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Autor(en): **Strebel, Ralph**

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Explicit resolutions for the binary polyhedral groups and for other central extensions of the triangle groups

RALPH STREBEL

Geometric background and algebraic results

This paper will exhibit two classes of finitely presented groups for which explicit free resolutions can be obtained by direct algebraic calculations. The resolutions will be either periodic of period 4, or of length 3.

1. The groups of the first class are *central extensions of the triangle groups*, admitting a 2-generator 2-relator presentation. Specifically, let l, m, n be integers with $\min(|l|, |m|, |n|) \geqq 2$ and define

$$G = G(l, m, n) = \langle \alpha, \beta; \alpha^l = \beta^m = (\alpha\beta)^n \rangle. \tag{1}$$

In the sequel the canonical images in G of α, resp. β, will be denoted by a, resp. b.

The element $(ab)^n$ of G is central, being a power of either generator, and the central quotient $G(l, m, n)/\langle (ab)^n \rangle$ is the triangle group

$$\begin{aligned} T = T(l, m, n) &= \langle \alpha, \beta; \alpha^l = \beta^m = (\alpha\beta)^n = 1 \rangle \\ &\cong \langle \alpha, \beta, \gamma; \alpha^l = \beta^m = \gamma^n = \alpha\beta\gamma = 1 \rangle. \end{aligned} \tag{2}$$

Every triangle group T can be realized faithfully as a group of isometries of the sphere $S^2 \subset \mathbb{R}^2$ when $|l|^{-1} + |m|^{-1} + |n|^{-1} > 1$ (or, equivalently, if T is finite), of the Euclidean plane if $|l|^{-1} + |m|^{-1} + |n|^{-1} = 1$, or of the hyperbolic plane if $|l|^{-1} + |m|^{-1} + |n|^{-1} < 1$. This action of T leads in all three cases to a tesselation of the space in question by pairs of adjacent triangles – whence the name "triangle group". (See, e.g., [10], and the references cited there for proofs and more details.)

2. Each of the groups $G(l, m, n)$ occurs as the *fundamental group of a suitable Seifert fiber space*. Such a space M is a compact 3-dimensional manifold equipped with a foliation by circles, called fibers. The set of all fibers forms the orbit space, which is a compact surface. The neighbourhoods of a fiber are fibered solid tori with the given fiber as their core, and depending on how these solid tori are fibered, the given fiber is called exceptional or ordinary. From the fact that M is compact it follows that there are only finitely many exceptional fibers.

In the special case where the orbit space of M is the 2-sphere and where there are three exceptional fibers, the fundamental group Γ of M has a presentation of the form

$$\Gamma = \langle \alpha, \beta, \gamma, \zeta;\ \alpha^l = \zeta^{l'},\ \beta^m = \zeta^{m'},\ \gamma^n = \zeta^{n'},\ \alpha\beta\gamma = \zeta^p \text{ and } \zeta \text{ is central}\rangle.$$

Here the pairs (l, l'), (m, m') and (n, n') are relatively prime, $\min(|l|, |m|, |n|) \geqq 2$ and p is an arbitrary integer. For $l' = m' = 1$, $n' = -1$ and $p = 0$, the group Γ is isomorphic with $G(l, m, n)$. (See H. Seifert [15], or [13], for more details and proofs.)

If the fundamental group Γ of a Seifert fiber space M is *finite*, it admits a faithful representation $\rho: \Gamma \rightarrowtail SO_4(\mathbb{R})$ for which the induced action on $S^3 \subset \mathbb{R}^4$ is fixpoint-free and the quotient space $\rho(\Gamma)\backslash S^3$ is homeomorphic to M (W. Threlfall and H. Seifert [20, Part II, p. 568, Hauptsatz]). From this fixpoint-free action of Γ on S^3 one can deduce (see, e.g., [2, p. 154]) that there exists a $\mathbb{Z}\Gamma$-free resolution $\mathbf{P} \twoheadrightarrow \mathbb{Z}$ which has period 4 and is finitely generated in each dimension; in particular, this is true for the finite groups $G(l, m, n)$.

If the fundamental group Γ of a Seifert fiber space M is *infinite*, M is in most cases aspherical, as can be deduced from the sphere theorem (see [13, p. 56, Satz 5] for a precise statement). In particular, the spaces with an infinite $G(l, m, n)$ are aspherical, whence the infinite groups $G(l, m, n)$ must be Poincaré-duality groups of dimension 3.

3. Our first result describes for each $G = G(l, m, n)$ an explicit $\mathbb{Z}G$-free resolution of $\mathbb{Z}$. These resolutions will have the same form up to dimension 3 for all groups, they be finite or infinite, and will permit one to read off many properties known on topological grounds. The proof will be uniform for all groups.

In order to define the resolution, choose the defining relators

$$r = (\alpha\beta)^n\beta^{-m} \quad \text{and} \quad s = (\beta\alpha)^n\alpha^{-l} \tag{3}$$

for $G(l, m, n)$; these relators do define $G(l, m, n)$, as can be checked speedily. Let F be the free group on $\{\alpha, \beta\}$, and let D_α, resp. D_β, denote the composite of the partial derivation $\partial/\partial\alpha: F \to \mathbb{Z}F$, resp. $\partial/\partial\beta: F \to \mathbb{Z}F$, and the canonical ring epimorphism $\mathbb{Z}F \twoheadrightarrow \mathbb{Z}G$.

THEOREM A. *If $G = G(l, m, n)$ and r, s are as before, the sequence of left $\mathbb{Z}G$-modules and $\mathbb{Z}G$-homomorphisms*

$$\mathbb{Z}G \xrightarrow{(1-b,\,1-a)} \mathbb{Z}G^2 \xrightarrow{\begin{pmatrix} D_\alpha r & D_\beta r \\ D_\alpha s & D_\beta s\end{pmatrix}} \mathbb{Z}G^2 \xrightarrow{\begin{pmatrix} 1-a \\ 1-b\end{pmatrix}} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \tag{4}$$

is exact. (*Here* $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ *takes* $g \in G$ *to* $1 \in \mathbb{Z}$.) *Furthermore:* (*i*) *If* G *is finite, the kernel of the left-most homomorphism is infinite cyclic, generated by the element* $\sum\{g \mid g \in G\}$; *therefore* (4) *leads by splicing to a periodic resolution with period* 4. (*ii*) *If* G *is infinite, the left-most homomorphism is injective and* (4) *is a* $\mathbb{Z}G$*-free resolution of* $\mathbb{Z}$; *moreover,* G *is an orientable Poincaré-duality group of dimension* 3.

4. The groups of the second class are *central extensions of* 1-*relator groups*. Let F be the free group on $\{\alpha_1, \ldots, \alpha_n\}$ and let ρ be a non-trivial element of F which is not a proper power. Define

$$L = \langle \alpha_1, \ldots, \alpha_n; [\rho^l, \alpha_1], \ldots, [\rho^l, \alpha_n] \rangle, \tag{5}$$

where $l \geqq 1$ and $[\rho^l, \alpha_i] := \rho^l \cdot \alpha_i \cdot \rho^{-l} \cdot \alpha_i^{-1}$. For $i = 1, \ldots, n$, let $a_i \in L$ denote the canonical image of $\alpha_i \in F$, and let D_{α_i} be the composite of the partial derivation $\partial/\partial\alpha_i : F \to \mathbb{Z}F$ and the canonical ring epimorphism $\mathbb{Z}F \twoheadrightarrow \mathbb{Z}L$.

