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Cheeger's inequality with a boundary term

M. N. Huxley

1. Introduction

Cheeger's inequality refers to the eigenvalues of the Laplacian on an oriented manifold M. For ease of exposition we consider two dimensional manifolds in this note, with local coordinates x, y and length l and measure μ given by

$$dl^2 = g(x, y)(dx^2 + dy^2), \qquad d\mu = g(x, y)\, dx\, dy,$$

where $g(x, y)$ is an analytic function of x and y. We say $f(x, y)$ is a Dirichlet eigenfunction on M if

$$\iint_M f^2(x, y) g(x, y)\, dx\, dy \text{ converges,}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -\lambda f(x, y) g(x, y)$$

(for some λ) on the interior of M, and if f is not identically zero, but $f(x, y) = 0$ on ∂M, the boundary of M.

CHEEGER'S THEOREM. *Let h be a constant such that*

$$l(\partial S) \geqslant h\mu(S)$$

for all subsets S of M with finite measure and piecewise smooth boundary ∂S. Then if λ is the eigenvalue of a Dirichlet eigenfunction,

$$2\sqrt{\lambda} > h.$$

For a compact manifold without boundary the constant h is zero, but [2] the infimum defining h may now be taken over subsets S whose measure is at most half that of M. In fact, the sets S considered are connected components of the inverse image of $[\delta, \infty)$ or of $(-\infty, -\delta]$ under f. We may replace h by the infimum $h(f)$ over such sets S; the constant h now furnishes a lower bound for $h(f)$.

If M is multiply connected, it is difficult to estimate the bound h, or even $h(f)$. Cutting the manifold introduces extra boundaries. We prove an appropriate extension of Cheeger's theorem:

Let N be a two-dimensional oriented manifold with metric given by

$$dl^2 = g(x, y)(dx^2 + dy^2), \qquad d\mu = g(x, y)\, dx\, dy,$$

and boundary $N = C \cup D$. Let f satisfy

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -\lambda f(x, y) g(x, y)$$

on the interior of N, and the mixed boundary conditions

$$f = 0 \text{ on } C, \qquad \frac{\partial f}{\partial n} = 0 \text{ on } D.$$

Let the manifold M be obtained by cutting N along E, a finite union of simple curves; the boundary of M thus consists of C, D and two copies E_1 and E_2 of E, with periodic boundary conditions identifying E_1 and E_2. Then if f is non-constant, we have

$$(2\lambda^{1/2} - h(f)) \iint_M f^2\, d\mu > - \oint_{\partial M} f^2\, dl.$$

2. Proof

We follow Buser's account [2] of Cheeger's theorem. By periodicity

$$\left(\int_{E_1} + \int_{E_2} \right) f \operatorname{grad} f \cdot ds = 0,$$

where grad f and ds are with respect to the coordinates x, y. We have $f = 0$ on C

and the normal derivative of f vanishes on D in both metrics. Hence

$$0=\oint_{\partial M} f\,\mathrm{grad}\,f\cdot ds=\iint_M \mathrm{div}\,(f\,\mathrm{grad}\,f)\,dx\,dy$$

$$=\iint_M |\mathrm{grad}\,f|^2\,dx\,dy+\iint_M f\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)dx\,dy$$

$$=\iint_M |\mathrm{grad}\,f|^2\,dx\,dy-\lambda\iint_M f^2 g\,dx\,dy.$$

It is convenient to suppose f normalised so that

$$\iint_M f^2 g\,dx\,dy=1.$$

Next we have

$$\iint_M |\mathrm{grad}\,f^2|\,g^{1/2}\,dx\,dy=2\iint_M |f|\,|\mathrm{grad}\,f|\,g^{1/2}\,dx\,dy$$

$$<\left\{4\iint_M f^2 g\,dx\,dy\iint_M |\mathrm{grad}\,f|^2\,dx\,dy\right\}^{1/2}=2\lambda^{1/2}$$

by the normalisation. Since f is non-constant we have grad f zero but f nonzero at the extrema of f, and so $f^2 g$ and $|\mathrm{grad}\,f|^2$ are not proportional. Thus the inequality is strict.

