{i \le j} Z_i^{(-s)} \\ \text{id} & \text{outside } Z_j^{(s)}. \end{cases}$$

Thus, in order to define $g(j)$ we must first extend $s(j)$ over the whole of W_1. Second, we must check that the extension can be chosen so that $gg(j)^{-1}$ has zero flux in $W_1 - \bigcup_{i \le j} Z_i^{(-s)}$. We will see that these two questions are related.

Now, $s(j)$ is defined on the complement of

$$Q_j^{(s)} = Z_j^{(s)} - \bigcup_{i \le j} Z_j^{(-s)}$$

and is injective there because of our assumption that $gg(j-1)^{-1}$ is in $\mathcal{N}_0$. If j corresponds to a p-simplex in T, that is, if $M_{p-1} < j \le M_p$, then one can easily check that $Q_j^{(s)} \cong S^{n-p-1} \times D^{p+1}$. Thus $Q_j^{(s)}$ is connected when $p < n-1$, and so $s(j)$ has an extension in $\mathscr{G}_1$. (See Remark (3.3).)

Let us now consider the second conditon. When $j \leq M_{n-3}$, the set $\bigcup_{i\leq j} Z_i^{(-s)}$ retracts onto part of the $(n-3)$-skeleton of T and so $H_c^{n-1}(\text{Int } W_1 - \bigcup_{i\leq j} Z_i^{(-s)}) = H_c^{n-1}(\text{Int } W_1)$. Therefore, for these values of j we just need $\Phi_{W_1}(gg(j)^{-1}) = 0$, which is true since g and $g(j)$ belong to $\mathscr{G}_1$. However, if $M_{n-3} < j \leq M_{n-2}$, the condition is significant. Let us suppose that $j = M_{n-3} + m$. Then $Z_j^{(s)}$ is a thickening of the $(n-2)$-simplex Δ_m^{n-2}, and $\bigcup_{i\leq j} Z_i^{(-s)}$ is a thickening of

$$T_j = ((n-3)\text{-skeleton of } T) \cup \{\Delta_l^{n-2} : l \leq m\}.$$

Let us denote the flux homomorphism relative to $W_1 - T_j$ by Φ_j. Then Φ_j is defined on those $g \in \mathscr{G}_1$ with support in $\text{Int } W_1 - T_j$, and it takes values in $H_c^{n-1}(\text{Int } W_1 - T_j)$. The inductive hypothesis implies that $\Phi_{j-1}(gg(j-1)^{-1}) = 0$, and we want to choose an extension $s(j)$ so that $\Phi_j(gg(j-1)^{-1}s(j)) = 0$. Consider the diagram

$$\begin{array}{ccccc} H^{n-2}(\Delta_m^{n-2}, \partial\Delta_m^{n-2}) & \xrightarrow{\delta} & H_c^{n-1}(\text{Int } W_1 - T_j) & \xrightarrow{j^*} & H_c^{n-1}(\text{Int } W_1 - T_{j-1}) \\ & \searrow & \nearrow{\scriptstyle i^*} & & \\ & & H_c^{n-1}(Q_j^{(s)}). & & \end{array} \qquad (*)$$

Note that the top row is exact, and that δ is either injective or zero. Let $s'(j) \in \mathscr{G}_1$ be any extension of $s(j)$. Since it has support in $Z_j^{(s)} = Z_j^{(s)} - T_{j-1}$, the element $\Phi_{j-1}(s'(j))$ is defined. Moreover $\Phi_{j-1}(s'(j)) = 0$ because $Z_j^{(s)}$ is contractible. Hence

$$j^*\Phi_j(gg(j-1)^{-1}s'(j)) = \Phi_{j-1}(gg(j-1)^{-1}) + \Phi_{j-1}(s'(j)) = 0.$$

Therefore, when $\delta = 0$, any extension $s'(j)$ will do. If $\delta \neq 0$ the possible choices for the extension of $s(j)$ have the form $s'(j)t(j)$ where $\text{supp } t(j) \subset Q_j^{(s)} \subset \text{Int } W_1 - T_j$. Let Φ_j' be the flux homomorphism relative to $Q_j^{(s)}$. Then $\Phi_j = i^*\Phi_j'$ in the diagram above. It is not hard to see that Φ_j' is surjective. Therefore, because $\text{Im } \delta = \text{Im } i^*$, one can choose $t(j)$ so that

$$\Phi_j(gg(j-1)^{-1}s'(j)t(j)) = \Phi_j(gg(j-1)^{-1}s'(j)) + \Phi_j(t(j)) = 0.$$

Thus a suitable extension of $s(j)$ can be found when $j \leq M_{n-2}$.