THEOREM B. *If the notation is as before and* $n \geqq 2$, *the sequence of left* $\mathbb{Z}L$*-modules and* $\mathbb{Z}L$*-homomorphisms*

$$\mathbb{Z}L \xrightarrow{(D_{\alpha_1}\rho, \ldots, D_{\alpha_n}\rho)} \mathbb{Z}L^n \xrightarrow{\begin{pmatrix} D_{\alpha_1}[\rho^l, \alpha_1] \cdots D_{\alpha_n}[\rho^l, \alpha_1] \\ \vdots \qquad\qquad \vdots \\ D_{\alpha_1}[\rho^l, \alpha_n] \cdots D_{\alpha_n}[\rho^l, \alpha_n] \end{pmatrix}} \mathbb{Z}L^n \xrightarrow{\begin{pmatrix} 1 - a_1 \\ \vdots \\ 1 - a_n \end{pmatrix}} \mathbb{Z}L \xrightarrow{\varepsilon} \mathbb{Z} \tag{6}$$

is a $\mathbb{Z}L$*-free resolution of* $\mathbb{Z}$.

The groups L are only in special cases Poincaré-duality groups of dimension 3. As we shall prove in Theorem 9 this happens if, and only if, F admits either a basis $\xi_1, \eta_1, \ldots, \xi_g, \eta_g$ such that $\rho = [\xi_1, \eta_1] \cdot \cdots \cdot [\xi_g, \eta_g]$, or a basis $\xi_1, \xi_2, \ldots, \xi_g$ for which $\rho = \xi_1^2 \cdot \cdots \cdot \xi_g^2$. It follows that the Poincaré-duality groups in the second class of groups are fundamental groups of Seifert fiber spaces which have at most one exceptional fiber and an orientable, or non-orientable, closed surface of genus $g > 0$ as their orbit space.

1. Some general facts about the groups $G(l, m, n)$

Let l, m, n be arbitrary integers and set

$$G = G(l, m, n) = \langle \alpha, \beta; \alpha^l = \beta^m = (\alpha\beta)^n \rangle.$$

As before we denote the canonical images in G of α, β by a, b.

Normalization of the parameters. The defining relations of G imply that $(ab)^n=(ba)^n$ and so the assignments $\alpha\mapsto b$, $\beta\mapsto a$ induce an isomorphism $G(l,m,n)\xrightarrow{\sim} G(m,l,n)$. Similarly, the assignments $\alpha\mapsto a^{-1}$, $\beta\mapsto ab$ lead to an isomorphism $G(l,m,n)\xrightarrow{\sim} G(-l,n,m)$. These two types of isomorphisms allow one to make any of $l,-l,m,-m,n$ or $-n$ the third parameter and so we can assume that $n=\min(|l|,|m|,|n|)$. By exchanging l and m, if need be, we arrive at the normalization

$$l\geqq m \quad\text{and}\quad \min(|l|,|m|)\geqq n\geqq 0.$$

The groups $G(l,m,0)$ are free products $\langle a\rangle*\langle b\rangle$, whereas the groups $G(l,m,1)$ are all cyclic; indeed one has

$$\begin{aligned}G(l,m,1)&=\langle\alpha,\beta;\ \alpha^l=\beta^m=\alpha\beta\rangle=\langle\alpha;\ \alpha^l=(\alpha^{l-1})^m\rangle\\&=\langle\alpha;\ \alpha^{(l-1)(m-1)-1}\rangle.\end{aligned}$$

As the results of this paper are of little interest when specialized to cyclic groups or to free products of cyclic groups, we shall henceforth assume, as we did in the introduction, that $\min(|l|,|m|,|n|)\geqq 2$. The previous normalization can then be sharpened to

$$l\geqq m \quad\text{and}\quad \min(|l|,|m|)\geqq n\geqq 2. \tag{7}$$

Isomorphic groups. Different triples satisfying this normalization condition yield in general non-isomorphic groups, as is revealed by our

THEOREM 1. *Let (l_1,m_1,n_1) and (l_2,m_2,n_2) be ordered triples of integers satisfying* (7). *If $G(l_1,m_1,n_1)$ and $G(l_2,m_2,n_2)$ are isomorphic, then either both ordered triples are equal, or the two triples are related to each other as are (l,n,n) and $(n,-l,n)$.*

For infinite groups $G(l_i,m_i,n_i)$ the assertion of Theorem 1 can be deduced from a far more general result of P. Orlik et al. on the fundamental groups of Seifert fiber spaces [13, p. 53, Satz 4]. However, whereas the proof of this more general result is quite complicated, Theorem 1 can be established by merely comparing the central quotients $G(l_i,m_i,n_i)/\zeta G(l_i,m_i,n_i)$ and the abelianizations $G(l_i,m_i,n_i)_{ab}$; for this reason we give an independent proof. Before embarking on it we determine the abelianization of the group $G(l,m,n)$.

Computation of the abelianized group. The relators $r=(\alpha\beta)^n\beta^{-m}$ and $s=(\beta\alpha)^n\alpha^{-l}$, which can be used to define G, lead to the relation matrix $R=$

$\begin{pmatrix} n & n-m \\ n-l & n \end{pmatrix}$ of G_{ab}. Put another way, G_{ab} is isomorphic to the cokernel of the homomorphism $\mathbb{Z}^2 \to Z^2$ that takes (x, y) to the matrix product $(x, y) \cdot R$. The determinant of R is

$$\det R = (l+m)n - lm = lmn(l^{-1} + m^{-1} - n^{-1}), \tag{8}$$

and so the theory of elementary divisors implies that

$$G_{ab} \cong Z/\mathbb{Z}e \times \mathbb{Z}/\mathbb{Z}e' \tag{9}$$

where $e = gcd(l, m, n)$ and $e' = \det R/e$.

Proof of Theorem 1. Set $G_i = G(l_i, m_i, n_i)$ for $i = 1$ or 2. We contend the element $(ab)^{n_i}$ generates the centre of G_i. To see this, let (l, m, n) be a triple satisfying (7) and let $T = T(l, m, n)$ be the corresponding triangle group. If T is infinite, its centre is trivial (see, e.g., [22, p. 126, **4.8.1**]). If T is finite, it is either dihedral, or it is polyhedral, i.e. isomorphic to $\mathfrak{A}_4$, $\mathfrak{S}_4$ or $\mathfrak{A}_5$, and ζT will be trivial except in case T is a dihedral group the order of which is divisible by 4. It follows that $\zeta G(l, m, n) = \langle (ab)^n \rangle$, except possibly if G is isomorphic to $G(k, \pm 2, 2)$ where $|k| \geqq 2$ is even. These exceptional groups have the alternative presentation

$$\begin{aligned} G(k, 2, 2) &= \langle \alpha, \beta; \alpha^k = \beta^2, \beta\alpha\beta^{-1} = \alpha^{-1} \rangle \\ &= \langle \alpha, \beta; \alpha^{2k} = 1, \alpha^k = \beta^2, \beta\alpha\beta^{-1} = \alpha^{-1} \rangle \end{aligned} \qquad (k \geqq 2)$$

and

$$\begin{aligned} G(k, -2, 2) &= \langle \alpha, \beta; \alpha^k = \beta^{-2}, \beta\alpha\beta^{-1} = \alpha^{-1}\beta^{-4} \rangle \\ &= \langle \alpha, \beta; \alpha^k = \beta^{-2}, \beta\alpha\beta^{-1} = \alpha^{2k-1} \rangle. \end{aligned}$$

Every element $g \in G(k, \pm 2, 2)$ is of one of the forms a^h or $a^h b$; the centrality condition $bgb^{-1} = g$ implies that $a^{2h} = 1$ in the first and that $a^{2h(k-1)} = 1$ in the second case. Now the order of a in $G(k, 2, 2)$ is $2k$, while the order of a in $G(k, -2, 2)$ is $2k(k-1)$; see, e.g., [5, §6.5, pp. 68–70], or the discussion below. So h is a multiple of k in either case. Since b is not central, we conclude that $\zeta G(k, \pm 2, 2) = \langle a^k \rangle = \langle (ab)^2 \rangle$.