Next we take curvilinear coordinates t, u on M, with $t=f(x,y)$, u running along the curves $f(x,y)=$ constant. These coordinates are orthogonal, with area element $dt\,du=|\mathrm{grad}\,f|\,dx\,dy$. We deduce that

$$2\lambda^{1/2}>\iint_M 2\,|f|\,|\mathrm{grad}\,f|\,g^{1/2}\,dx\,dy=\iint_M 2\,|t|\,g^{1/2}\,du\,dt.$$

Next we let $M(t)$ be the set of points with $f(x,y)\geqslant t$, $M'(t)$ be the set of points with $f(x, y)\leq t$, and $L(t)$ be the curve $f(x, y)=t$ of length

$$l(t)=\int_{L(t)} g^{1/2}\,du$$

in the metric on M. The boundary of $M(t)$ consists of $L(t)$ and that part of ∂M that lies in $M(t)$, so that

$$\int_{L(t)} g^{1/2}\,du+\int_{M(t)\cap\partial M} g^{1/2}\,ds\geq h(f)\mu(M(t)),$$

where du and ds are Euclidean lengths. Hence

$$2\lambda^{1/2}+2\int_{t>0} t\int_{M(t)\cap\partial M} g^{1/2}\,ds\,dt+2\int_{t<0}|t|\int_{M'(t)\cap\partial M} g^{1/2}\,ds\,dt$$

$$\geq h(f)\int_{t>0} 2t\iint_{M(t)} g\,dx\,dy\,dt+h(f)\int_{t<0} 2|t|\iint_{M'(t)} g\,dx\,dy\,dt.$$

Interchanging the order of integration, we have

$$2\lambda^{1/2}+\int_{M(0)\cap\partial M} t^2 g^{1/2}\,ds+\int_{M'(0)\cap\partial M} t^2 g^{1/2}\,ds$$

$$\geq h(f)\left\{\iint_{M(0)} g\,t^2\,dx\,dy+\iint_{M'(0)} g\,t^2\,dx\,dy\right\},$$

so that

$$2\lambda^{1/2}+\oint_{\partial M} f^2\,dl\geq h(f),$$

where dl is the differential of distance in the metric.

3. Applications

Consider the upper half plane H as hyperbolic space of curvature -1, with $g(x, y)=1/y^2$. For sets S of finite measure the isoperimetric inequality [1] states

$$l^2(\partial S)\geq\mu^2(S)+4\pi\mu(S).$$

Let Γ be a group acting discontinuously on H, for which the quotient space $\Gamma\backslash H$ has finite measure. If Γ contains no rotations or transvections (limit rotations about points at infinity, the 'cusps'), then $\Gamma\backslash H$ is a compact Riemann surface, and every simply connected subset carries the hyperbolic metric. The compact case has been studied extensively, cf. Elstrodt's survey [4]. If Γ does contain transvections, there is a continuous spectrum $\lambda > \frac{1}{4}$ whose multiplicity is the number of inequivalent cusps. The generalised eigenfunctions of the continuous spectrum are given by the values of the Eisenstein series introduced by Maass [8] on the line Re $s = \frac{1}{2}$, with $\lambda = s(1-s)$. The continuation of the Eisenstein series to Re $s = \frac{1}{2}$ is difficult (except in special cases); see [5, 9, 10, 11]. The Eisenstein series has a pole at $s = 1$ with constant residue f_0, the trivial constant eigenfunction, and any other poles in $\frac{1}{2} \leqslant s \leqslant 1$ on the real axis correspond to square-integrable eigenfunctions, again by $\lambda = s(1-s)$. All other eigenfunctions are 'cusp forms', zero at all cusps of $\Gamma\backslash H$.