Now consider j in the range $M_{n-2} < j \leq M_{n-1}$. Notice that $\text{supp } (gg(M_{n-2})^{-1}) \subset W_1 - T^{(n-2)}$, where $T^{(n-2)}$ is the $(n-2)$-skeleton of T. Using the definition of the flux homomorphism given in §2 above, one can check that for each $j = M_{n-2} + m$ the two components of $Z_j^{(s)} - gg(M_{m-2})^{-1}(\Delta_m^{n-1})$ have the same volume as the corresponding components of $Z_j^{(s)} - \Delta_m^{n-1} \cong Q_j^{(s)}$. (This holds because $gg(M_{n-2})^{-1}$

satisfies condition (iii).) Thus $s(j)$ can be extended for each such j. But for these j condition (iii) is automatically satisfied. Hence the $g(j)$ can be defined for these j. The $g(j)$ for $j > M_{n-1}$ are now uniquely determined, since $Q_j^{(s)} = \varnothing$ in this case. □

We will say that the elements $g(j)$, $0 \leqslant j \leqslant M$, of Lemma 4.8 form a *canonical decomposition of* g *with respect to* $Z^{(s)}$. We will need the following sharpened version of this lemma.

LEMMA 4.9. *Given any neighbourhood* $\mathcal{M}_1$ *there is a neighbourhood* $\mathcal{M}_0 \subset \mathcal{M}_1$ *such that if* $g \in \mathcal{M}_0$ *and if*

$$\mathbf{supp}\, g \subset Q_k^{(s)} = Z_k^{(s)} - \bigcup_{i \leqslant k} Z_i^{(-s)}$$

for some k *and* s *then* g *has a canonical decomposition with respect to* $\mathscr{Z}^{(s+1)}$ *such that*

(i) $g(j) = \mathrm{id}$ *for* $j \leqslant M_p$, *where* $M_{p-1} < k \leqslant M_p$; *and*
(ii) *each* $g(j) \in \mathcal{M}_1$ *and has support in* $Z_k^{(s+1)}$.

Proof. This is a straightforward generalization of Lemma 4.8 and is proved in the same way. Condition (i) is possible because $Q_k^{(s)}$ is covered by the sets $Z_j^{(s+1)}$, $j > M_p$. Since there are only a finite number of s and k, the neighbourhood $\mathcal{M}_0$ clearly exists. □

Remark 4.10. If $\mathbf{supp}\, g \subset Z'^{(s)} \subset W'$, one can apply Lemmas 4.8 and 4.9 using the cover $\mathscr{Z}'$ of W' instead of $\mathscr{Z}$. Hence one can assume in addition that $g(j) = g(j-1)$ for $j \notin J'$.

(4.11) *Proof of Lemma* 4.4 *for* $d = 2$

When $\dim \kappa = 1$, the integer points in Λ_κ are j, $0 \leqslant j \leqslant M$, and we define the $\psi_\kappa(j)$ to be a canonical decomposition of $f_\kappa(M)$ with respect to $\mathscr{Z}^{(1)}$. If $\kappa \in K'$ we may by (4.10) assume that $\psi_\kappa(j) = \psi_\kappa(j-1)$ if $j \notin J'$. Thus (4.4)(d) is satisfied.

Let us now consider a 2-cube κ. Its integer points are (j, k), $0 \leqslant j, k \leqslant M$. The compatibility conditions (4.4)(a) determine ψ_κ on $\partial\kappa$. We will also suppose that ψ_κ is defined along the diagonal (j, j), $0 \leqslant j \leqslant M$, to be a canonical decomposition of $\psi_\kappa(M, M) = f_\kappa(M, M)$ with respect to $\mathscr{Z}^{(1)}$. We will then show how to define the $\psi_\kappa(j, k)$ for $j \geqslant k$. The case $j \leqslant k$ may be obtained by symmetry.