Assume now that $G_1 = G(l_1, m_1, n_1)$ and $G_2 = G(l_2, m_2, n_2)$ are isomorphic. Then so are $T_1 = T(l_1, m_1, n_1) \cong G_1/\zeta G_1$ and $T_2 = T(l_2, m_2, n_2)$. But T_1 and T_2 can only be isomorphic if the unordered triples $\{|l_1|, |m_1|, |n_1|\}$ and $\{|l_2|, |m_2|, |n_2|\}$ coincide; this assertion is obvious if the T_i are finite and it follows for the infinite triangle groups from the fact that the elements of finite order of T_i have order dividing $|l_i|, |m_i|, |n_i|$ and that the orders $|l_i|$, $|m_i|$ and $|n_i|$ occur (cf. [22, p. 126, Thm. 4.8.1.a)]). The normalization (7) now implies that $n_1 = n_2 = n$. In addition, we have that

$$\{|l_1|, |m_1|\} = \{|l_2|, |m_2|\}. \tag{10}$$

Next we exploit the fact that $(G_1)_{ab}$ and $(G_2)_{ab}$ are isomorphic. In view of (8), (9) and (10) this fact leads to the equation

$$l_1^{-1} + m_1^{-1} - n^{-1} = \varepsilon(l_2^{-1} + m_2^{-1} - n^{-1}),$$

where $\varepsilon = \pm 1$. Suppose first that $\varepsilon = 1$ and set $\mu := l_1^{-1} + m_1^{-1} = l_2^{-1} + m_2^{-1}$. Clearly

$$\mu = 0 \Leftrightarrow l_i = -m_i$$

$$\mu > 0 \Leftrightarrow l_i \geqq m_i > 0, \quad \text{or} \quad l_i > 0, m_i < 0 \quad \text{and} \quad l_i < |m_i|$$

$$\mu < 0 \Leftrightarrow 0 > l_i \geqq m_i, \quad \text{or} \quad l_i > 0, m_i < 0 \quad \text{and} \quad l_i > |m_i|$$

Making use of these case distinctions and of (7), (10) one verifies quickly that $l_1^{-1} + m_1^{-1} = l_2^{-1} + m_2^{-1}$ implies that $l_1 = l_2$ and $m_1 = m_2$.

Finally let $\varepsilon = -1$. Then

$$l_1^{-1} + l_2^{-1} + m_1^{-1} + m_2^{-1} = 2n^{-1} > 0.$$

If all four summands of the left hand side are positive, (7) and (10) imply that $l_1 = l_2$ and $m_1 = m_2$. If one summand of the left hand side is negative, it follows from (10) and (7) that two summands must be the negative of each other, whence (7) implies that the two remaining summands are equal to n^{-1}. So we are in the special case where the two ordered triples are of the form (l, n, n) and $(n, -l, n)$. □

We proceed to determine when the abelianized group G_{ab} is infinite and when it is trivial.

Groups with infinite abelianization. Assume the parameters are normalized as in (7) and G_{ab} is infinite. Then $l^{-1} + m^{-1} = n^{-1}$ by (8) and (9), and l and m will be positive. Set $d = gcd(l, m) > 0$ and $f = l/d$, resp. $g = m/d$. From $n^{-1} = l^{-1} + m^{-1} = (f+g)/dfg$ one sees that n is a multiple of fg, say $n = e \cdot fg$, and so d becomes $e(f+g)$. This shows that l, m, n are given by

$$\begin{cases} l = e \cdot f(f+g), \qquad m = e \cdot (f+g)g, \qquad n = e \cdot fg, \\ \text{where } e \geqq 1 \text{ and } f, g \text{ are relatively prime positive integers.} \end{cases} \tag{11}$$

Conversely, every triple (l, m, n) given by (11) satisfies $l^{-1}+m^{-1}=n^{-1}$. Therefore (11) characterizes the normalized triples leading to an infinite abelianization G_{ab}. Note that $G_{ab}\cong\mathbb{Z}/\mathbb{Z}e\times\mathbb{Z}$ by (9).

Perfect groups. If G_{ab} is trivial, i.e. if G is *perfect*, the parameters l, m, n satisfy by (8) the equation $(l+m)n-lm=\pm1$, which can be rewritten as

$$(l-n)(m-n)=n^2\mp1. \tag{12}$$

The integral solutions of (12) are easily surveyed; an infinite sequence of solutions is, e.g., given by the formula

$$(l, m, n)=(2n+1, 2n-1, n) \quad \text{where} \quad n\geqq2.$$

For $n=2$ equation (12) has the normalized solutions $(5, 3, 2)$ and $(7, 3, 2)$. The first gives the binary icosahedral group, the second the infinite group

$$\begin{aligned} G(7,3,2)&=\langle\alpha,\beta;\alpha^7=\beta^3=(\alpha\beta)^2\rangle\cong\langle\alpha,\beta,\gamma;\alpha^7=\beta^3=(\alpha\beta)^2,\gamma:=\alpha\beta\rangle\\ &=\langle\alpha,\beta,\gamma;\ \alpha^7=\beta^3=\gamma^2=\alpha\beta\gamma\rangle\end{aligned}$$

discussed in [7].

Analysis of the finite groups $G(l, m, n)$. We begin with the question which triangle groups $T(l, m, n)$ are finite. The answer is that this happens if, and only if, $|l|^{-1}+|m|^{-1}+|n|^{-1}>1$. This answer is usually justified by letting T act on a suitable space in the way indicated in **1** of the introduction. There is also a little-known algebraic argument due to P. M. Curran [6]; it is in the spirit of the proofs of this paper and runs briefly like this: Verify first that the canonical images of α, β, γ in

$$T(l, m, n)=\langle\alpha,\beta,\gamma;\alpha^l=\beta^m=\gamma^n=\alpha\beta\gamma=1\rangle \tag{14}$$

have orders l, m and n by constructing suitable quotient groups in which the canonical images of α, β, γ have the desired order (see, e.g. [22, p. 135]). Then use the following

LEMMA 2 (P. M. Curran [6, p. 620]). *Let l, m, n be positive integers and set*

$$S=\langle\alpha,\beta,\gamma;\alpha^l=\beta^m=\gamma^n=\rho=1\rangle, \tag{15}$$

where ρ is an arbitrary element of the free group on α, β, γ. Denote the canonical images of α, β, γ by a, b, c, and let $\bar{l}, \bar{m}, \bar{n}$ be their orders. If S is finite then $\bar{l}^{-1}+\bar{m}^{-1}+\bar{n}^{-1}>1$.