The modular group PSL(2, Z) and its congruence subgroups are of particular interest [8, 9]. Recently Kuznietsov [6] and Deshouillers and Iwaniec [3] have used the Kuznietsov Trace Formulae to study them. These formulae differ from that of Selberg by taking the group elements not in conjugacy classes, but in double cosets of the Borel subgroup of upper triangular matrices, and using the Fourier theory for the transvection group at the cusp ∞. Eigenfunctions with $\lambda < \frac{1}{4}$ complicate asymptotic formulae as in the Selberg theory [4]. The Linnik-Selberg conjecture on averages of Kloosterman sums [7] would imply $\lambda \geqslant \frac{1}{4}$, and Kuznietsov [6] has shown that an averaged form of the conjecture holds in the absence of such exceptional eigenvalues. For congruence subgroups of the modular group the Eisenstein series $E(z, s)$ is a linear combination of Epstein zeta-functions in the variable s, and $E(z, s)$ is easily seen to be regular for Re $s > \frac{1}{2}$ except for the pole at $s = 1$; cf. [8]. Accordingly exceptional eigenfunctions, if any, must be cusp forms.

Congruence subgroups of the modular group of level N are those subgroups containing $\Gamma(N)$, the principal subgroup of level N, which consists of matrices congruent mod N to the identity. The lengths of translations on $\Gamma(N)$ tend to infinity with N, and for $N \geqslant 2$ $\Gamma(N)$ contains no rotations. For $N \geqslant 3$ $\Gamma(N)$ has index

$$I(N) = \tfrac{1}{2}N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right),$$

with $I(N)/N$ distinct cusps, and genus

$$1 + \frac{(N-6)I(N)}{12N}.$$

Other interesting subgroups consist of the matrices which become upper triangular ($\Gamma_0(N)$), lower triangular ($\Gamma^0(N)$) or diagonal ($\Gamma_0^0(N)$) when reduced mod N. We note that $\Gamma_0(N)$ and $\Gamma^0(N)$ are conjugate in PSL $(2, Z)$, and that $\Gamma_0^0(N)$ is conjugate in $\mathrm{PSL}(2, R)$ to $\Gamma_0(N^2)$ and to $\Gamma^0(N^2)$. The eigenfunctions on conjugate groups differ only by a rigid motion in H.

For certain small values of N we can rule out exceptional eigenvalues by purely combinatorial arguments.

THEOREM. *Let the group Γ act discontinuously on the upper half plane H, and let $\Gamma\backslash H$ have finite area. Let f be a non-constant eigenfunction of the Laplacian that vanishes at all the cusps. If either*

(1) *$\Gamma\backslash H$ has genus zero, and f vanishes at all the fixed points of rotations with at most three exceptions, or* (2) *$\Gamma\backslash H$ has genus one, and f vanishes at all the fixed points of rotations with at most one exception, then the corresponding eigenvalue satisfies $\lambda > \frac{1}{4}$.*

Proof. Since f is non-constant, it has at least two nodal domains. When $\Gamma\backslash H$ has genus zero, at least two of them are topological discs, and one of these contains at most one fixed point in its interior. When $\Gamma\backslash H$ has genus one, either one nodal domain is a disc (which may contain a fixed point), or at least two nodal domains are topological annuli, and one of these contains no fixed point in its interior.

A disc on $\Gamma\backslash H$ containing one fixed point, that of a rotation group of order n, lifts to a disc on H that covers it n times. The cusps are also singularities of the hyperbolic metric on $\Gamma\backslash H$, but they lie on nodal lines—this may be verified directly from the Fourier series expansion of f—and so cannot lie inside a nodal domain. Hence Cheeger's theorem applies, and $\lambda > \frac{1}{4}$.