Let $g = f_\kappa(M, M)$. then by (4.8)(iii) both $\mathbf{supp}\,(g\psi_\kappa(k, k)^{-1})$ and

supp $(g\psi_\kappa(M,k)^{-1})$ are contained in $W_1 - \bigcup_{i\leq k} Z_i^{(-1)}$. Hence

(i) $\quad \mathbf{supp}\,(\psi_\kappa(M,k)\psi_\kappa(k,k)^{-1}) \subset W_1 - \bigcup_{i\leq k} Z_i^{(-1)}$

Let us suppose that the $\psi_\kappa(j,k)$ have been defined for all (j,k) where $j \geq k$ and $k < l$ in such a way that:

(ii) $\quad \operatorname{supp}(\psi_\kappa(j,k)\psi_\kappa(j-1,k)^{-1}) \subset Z_j^{(2k+1)}$,

$\quad\quad \operatorname{supp}(\psi_\kappa(j,k)\psi_\kappa(j,k-1)^{-1}) \subset Z_k^{(2k+1)}$,

(iii) $\quad \mathbf{supp}\,(\psi_\kappa(M,k)\psi_\kappa(j,k)^{-1}) \subset W_1 - \bigcup_{i\leq j} Z_i^{(-2k-1)}$.

(These conditions are satisfied when $l = 1$.) Put

$$h_{0l} = \psi_\kappa(l,l)\psi_\kappa(l,l-1)^{-1}, \qquad h_{1l} = \psi_\kappa(M,l)\psi_\kappa(M,l-1)^{-1}.$$

Then we claim that

(iv) $\quad \operatorname{supp}(h_{1l}h_{0l}^{-1}) \subset Q_l^{(2l)}$.

To see this, note first that $\operatorname{supp} h_{1l} \subset Z_l^{(1)}$ by (4.8)(ii). Also,

$$\begin{aligned} \operatorname{supp} h_{0l} &\subset \operatorname{supp}(\psi_\kappa(l,l)\psi_\kappa(l-1,l-1)^{-1}) \cup \operatorname{supp}(\psi_\kappa(l-1,l-1)\psi_\kappa(l,l-1)^{-1}) \\ &\subset Z_l^{(1)} \cup Z_l^{(2l-1)} = Z_l^{(2l-1)} \text{ by (ii) above.} \end{aligned}$$

Further

$$\begin{aligned} h_{1l}h_{0l}^{-1} &= \psi_\kappa(M,l)\psi_\kappa(M,l-1)^{-1}\psi_\kappa(l,l-1)\psi_\kappa(l,l)^{-1} \\ &= (\psi_\kappa(M,l)\psi_\kappa(M,l-1)^{-1}\psi_\kappa(l,l-1)\psi_\kappa(M,l)^{-1})(\psi_\kappa(M,l)\psi_\kappa(l,l)^{-1}) \\ &= \mathrm{id} \text{ on } \psi_\kappa(M,l)Z_i^{(-2l+1)} \cap Z_i^{(-1)} \quad \text{for} \quad i \leq l \end{aligned}$$

by (iii) above and (4.8)(iii). But $\psi_\kappa(M,l)Z_i^{(-2l+1)} \supset Z_i^{(-2l)}$ since $\psi_\kappa(M,l) \in \mathcal{N}_0 \subset \mathcal{M}$. Therefore (iv) holds.

When $M_{p-1} < l \leq M_p$ for $p \neq n-2$, then $H_c^{n-1}(Q_l^{(2l)}) = 0$. In this case (iv) is equivalent to

(iv)′ $\quad \mathbf{supp}\,(h_{1l}h_{0l}^{-1}) \subset Q_l^{(2l)}$.

Hence Lemma 4.9 imples that $h_{1l}h_{0l}^{-1}$ has a canonical decomposition $h(j)$, $j > M_p$, with respect to $\mathscr{X}^{(2l+1)}$ and such that supp $h(j) \subset Z_l^{(2l+1)}$ for all j. Now set

(v) $\psi_\kappa(j, l) = h(j)h_{0l}\psi_\kappa(j, l-1)$ for $l \leq j \leq M$.