Proof. Since S is finite, $H^1(S, \mathbb{Z}S) = 0$. On the other hand, $H^1(S, \mathbb{Z}S) = \ker \partial_2^*/\operatorname{im} \partial_1^*$ can be computed by means of the exact complex of left $\mathbb{Z}S$-modules and $\mathbb{Z}S$-homomorphisms associated with the presenta-

$$\mathbb{Z}S^4 \xrightarrow{\begin{pmatrix} D_\alpha\alpha^l & 0 & 0 \\ 0 & D_\beta\beta^m & 0 \\ 0 & 0 & D_\gamma\gamma^n \\ D_\alpha\rho & D_\beta\rho & D_\gamma\rho \end{pmatrix}} \mathbb{Z}S^3 \xrightarrow{\begin{pmatrix} 1-a \\ 1-b \\ 1-c \end{pmatrix}} \mathbb{Z}S \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

tion (15); cf. [2, p. 45, Ex. 3(d), or p. 90, Ex. 4(c) and (d)]). Clearly $\operatorname{im} \partial_1^* \cong \mathbb{Z}S/\sum\{s \mid s \in S\} \cdot \mathbb{Z}S$ and so $\operatorname{rank}(\operatorname{im} \partial_1^*) = |S| - 1$. Next ker $(\mathbb{Z}S \xleftarrow{1+a+\cdots+a^{l-1}} \mathbb{Z}S) = (1-a) \cdot \mathbb{Z}S$ and hence rank $((1-a) \cdot \mathbb{Z}S) = |S|(1 - 1/\bar{l})$; similar statements hold for the two other, analogous maps. It follows that ker ∂_2^* equals the kernel of

$$\mathbb{Z}S \xleftarrow{(D_\alpha\rho,\, D_\beta\rho,\, D_\gamma\rho)} ((1-a) \cdot \mathbb{Z}S \oplus (1-b) \cdot \mathbb{Z}S \oplus (1-c) \cdot \mathbb{Z}S)^{\text{transposed}}$$

and thus $\operatorname{rank}(\ker \partial_2^*) \geqq |S|\,((1 - 1/\bar{l}) + (1 - 1/\bar{m}) + (1 + 1/\bar{n}) - 1)$. The claim then follows from $|S| - 1 = \operatorname{rank}(\operatorname{im} \partial_1^*) = \operatorname{rank}(\ker \partial_2^*)$. □

To complete the determination of the finite triangle groups $T(l, m, n)$, use that a change of the order or the signs of the parameters l, m, n does not influence the isomorphism type of the group; so one can assume that $l \geqq m \geqq n = 2$. The solutions $(l, 2, 2)$ of $l^{-1} + m^{-1} + n^{-1} > 1$ yield the dihedral groups of order $2l$; the remaining three solutions $(l, 3, 2)$, where $l = 3, 4, 5$, give the polyhedral groups $\mathfrak{A}_4$, $\mathfrak{S}_4$ and $\mathfrak{A}_5$ (cf. [5, p. 7 and p. 67]).

We now pass to the finite groups $G(l, m, n)$ and begin with the simple

LEMMA 3. *If* $\min(|l|, |m|, |n|) \geqq 2$ *and* $T = T(l, m, n)$ *is finite then* $G = G(l, m, n)$ *is likewise finite.*

Proof. Assume the parameters are normalized as in (7). As T is finite one has $n = 2$ and $|l|^{-1} + |m|^{-1} > 1/2$; a comparison with (11) discloses that G_{ab} is finite. The central extension $\langle a^l \rangle \lhd G \twoheadrightarrow T$ gives rise to the exact sequence

$$H_2G \to H_2T \to \langle a^l \rangle \to G_{ab} \to T_{ab} \to 0. \tag{16}$$

Since T is finite, H_2T is so, and then (16) shows that $\langle a^l \rangle$ and hence G are finite. □

The determination of the finite groups $G(l, m, n)$ can be completed as follows: By (7) and the previous reasonings n can be assumed to be 2. Since G is a finite

2-generator, 2-relator group, its multiplicator H_2G is trivial (see, e.g., [2, p. 46, Ex. 5]). The central extension $\langle a^l\rangle \lhd G \twoheadrightarrow T$ gives rise to the exact sequence (16), the exactness of which implies that

$$|\langle a^l\rangle| = |G_{ab}| \cdot |H_2T| \cdot |T_{ab}|^{-1}. \tag{17}$$

The quotient $|H_2T| \cdot |T_{ab}|^{-1}$ can be shown to be 2/4, resp. 2/3, 2/2 or 2/1 for $T(l, 2, 2)$, resp. $T(l, 3, 2)$. On the other hand, G_{ab} is by (8) and (9) equal to 4, resp. 3, 2 or 1 for the special cases $G(l, 2, 2)$, resp. $G(l, 3, 2)$, the parameter l being positive.

It follows that $|\langle a^l\rangle| = 2$ in all these special cases and so these groups are the binary dihedral, resp. polyhedral groups (cf. [5, §6.5]).

Now let $G(l, m, 2)$ be an arbitrary finite group and set

$$G_0 := G(|l|, |m|, 2) = \langle \alpha_0, \beta_0; \alpha_0^{|l|} = \beta_0^{|m|} = (\alpha_0\beta_0)^2\rangle$$

Because $a_0^{|l|} = b_0^{|m|} = (a_0b_0)^2$ has order 2, the assignments $\alpha \mapsto a_0$, $\beta \mapsto b_0$ extend to an epimorphism $G \twoheadrightarrow G_0$ and give rise to a central extension

$$\langle a^{2l}\rangle \lhd G \twoheadrightarrow G_0. \tag{18}$$

By (17) and (8), (9) the kernel $\langle a^{2l}\rangle$ has order

$$u(l, m) = |l^{-1} + m^{-1} - \tfrac{1}{2}| \cdot (|l|^{-1} + |m|^{-1} - \tfrac{1}{2})^{-1}.$$

The sequence (18) will split whenever $u(l, m)$ and the order of $G_0 = G(|l|, |m|, 2)$ are relatively prime. Upon computation one finds that the values of $u(l, m)$ corresponding to the sequence of signs $--$, $+-$, $-+$, $++$ are

31, 19, 11 and 1 for the triple (5, 3, 2) with $|G_0| = 120$

13, 7, 5 and 1 for the triple (4, 3, 2) with $|G_0| = 48$

7, 3, 3 and 1 for the triple (3, 3, 2) with $|G_0| = 24$

$l+1$, $l-1$, 1 and 1 for the triples $(l, 2, 2)$ with $|G_0| = 4.l$

Hence the only extensions (18) which may not split correspond to the groups $G(-3, 3, 2) \cong G(3, -3, 2)$ and $G(l, -2, 2)$ for odd $l \geqq 3$. These extensions do in fact not split, as can be seen from the 5-term sequence

$$0 \to 0 \to \langle a^{2l}\rangle \to G_{ab} \to (G_0)_{ab} \to 0$$

induced by (18) and the facts that G_{ab} is cyclic in all these cases, while $|\langle a^{2l}\rangle|$ and $|(G_0)_{ab}|$ are not relatively prime.

2. Proof of Theorem A

The proof relies on two auxiliary results. The first of them asserts that sequence (4), occurring in the statement of Thm. A is always a complex and that it is exact if $H^1(G, \mathbb{Z}G)$ is trivial; it can be established by an easy calculation. The second result shows that $H^1(G, \mathbb{Z}G)$ is trivial for our groups G; its proof makes use of an argument of Serre's and Stallings' characterization of finitely generated groups with infinitely many ends.

Let $F = F(\{\alpha, \beta\})$ be the free group on α and β. As in the introduction set $r = (\alpha\beta)^n\beta^{-m}$ and $s = (\beta\alpha)^n\alpha^{-l}$, and define $G = G(l, m, n) := \langle \alpha, \beta; r, s\rangle$. Assume $lmn \neq 0$.