An annulus on $\Gamma\backslash H$ may lift to an annulus on H. In this case we may apply Cheeger's theorem at once. Otherwise we must make a cut, and lift to a disc D in H for which two connected arcs E and τE of the boundary are identified by some τ in Γ. If τ is a rotation of order n, then n copies of D fit together to form an annulus in H, and $\lambda > \frac{1}{4}$ again by Cheeger's theorem. If τ has infinite order, we unite n copies $D, \tau D, \ldots, \tau^{n-1}D$ and the arcs $\tau E, \ldots, \tau^{n-1}E$ to form a simply connected region whose boundary consists of E, $\tau^n E$ and two nodal lines of f. The cut E can be taken away from the cusps, so that

$$\int_E f^2 \, dl < \infty,$$

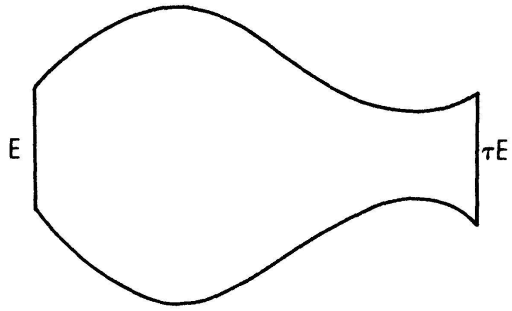

Figure 1

and using the value $h=1$ appropriate to hyperbolic space, we have

$$(2\lambda^{1/2}-1)n\iint_D f^2\,d\mu>-2\int_E f^2\,dl.$$

Hence $\lambda\geqslant\frac{1}{4}$, since n can be taken arbitrarily large. This inequality is in fact strict, as we have shown

$$\iint_D |f\operatorname{grad} f|\frac{dx\,dy}{y}\geqslant\tfrac{1}{4}\iint_D f^2\,d\mu,$$

and the left hand side is strictly less than

$$2\lambda^{1/2}\iint_D f^2\,d\mu.$$

COROLLARY. *We have* $\lambda>\frac{1}{4}$ *on* $\Gamma(N)\backslash H$ *for* $N=1,\ldots,5$ (*genus* 0), *and* $N=6$ (*genus* 1), *there being no rotations in the group for* $N\geqslant 2$. *We have* $\lambda>\frac{1}{4}$ *on* $\Gamma^0(N)\backslash H$ *for* $N=1,\ldots,12,14,15,16,18,20,24,25,27,32$, *and* 36. (*The genus is* 0 *for* $N=1,\ldots,5,7,\ldots,10,12,16,18$ *and* 25. *There are two fixed points for* $N=1,3,5,7$ *and* 10, *and one for* $N=2$. *The other values of N listed give genus* 1 *and no rotations.*)

We remark that if Γ has neither cusps nor rotations, then $\Gamma\backslash H$ has genus at least two. The rotations all lie outside a subgroup of finite index in Γ (index at most six for congruence subgroups of the modular group), but the corresponding surface has a larger genus. One might hope that any eigenfunction vanishing at all cusps would have $\lambda\geqslant\frac{1}{4}$; but using ideas of Buser we can construct a group Γ for which $\Gamma\backslash H$ has one cusp, four fixed points of rotations and genus zero, λ_1 is arbitrarily small, and the first non-constant eignefunction is skewsymmetric about an axis through the cusp, and so is zero there.

Our Theorem can be generalised by allowing some of the singular points of the metric at which f does not vanish to be cusps, not fixed points. This uses a different argument to deal with a cusp in the interior of a nodal domain.

For the congruence subgroups the fact that $\lambda > \frac{1}{4}$ for the modular group and for $\Gamma^0(2)$ is implicit in Maass [8], who was interested mainly in the value $\lambda = \frac{1}{4}$. It was proved explicitly by Roelcke [9]. For the modular group Roelcke showed $\lambda > 3\pi^2/2$, better than the bound $\lambda > 25/4$ obtained from estimating $h(f)$ in Cheeger's theorem. We shall discuss numerical estimates and particular examples more fully elsewhere.

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