It is not hard to check that (ii) and (iii) above hold. For example

$$\psi_\kappa(M, l)\psi_\kappa(j, l)^{-1} = (h_{1l}h_{0l}^{-1}h(j)^{-1})h(j)h_{0l}(\psi_\kappa(M, l-1)\psi_\kappa(j, l-1)^{-1})h_{0l}^{-1}h(j)^{-1}$$

is the identity on $Z_i^{(-2l-1)} \cap h(j)h_{0l}(Z_i^{(-2l+1)}) = Z_i^{(-2l-1)}$ for all $i \leq j$.

It remains to consider the rows $\psi_\kappa(\cdot, l)$ where $M_{n-3} < l \leq M_{n-2}$. We have to ensure that (iv)$'$ holds. Consider diagram (*) in Lemma 4.8. If $\delta \neq 0$, then i^* is injective and so it suffices to prove that $\Phi_l(h_{1l}h_{0l}^{-1}) = 0$. If $g = \psi_\kappa(M, M)$, we know by (4.8)(iii) that

$$0 = \Phi_l(g\psi_\kappa(M, l)^{-1}) = \Phi_l(g\psi_\kappa(M, l-1)^{-1}h_{1l}^{-1})$$

and

$$0 = \Phi_l(g\psi_\kappa(l, l-1)^{-1}h_{0l}^{-1}).$$

But $\Phi_l(\psi_\kappa(M, l-1)\psi_\kappa(l, l-1)^{-1}) = 0$ by (iii) above. Hence $\Phi_l(h_{1l}h_{0l}^{-1}) = 0$ as required.

For those l for which $\delta = 0$, one argues rather differently. Notice that in the above construction the elements $\psi_\kappa(j, k)$ in the triangle $M_{p-1} < k \leq j \leq M_p$ depend only on the diagonal elements $\psi_\kappa(k, k)$ and the elements $\psi_\kappa(j, M_{p-1})$ in the M_{p-1}th row. (This is true because we chose the $h(j)$ in (v) so that $h(j) = \mathrm{id}$ for $j \leq M_p$.) Therefore, once the elements $\psi_\kappa(j, M_{n-3})$, $M_{n-3} < j \leq M_{n-2}$ are chosen, the elements $h_{1k}h_{0k}^{-1}$, $M_{n-3} < k \leq M_{n-2}$, are determined. Two different choices of $\psi_\kappa(l, M_{n-3})$ differ by an element $t(l)$ with support in $Q_l^{(2l)}$. Further, if $\delta = 0$ for l, then $\Phi_l'(t(l))$ can be arbitrary. It is not hard to check that if one changes $\psi_\kappa(l, M_{n-3})$ by $t(l)$ then h_{0l} changes by a conjugate of $t(l)$ and so $\Phi_l'(h_{1l}h_{0l}^{-1})$ changes by $\Phi_l'(t(l))$. Hence one can choose the row $\psi_\kappa(\cdot, M_{n-3})$ so that (iv)$'$ is satisfied for all l with $\delta = 0$. This argument does not make sense when $n = 2$, but fortunately the map δ is never 0 in this case.

This completes the construction of the $\psi_\kappa(j, k)$, $j \geq k$. The $\psi_\kappa(j, k)$, $j \leq k$, are defined symmetrically. It remains to check that the conditions of Lemma 4.4 are satisfied. Now, conditions (a) and (b) are clear from the construction, (c) follows from (ii) above, and (d) follows by Remark 4.10. Finally, notice that in the construction of a particular $\psi_\kappa(j, k)$ we apply Lemma 4.8 three times to define ψ_κ on the edges of the 2-simplex which contains (j, k) and then apply Lemma 4.9

exactly $k-1$ times. It follows easily that one can choose the initial neighbourhood $\mathcal{N}$ which contains the $f_\kappa(v)$ to be so small that all the $\psi_\kappa(j,k)$ lie in $\mathcal{N}_0$. This completes the proof of Lemma 4.4 when $d=2$. $\square$