LEMMA 4. *Let* $\partial_3: \mathbb{Z}G \to \mathbb{Z}G^2$ *and* $\partial_2: \mathbb{Z}G^2 \to \mathbb{Z}G^2$ *be the homomorphisms of left* $\mathbb{Z}G$*-modules given by multiplying on the right by the matrices*

$$(1-b, 1-a) \quad \textit{and} \quad \begin{pmatrix} D_\alpha r & D_\beta r \\ D_\alpha s & D_\beta s \end{pmatrix}.$$

Then $\operatorname{im} \partial_3 \subseteq \ker \partial_2$ *and the homology group* $\ker \partial_2/\operatorname{im} \partial_3$ *is* $\mathbb{Z}$*-isomorphic with the first cohomology group* $H^1(G, \mathbb{Z}G)$. *Moreover, if* $H^1(G, \mathbb{Z}G)$ *is trivial so is* $H^2(G, \mathbb{Z}G)$.

Proof. We shall compute $H^1(G, \mathbb{Z}G)$ by means of the well-known exact sequence

$$\mathbb{Z}G^2 \xrightarrow{\partial_2} \mathbb{Z}G^2 \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0, \tag{19}$$

where ∂_2 is as above, $\partial_1(\lambda, \mu) = \lambda(1-a) + \mu(1-\mathrm{b})$ and $\varepsilon: \mathbb{Z}G \to \mathbb{Z}$ is the unit augumentation (cf. [2, p. 90, Ex. 4]). For reasons that will become clear at a later stage of the proof we extend (19) by adding ∂_3 to the left ending up with the sequence

$$\mathbf{P} \twoheadrightarrow \mathbb{Z}: \mathbb{Z}G \xrightarrow{(1-b,\, 1-a)} \mathbb{Z}G^2 \xrightarrow{\begin{pmatrix} D_\alpha r & D_\beta r \\ D_\alpha s & D_\beta s \end{pmatrix}} \mathbb{Z}G^2 \xrightarrow{\begin{pmatrix} 1-a \\ 1-b \end{pmatrix}} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0. \tag{20}$$

Note that at this stage the sequence $\mathbf{P}$ is not known to be a complex.

The dual sequence $\mathbf{P}^* = \operatorname{Hom}_{\mathbb{Z}G}(\mathbf{P}, \mathbb{Z}G)$ can be described in dual bases as

$$\mathbb{Z}G \xleftarrow{(1-b,\, 1-a)} \mathbb{Z}G^2 \xleftarrow{\begin{pmatrix} D_\alpha r & D_\beta r \\ D_\alpha s & D_\beta s \end{pmatrix}} \mathbb{Z}G^2 \xleftarrow{\begin{pmatrix} 1-a \\ 1-b \end{pmatrix}} \mathbb{Z}G, \tag{21}$$

where all modules are right $\mathbb{Z}G$-modules and the matrices describe $\mathbb{Z}G$-linear homomorphisms by multiplication on the left. In order to revert to left $\mathbb{Z}G$-modules we use the ring antiautomorphism $\tau:\mathbb{Z}G \xrightarrow{\sim} \mathbb{Z}G$, obtained by extending the inversion $g \mapsto g^{-1}$ in the group $\mathbb{Z}$-additively. This transforms (21) into the sequence of left $\mathbb{Z}G$-modules

$$\mathbb{Z}G \xrightarrow{(1-a^{-1},\,1-b^{-1})} \mathbb{Z}G^2 \xrightarrow{\begin{pmatrix}\tau(D_\alpha r) & \tau(D_\alpha s)\\ \tau(D_\beta r) & \tau(D_\beta s)\end{pmatrix}} \mathbb{Z}G^2 \xrightarrow{\begin{pmatrix}1-b^{-1}\\ 1-a^{-1}\end{pmatrix}} \mathbb{Z}G. \tag{22}$$

Also $H^1(G,\mathbb{Z}G) \cong \ker \partial_2^*/\mathrm{im}\,\partial_1^*$ is $\mathbb{Z}$-isomorphic to the homology of (22) at the left middle module $\mathbb{Z}G^2$.

So far only the fact that G is given by a 2-generator 2-relator presentation has been used. We now bring into play the pecularity of G that the assignments $\alpha \mapsto a^{-1}$, $\beta \mapsto b^{-1}$ induce an automorphism $\sigma: G \xrightarrow{\sim} G$; indeed

$$(a^{-1}b^{-1})^n \cdot (b^{-1})^{-m} = (ba)^{-n}b^m = b^m(ab)^{-n} = ((ab)^n b^{-m})^{-1} = 1$$

and, similarly, $(b^{-1}a^{-1})^n \cdot (a^{-1})^{-l} = 1$. By means of this isomorphism σ and by exchanging the two summands of both middle modules $\mathbb{Z}G^2$ in (22) we arrive at the isomorphic sequence

$$\mathbb{Z}G \xrightarrow{(1-b,\,1-a)} \mathbb{Z}G^2 \xrightarrow{\begin{pmatrix}\sigma\tau(D_\beta s) & \sigma\tau(D_\beta r)\\ \sigma\tau(D_\alpha s) & \sigma\tau(D_\alpha r)\end{pmatrix}} \mathbb{Z}G^2 \xrightarrow{\begin{pmatrix}1-a\\ 1-b\end{pmatrix}} \mathbb{Z}G \tag{23}$$

Observe that we know at this stage that the homology is defined at the middle left term and that it is isomorphic to $H^1(G,\mathbb{Z}G)$. Our aim is to verify that the (2×2)-matrix displayed in (23) is identical with the (2×2)-matrix of the sequence $\mathbf{P}$ defined in (20). Once this is known, $\mathbf{P} \twoheadrightarrow \mathbb{Z}$ must be a complex and its homology in dimension 2 equals $H^1(G,\mathbb{Z}G)$. In addition, if $H^1(G,\mathbb{Z}G) = 0$ the complex $\mathbf{P} \twoheadrightarrow \mathbb{Z}$ is exact and can be used to compute $H^2(G,\mathbb{Z}G)$. If this is done, the above manipulations show that $H^2(G,\mathbb{Z}G)$ is isomorphic to the homology of (23) at the right middle term, i.e. to the homology of $\mathbf{P} \twoheadrightarrow \mathbb{Z}$ in dimension 1. As this homology is a priori known to be trivial, $H^2(G,\mathbb{Z}G)$ must be trivial and all assertions of Lemma 4 will be established.

The entries $D_\alpha r$ and $D_\beta r$ of the (2×2)-matrix displayed in (20) are:

$$D_\alpha r = D_\alpha((\alpha\beta)^n\beta^{-m}) = (\mathrm{sign}\, n)\cdot(1 + ab + \cdots + (ab)^{|n|-1})\cdot(ab)^{(n-|n|)/2}$$

and

$$D_\beta r = D_\beta((\alpha\beta)^n\beta^{-m}) = (\text{sign } n)\cdot(a+aba+\cdots+(ab)^{|n|-1}a)\cdot(ab)^{(n-|n|)/2}$$
$$-(\text{sign } m)\cdot(1+b+\cdots+b^{|m|-1})\cdot b^{(m-|m|)/2}.$$

The entry $D_\beta s$ arises from $D_\alpha r$ by exchanging throughout a and b, while $D_\alpha s$ arises from $D_\beta r$ by exchanging a and b, and replacing m by l. The composite $\sigma \circ \tau$ transforms a product $x_1x_2\cdots x_k$, where each factor is one of a, a^{-1}, b or b^{-1}, into the reversed product $x_k\cdots x_2x_1$, and it is $\mathbb{Z}$-linear. Using these facts one checks easily that $\sigma\circ\tau$ exchanges $D_\alpha r$ and $D_\beta s$, while it fixes $D_\beta r$ and $D_\alpha s$, these group ring elements being $\mathbb{Z}$-linear combinations of palindroms. It follows that the (2×2)-matrices displayed in (20) and (23) are identical. □

LEMMA 5. *If* $\min(|l|,|m|,|n|)\geqq 2$ *then* $H^1(G,\mathbb{Z}G)=0$.

Proof. The claim is true for finite groups on general grounds; but the following argument takes care of them at no extra expense.

Let l_1, m_1, n_1 be integers greater than 1 and let $T(l_1, m_1, n_1)$ be the corresponding triangle group. By a result of Serre's [16, p. 85, **6.3.5**] every inversion-free action of $T(l_1, m_1, n_1)$ on a tree has a fixed point; in particular, no homomorphic image of $T(l_1, m_1, n_1)$ can be a non-trivial amalgam $A *_C B$.

Let us go back to the given group G. If $a^l = b^m$ has finite order, say k, then G is a homomorphic image of $T(l_1, m_1, n_1)$, where $l_1 = k\,|l|$, $m_1 = k\,|m|$ and $n_1 = k\,|n|$, and so it is not a non-trivial amalgam. Moreover, G_{ab} is finite. Stallings' structure theorem (e.g. [17, p. 38, **4.A.6.5** and p. 57, **5.A.10**]) therefore implies that $H^1(G,\mathbb{Z}G)$ is trivial.

If $a^l = b^m$ has infinite order, consider the central extension $Z=\langle a^l\rangle \lhd G \twoheadrightarrow T$. It leads in cohomology to the exact sequence

$$H^1(T, H^0(Z,\mathbb{Z}G)) \rightarrowtail H^1(G,\mathbb{Z}G) \to H^0(T, H^1(Z,\mathbb{Z}G)) \to H^2(T, H^0(Z,\mathbb{Z}G)) \to\cdots. \tag{24}$$

Since Z is infinite, $H^0(Z,\mathbb{Z}G)=(\mathbb{Z}G)^Z$ is trivial; because Z is an orientable Poincare-duality group of dimension one, $H^1(Z,\mathbb{Z}G)=\mathbb{Z}\otimes_{\mathbb{Z}Z}\mathbb{Z}G\cong\mathbb{Z}T$. Finally, since T is infinite by Lemma 3, the exactness of (24) and the previous reasoning imply that $H^1(G,\mathbb{Z}G)\xrightarrow{\sim} H^0(T,\mathbb{Z}T)=0$, as asserted. □

The *proof of Theorem A* is now quickly completed. By assumption $\min(|l|,|m|,|n|)\geqq 2$ and so Lemmata 3, 4 and 5 apply. They show that the sequence (4), which is identical with (20), is an exact complex and that $H^1(G,\mathbb{Z}G)$

and $H^2(G, \mathbb{Z}G)$ are both trivial. The kernel of $\partial_3: \mathbb{Z}G \to \mathbb{Z}G^2$, taking $\lambda \in \mathbb{Z}G$ to $(\lambda(1-b), \lambda(1-a))$, consists of all $\lambda \in \mathbb{Z}G$ which are fixed by the generators b and a, hence by all of G.

If G is finite, ker ∂_3 is therefore generated by $\sum\{g \mid g \in G\}$ and the sequence

$$\mathbb{Z}G^2 \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\sum\{g \mid g \in G\}} \mathbb{Z}G \xrightarrow{\partial_3} \mathbb{Z}G^2$$

is exact. This proves that (4) leads by splicing to a periodic $\mathbb{Z}G$-free resolution of $\mathbb{Z}$ having period 4.

If, on the other hand, G is infinite, ker ∂_3 is trivial and (4) is a finite $\mathbb{Z}G$-free resolution of $\mathbb{Z}$. Moreover, G is an orientable Poincaré-duality group of dimension 3. Indeed, $H^1(G, \mathbb{Z}G)$ and $H^2(G, \mathbb{Z}G)$ are trivial by the previous remarks. Next, if $H^3(G, \mathbb{Z}G)$ is computed by means of (4), one obtains the following chain of isomorphisms of right $\mathbb{Z}G$-modules:

$$H^3(G, \mathbb{Z}G) \cong \mathbb{Z}G/(1-b)\mathbb{Z}G + (1-a)\mathbb{Z}G = \mathbb{Z}G/IG \xrightarrow{\varepsilon_\sim} \mathbb{Z}.$$

The claim then follows from well-known results about duality groups (see, e.g., [1, p. 140, Thm. 9.2 and p. 173], or [2, p. 220, Thm. 10.1, and definition on p. 221]). □

Remark 6. If G is a binary dihedral group $G(l, 2, 2)$, the periodic resolution $\mathbf{P} \twoheadrightarrow \mathbb{Z}$ obtained from sequence (20) by splicing, is isomorphic to the resolution $\mathbf{P}' \twoheadrightarrow \mathbb{Z}$ described by Cartan-Eilenberg [3, p. 252]. In order to see this, identify $G(l, 2, 2)$ and $\pi = \langle x, y;\, x^l = y^2 = (xy)^2 \rangle$ in the obvious way, and note that

$$D_\beta r = D_\beta(\alpha\beta\alpha \cdot \beta^{-1}) = a - 1.$$

The function $\Phi: \mathbf{P} \to \mathbf{P}'$ which respects the dimensions of the chain groups, is the identity in dimensions different from $2+4p$ and sends $(\lambda, \mu) \in P_{2+4p}$ to $(-\mu, \lambda + \mu b) \in P'_{2+4p}$, can then easily be verified to be a chain isomorphism. □

3. Proof of Theorem B

Let F be free on $\{\alpha_1, \ldots, \alpha_n\}$, let ρ be a non-trivial element of F which is not a proper power, let $l \geqq 1$ and set

$$L = \langle \alpha_1, \ldots, \alpha_n;\, [\rho^l, \alpha_1], \ldots, [\rho^l, \alpha_n] \rangle,$$

where $[\rho^l, \alpha_i] := \rho^l \cdot \alpha_i \cdot \rho^{-l} \cdot \alpha_i^{-1}$. If $n=1$, the group L is infinite cyclic and $\mathbb{Z}L \xrightarrow{1-a_1} \mathbb{Z}L \xrightarrow{\varepsilon} \mathbb{Z}$ is an explicit $\mathbb{Z}L$-free resolution of $\mathbb{Z}$. So assume $n \geqq 2$. Denote the canonical image in L of α_i by a_i, that of ρ by r and write D_{α_i} for the composite $F \xrightarrow{\partial/\partial\alpha_i} \mathbb{Z}F \xrightarrow{\text{can}} \mathbb{Z}G$. We aim at verifying that the sequence of left $\mathbb{Z}L$-modules and $\mathbb{Z}L$-homomorphisms

$$\mathbb{Z}L \xrightarrow{(D_{\alpha_1}\rho, \ldots, D_{\alpha_n}\rho)} \mathbb{Z}L^n \xrightarrow{\begin{pmatrix} D_{\alpha_1}[\rho^l, \alpha_1] \cdots D_{\alpha_n}[\rho^l, \alpha_1] \\ \vdots \qquad \vdots \\ D_{\alpha_1}[\rho^l, \alpha_n] \cdots D_{\alpha_n}[\rho^l, \alpha_n] \end{pmatrix}} \mathbb{Z}L \xrightarrow{\begin{pmatrix} 1-a_1 \\ \vdots \\ 1-a_n \end{pmatrix}} \mathbb{Z}L \xrightarrow{\varepsilon} \mathbb{Z} \tag{25}$$

is an exact complex. Our verification will be based on Lemma 7 below and Lyndon's Identity Theorem (cf. [9, pp. 158+161]) which asserts that

$$\mathbb{Z}\bar{L} \xrightarrow{1-\bar{r}} \mathbb{Z}\bar{L} \xrightarrow{(D_{\alpha_1}\rho^l, \ldots, D_{\alpha_n}\rho^l)} \mathbb{Z}\bar{L} \xrightarrow{\begin{pmatrix} 1-a_1 \\ 1-a_n \end{pmatrix}} \mathbb{Z}\bar{L} \xrightarrow{\varepsilon} \mathbb{Z} \tag{26}$$

is exact; here $\bar{L}$ denotes the 1-relator group $\langle \alpha_1, \ldots, \alpha_n; \rho^l \rangle$.