In the above proof we constructed the $\psi_\kappa(j,k)$ for $0\leq k\leq j\leq M$ from three given elements: $\psi_\kappa(0,0)$, $\psi_\kappa(M,0)$ and $\psi_\kappa(M,M)$. This may be thought of as a "two-dimensional" version of Lemma 4.8. In dimension p we want to define elements $\psi_\kappa(\iota)$, for all ι in the integer lattice of a p-simplex, given the values of ψ_κ at the vertices of that simplex. These $\psi_\kappa(\iota)$ should have certain properties which are formulated in the following definition.

DEFINITION 4.12. Let σ_p be the p-simplex $\{x : 0\leq x_p\leq\cdots\leq x_1\leq M\}$ with set of vertices V_p and integer lattice Λ_p, and let $\mathcal{P}$ be a neighbourhood of the identity in $\mathcal{G}_1$. Suppose elements $\psi_\sigma(v)$, $v\in V_p$, are given where $\psi_\sigma(0,\ldots,0)=\mathrm{id}$. Then a *canonical decomposition of the* $\psi_\sigma(v)$, $v\in V_p$ *in* $\mathcal{P}$ *and with respect to* $\mathcal{Z}^{(s)}$ is a collection

$$\psi_\sigma(j_1,\ldots,j_p) : (j_1,\ldots,j_p)\in\Lambda_p,$$

of elements of $\mathcal{P}$ satisfying the conditions:

(i) $\mathrm{supp}\,(\psi_\sigma(j_1,\ldots,j_l,\ldots,j_p)\psi_\sigma(j_1,\ldots,j_l-1,\ldots,j_p)^{-1})\subseteq Z_{j_l}^{(s)}$
(ii) for each l, $1\leq l\leq p$,

$$\mathbf{supp}\,(\psi_\sigma(M,\ldots,M,M,j_{l+1},\ldots,j_p)\psi_\sigma(M,\ldots,M,j_l,j_{l+1},\ldots,j_p)^{-1})$$
$$\subset W_1-\bigcup_{i\leq j_l} Z_i^{(-s)}.$$

This decomposition will be said to be *subordinate to* $Q_k^{(s)}$ if, in addition,

(iii) $\psi_\sigma(j_1,\ldots,j_p)=\mathrm{id}$ for $j_p\leq\cdots\leq j_1\leq N_q$ where $N_{q-1}<k\leq N_q$
(iv) $\mathrm{supp}\,\psi_\sigma(j_1,\ldots,j_p)\subset Z_k^{(s)}$ for all $(j_1,\ldots,j_p)\in\Lambda_p$.

LEMMA 4.13. (a) *For any neighbourhood of the identity* $\mathcal{M}_1$ *there is a neighbourhood* $\mathcal{M}_0$ *such that, if* $\psi_\sigma(\iota)$, $\iota\in\Lambda_p\cap\partial\sigma_p$, *are any elements of* $\mathcal{M}_0$ *which satisfy* (4.12)(i), (ii) *for some* s' *wherever this makes sense, then one can define* $\psi_\kappa(\iota)$ *for the other* $\iota\in\Lambda_p$ *so that the* $\psi_\kappa(\iota)$ *form a canonical decomposition of the* $\psi_\sigma(v)$ *in* $\mathcal{M}_1$ *with respect to* $\dot{\mathcal{Z}}^{(r)}$, *with* $r=s'+2M$.

(b) *If* $\psi_\sigma(v)$, $v\in V_p$, *are any elements of* $\mathcal{M}_0$ *such that*

$$\mathbf{supp}\,\psi_\sigma(v)\subset Q_k^{(s)}\quad \textit{for all}\quad v\in V_p,$$

then there is a canonical decomposition of the $\psi_\sigma(v)$ *in* $\mathcal{M}_1$ *which is subordinate to* $Q_k^{(s+1)}$.

Proof. Let us suppose that (a) and (b) have been proved for all $p' < p$, where $p \geq 3$. Further, we will suppose that $\psi_\sigma(j_1, \ldots, j_p)$ has been defined for all $(j_1, \ldots, j_p)$ with $j_p < l$ in such a way that (4.12)(i), (ii) is satisfied with $s = s' + 2m$ when $j_p = m$. Consider the level $j_p = l$. The $(p-1)$-simplex $\sigma_p \cap (j_p = l)$ has vertices $v_i(l)$, $0 \leq i \leq p-1$, where

$$v_i(j) = (M, \ldots, M, l, \ldots, l, j), \quad \text{with } i \text{ factors of } M.$$

Put $h_{il} = v_i(l)v_i(l-1)^{-1}$. For each i, the four elements $v_i(l)$, $v_i(l-1)$, $v_{p-1}(l)$ and $v_{p-1}(l-1)$ are contained in a 2-dimensional face of σ_p. (Here we need $p \geq 3$.) Therefore, our assumption that the $\psi_\sigma(\iota)$ satisfy (4.12)(i), (ii) on $\partial\sigma_p$, together with the calculation of (4.11), shows that

$$\textbf{supp}\ h_{il}h_{p-1l}^{-1} \subset Q_l^{(r)} \quad \text{for} \quad r = s' + 2l - 1.$$

Hence

$$\textbf{supp}\ h_{il}h_{0l}^{-1} \subset Q_l^{(r)} \quad \text{for} \quad 0 \leq i \leq p-1.$$

Since (b) holds when $p' = p-1$, one can therefore find a canonical decomposition $h(j_1, \ldots, j_{p-1})$ of the $h_{il}h_{0l}^{-1}$ which is subordinate to $Q_l^{(r)}$, for $r = s' + 2l$. One now checks as in (4.11) that the elements

$$\psi_\sigma(j_1, \ldots, j_{p-1}, l) = h(j_1, \ldots, j_{p-1})h_{0l}\psi_\sigma(j_1, \ldots, j_{p-1}, l-1)$$

satisfy the inductive hypothesis.

The proof of (b) is similar. One should insert an appropriate number of auxiliary covers in between $\mathcal{X}^{(s)}$ and $\mathcal{X}^{(s+1)}$, and then should choose the $\psi_\rho(\iota)$, where ρ is a q-dimensional face of σ, in order of increasing q. □

(4.14) *Proof of Lemma* 4.4 (*general case*)

One constructs the $\psi_\kappa(\iota)$ inductively over the skeleta of K using Lemma 4.13(a). The argument is just like that used when $d = 2$, and its details will be left to the reader. □

(4.15) *Remark*

In the proof of Lemma 2.5 we have used two properties of the group $\mathcal{G}_1$: first, that it is locally contractible, so that the neighbourhoods of (4.2) exist, and second, that it has an appropriate isotopy extension theorem, so that the fragmentation Lemma 4.8 holds. Fathi shows in [3]§4 that the group of all homeomorphisms of a compact manifold which preserve a good measure has these properties. Hence Lemma 2.5 holds for this group. Indeed all the results of this paper are valid for this group.

Lemma 2.5 also holds for the group of all homeomorphisms of a compact manifold by [2]. Using this, one can presumably extend the proof given by Mather in [8] of the Mather–Thurston theorem to the C^0-case. See [8] §6.

Finally, note that Lemma 2.5 holds in the symplectic case. For the group of all symplectic diffeomorphisms of a symplectic manifold is locally contractible by [12], and Banyaga proves the equivalent of Lemma 4.8 in [1] III.3.2. In fact, let us say in this case that **supp** $g \subset Y$ if supp $g \subset \text{Int } Y$ and if $S(g) = 0$ in $H_c^1(\text{Int } Y)$, where S is the homomorphism defined by Banyaga in [1] II.1. Then Lemma 4.8 holds as stated, and the proof is the same, except that the obstruction to extending $s(j)$ in the required fashion now occurs for $j \leqslant M_0$ instead of for $M_{n-3} < j \leqslant M_{n-2}$. Therefore, just as when $n = 2$ in the volume preserving case, the map δ in the diagram corresponding to (*) is never zero. This means that condition (iv)′ in (4.11) is always satisfied, which slightly simplifies that proof.

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State University of New York
Department of Mathematics
Stony Brook
N.Y. 11794

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