LEMMA 7. *The canonical image $r \in L$ of $\rho \in F \setminus \{1\}$ has infinite order.*

Proof. Since free groups are residually nilpotent there exists $c \geqq 1$ such that $\rho \in \gamma_c F \setminus \gamma_{c+1} F$. The obvious epimorphism $L \twoheadrightarrow F/\gamma_{c+1}F$ sends r to a non-trivial element of the central subgroup $\gamma_c F/\gamma_{c+1}F$, which is known to be free abelian (cf. [11, p. 341, Cor. 5.12(iv)]). □

We are now ready to prove that (25) is an exact complex. We begin by verifying that the left-most homomorphism $\partial_3 : \mathbb{Z}L \to \mathbb{Z}L^n$ is injective. If $\lambda \in \mathbb{Z}L$ and $\lambda \cdot D_{\alpha_j}\rho = 0$ for all j, then

$$0 = \sum_j (\lambda \cdot D_{\alpha_j}\rho)(1-a_j) = \lambda \sum_j (D_{\alpha_j}\rho \cdot (1-a_j)) = \lambda \cdot (1-r).$$

As r has infinite order by Lemma 7, $(1-r)$ is not a zero-divisor and hence $\lambda = 0$.

Let $\partial_2 : \mathbb{Z}L^n \to \mathbb{Z}L^n$ be the differential of (25) given by the $(n \times n)$-matrix with entries $D_{\alpha_i}[\rho^l, \alpha_j]$. These entries can be described more explicitly; indeed:

$$\begin{aligned} D_{\alpha_i}[\rho^l, \alpha_j] &= D_{\alpha_i}(\rho^l \cdot \alpha_j \cdot \rho^{-l} \cdot \alpha_j^{-1}) = D_{\alpha_i}\rho^l + r^l \cdot D_{\alpha_i}\alpha_j - a_j D_{\alpha_i}\rho^l - D_{\alpha_i}\alpha_j \\ &= (1-a_j) \cdot D_{\alpha_i}\rho^l + (r^l - 1) \cdot \delta_{ij}. \end{aligned}$$

Here δ_{ij} is the Kronecker symbol. A row vector $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}L^n$ lies in the kernel of ∂_2 if, and only if,

$$0=\sum_j \lambda_j \cdot \{(1-a_j)\cdot D_{\alpha_i}\rho^l+(r^l-1)\delta_{ij}\}$$

or, equivalently, if

$$\lambda_i \cdot (1-r^l)=\Big(\sum_j \lambda_j(1-a_j)\Big)\cdot D_{\alpha_i}\rho^l \tag{27}$$

for every $i=1,2,\ldots,n$.

Assume $(\lambda_1,\ldots,\lambda_n)\in\ker\partial_2$ and set $\mu:=\sum_j \lambda_j(1-a_j)$. By applying the canonical ring epimorphism

$$^{-}:\mathbb{Z}\mathrm{L}\twoheadrightarrow \mathbb{Z}\bar{L}:=\mathbb{Z}\langle\alpha_1,\ldots,\alpha_n;\rho^l\rangle$$

to (27) one obtains the equations $0=\bar{\mu}\cdot\bar{D}_{\alpha_i}\rho^l$, where i ranges over $1,2,\ldots,n$; they show that $\bar{\mu}$ is in the kernel of the second differential $\bar{\partial}_2$ of the exact sequence (26). There exists therefore $\nu\in\mathbb{Z}L$ with $\bar{\mu}=\bar{\nu}(1-\bar{r})$. The kernel of the canonical projection $L\twoheadrightarrow\bar{L}$ is the cyclic, *central* subgroup generated by r^l, whence $\ker(\mathbb{Z}L\twoheadrightarrow\mathbb{Z}\bar{L})$ is the ideal generated by the central element $1-r^l$. So there will exist $\nu_1\in\mathbb{Z}L$ with $\mu=\nu_1(1-r)$. The equations (27) now imply that

$$\begin{aligned}\lambda_i\cdot(1-r^l)&=\mu\cdot D_{\alpha_i}\rho^l=\nu_1(1-r)(1+r+\cdots+r^{l-1})D_{\alpha_i}\rho\\&=\nu_1(1-r^l)D_{\alpha_i}\rho=\nu_1\cdot D_{\alpha_i}\rho\cdot(1-r^l)\end{aligned}\tag{28}$$

for $i=1,2,\ldots,n$. Since r^l has infinite order by Lemma 7, $1-r^l$ is not a zero-divisor of $\mathbb{Z}L$. Therefore (28) implies that $\lambda_i=\nu_1\cdot D_{\alpha_i}\rho$ for all i, i.e. that $(\lambda_1,\ldots,\lambda_n)$ belongs to $\operatorname{im}\partial_3$. Conversely, if $(\lambda_1,\ldots,\lambda_n)$ is in $\operatorname{im}\partial_3$, i.e., if $\lambda_i=\nu_1 D_{\alpha_i}\rho$ for some $\nu_1\in\mathbb{Z}L$, one sees by reversing part of (28) that

$$(\nu_1 D_{\alpha_i}\rho)(1-r^l)=\nu_1\cdot(1-r)\cdot D_{\alpha_i}\rho^l,$$

while it is quite generally true that

$$\nu_1\cdot(1-r)=\nu_1\sum_j(D_{\alpha_j}\rho(1-a_j))=\sum_j(\nu_1 D_{\alpha_j}\rho)(1-a_j).$$

Put together, these equations give

$$(\nu_1 D_{\alpha_i}\rho)(1-r^l)=\sum_j((\nu_1 D_{\alpha_j}\rho)(1-a_j))\cdot D_{\alpha_i}\rho^l$$

for every i; they show that the row vector $\partial_3(\nu_1)$ satisfies the equations (27) or, equivalently, that $\partial_2(\partial_3(\nu_1))=0$. The verification that (25) is exact at the left middle module $\mathbb{Z}L^n$ is now complete.

As the right half of (25) is exact on general grounds, the exactness of the entire sequence (25) is established. □

4. Poincaré-duality groups among the groups L

In this section we shall determine which of the groups

$$L=\langle\alpha_1,\ldots,\alpha_n;[\rho^l,\alpha_1],\ldots,[\rho^l,\alpha_n]\rangle$$

are 2-dimensional and which of them are Poincaré-duality-groups of dimension 3. The notation will be as in Section 3; in particular, $\rho\in F=F(\{\alpha_1,\ldots,\alpha_n\})$ is a non-trivial element which is not a proper power, $l\geqq 1$ and $n\geqq 2$. Let $\bar{L}$ and $\bar{\bar{L}}$ designate the 1-relator groups

$$\bar{L}=\langle\alpha_1,\ldots,\alpha_n;\rho^l\rangle \quad\text{and}\quad \bar{\bar{L}}=\langle\alpha_1,\ldots,\alpha_n;\rho\rangle.$$

We begin with the easy

LEMMA 8. (i) $H^1(L,\mathbb{Z}L)=0$.
(ii) $H^{j+1}(L,\mathbb{Z}L)$ *and* $H^j(\bar{L},\mathbb{Z}\bar{L})$ *are* $\mathbb{Z}$*-isomorphic for all* $j\geqq 1$.
(iii) $H^2(\bar{\bar{L}},\mathbb{Z}\bar{\bar{L}})$ *is a* $\mathbb{Z}L$*-homomorphic image of* $H^3(L,\mathbb{Z}L)$.

Proof. Spectral sequences, applied to the central extension $\langle r^l\rangle\lhd L\twoheadrightarrow\bar{L}$, and the fact that $\langle r^l\rangle$ is infinite cyclic by Lemma 7, imply that

$$H^{j+1}(L,\mathbb{Z}L)\cong H^j(\bar{L},H^1(\langle r^l\rangle,\mathbb{Z}L))\cong H^j(\bar{L},\mathbb{Z}\bar{L})$$

for all $j\geqq 0$. These isomorphisms, together with the fact that $\bar{L}$ is infinite and hence $H^0(\bar{L},\mathbb{Z}\bar{L})=0$, establish (i) and (ii). Claim (iii) is a consequence of the facts that the second cohomology group $H^2(\bar{\bar{L}},\mathbb{Z}\bar{\bar{L}})$ of the torsion-free 1-relator group $\bar{\bar{L}}$ is by Lyndon's Identity Theorem isomorphic with $\mathbb{Z}\bar{\bar{L}}/\sum_j(\bar{\bar{D}}_{\alpha_j}\rho\cdot\mathbb{Z}\bar{\bar{L}})$, whereas $H^3(L,\mathbb{Z}L)$, when computed by the resolution (25), is given by $\mathbb{Z}L/\sum_j(D_{\alpha_j}\rho\cdot\mathbb{Z}L)$. □

2-dimensional groups. Assume the cohomological dimension of L is less than 3. Then $H^3(L,\mathbb{Z}L)=0$ and so $H^2(\bar{\bar{L}},\mathbb{Z}\bar{\bar{L}})$ will be trivial by Lemma 8(iii). But $\bar{\bar{L}}$ is a torsion-free 1-relator group, hence of cohomological dimension at most 2; it is also of type (FP). Therefore the vanishing of $H^2(\bar{\bar{L}}, \mathbb{Z}\bar{\bar{L}})$ implies that $\bar{\bar{L}}$ is at most 1-dimensional (cf. [1, p. 137, Lemma 9.1]), and so $\bar{\bar{L}}$ is free by Stallings' theorem. By a result of J. H. C. Whitehead's (see, e.g., [9, p. 106, Prop. 5.10]) a 1-relator group can only be free if the defining relator is either trivial or primitive. Since $\rho \neq 1$, we conclude that ρ must be a member of some basis of $F = F(\{\alpha_1, \ldots, \alpha_n\})$.

After a change of notation, we will have $\rho = \alpha_1$ and

$$L = \langle \alpha_1, \ldots, \alpha_n; [\alpha_1^l, \alpha_2], \ldots, [\alpha_1^l, \alpha_n] \rangle. \tag{29}$$

This group can ve viewed as an HNN-extension with base group $\langle a_1 \rangle$, associated subgroups both equal to $\langle a_1^l \rangle$ and stable letters $\alpha_2, \ldots, \alpha_n$; it can also be obtained from the direct product $\mathbb{Z} \times F(\{\alpha_2, \ldots, \alpha_n\})$ by adjoining an l-th root. Both descriptions allow to infer that every group of the form (29) has cohomological dimension precisely 2 (recall that $n \geqq 2$). Incidentally, the second description of L reveals also that if $l > 1$ or $n > 2$, the group L has a subgroup of infinite index which is free abelian of rank 2, hence, in particular, 2-dimensional, and therefore L cannnot be a Poincaré-duality group of dimension 2 (by [18]). So the only Poincaré-duality group of the form (29) is $L = \langle \alpha_1, \alpha_2; [\alpha_1, \alpha_2] \rangle$.

3-dimensional Poincaré-duality groups. A second application of Lemma 8 will be made in the proof of

THEOREM 9. *The following statements are equivalent:*

(i) *There exists either a basis* $\xi_1, \eta_1, \ldots, \xi_g, \eta_g$ *of* $F(\{\alpha_1, \ldots, \alpha_n\})$ *with* $\rho = [\xi_1, \eta_1] \cdot \cdots [\xi_g, \eta_g]$, *or a basis* $\xi_1, \xi_2, \ldots, \xi_g$ *with* $\rho = \xi_1^2 \cdot \xi_2^2 \cdot \cdots \cdot \xi_g^2$.

(ii) $H^3(L, \mathbb{Z}L)$ *is infinite cyclic.*

(II) L *is a 3-dimensional Poincaré-duality group.*

Proof. If (i) is true L has a presentation

$$\left\langle \xi_1, \eta_1, \ldots, \xi_g, \eta_g; \left(\prod_i [\xi_i, \eta_i]\right)^l \text{ commutes with all } \xi_j, \eta_j \right\rangle \tag{30}$$

or a presentation

$$\langle \xi_1, \ldots, \xi_g; (\xi_1^2 \cdot \xi_2^2 \cdot \cdots \cdot \xi_g^2)^l \text{ commutes with all } \xi_j \rangle. \tag{31}$$

If $H^3(L, \mathbb{Z}L)$ is computed by means of the resolution (25) one finds that $H^3(L, \mathbb{Z}L) \cong \mathbb{Z}L / \sum_\beta (D_\beta \rho \cdot \mathbb{Z}L)$, where β ranges over the basis displayed in (30) or (31).

Assume first L has a presentation of the form (30); we contend that $\sum_\beta (D_\beta\rho \cdot \mathbb{Z}L)$ is the augmentation ideal IL of $\mathbb{Z}L$. In order to verify this we shall prove more, namely that

$$I:=\sum_j (\partial/\partial\xi_j(\rho)) \cdot \mathbb{Z}F + \sum_j (\partial/\partial\eta_j(\rho)) \cdot \mathbb{Z}F$$

is the augmentation ideal IF of $\mathbb{Z}F$. The key to this is the fact that the partial derivatives of $\rho' := [\xi_1, \eta_1] \cdots [\xi_{g-1}, \eta_{g-1}]$ with respect to $\xi_1, \eta_1, \ldots, \xi_{g-1}$ and η_{g-1} agree with those of ρ. Hence we can assume inductively that

$$J:=\sum_{1=j}^{g-1} \frac{\partial\rho}{\partial\xi_j} \cdot \mathbb{Z}F + \sum_{1=j}^{g-1} \frac{\partial\rho}{\partial\eta_j} \cdot \mathbb{Z}F = \sum_{1=j}^{g-1} (1-\xi_j) \cdot \mathbb{Z}F + \sum_{1=j}^{g-1} (1-\eta_j) \cdot \mathbb{Z}F.$$

In particular, $(1-\rho')$ belongs to J. Let ξ, resp. η be short for ξ_g, resp. η_g. From

$$\partial/\partial\xi(\rho) = \rho' \cdot (1-\xi\eta\xi^{-1}) \quad \text{and} \quad \partial/\partial\eta(\rho) = \rho' \cdot (\xi - [\xi, \eta])$$

one deduces that $I \subseteq IF$, and that $(1-\xi\eta\xi^{-1})$ and $(\xi-[\xi, \eta])$ belong to I. Hence so do

$$(1-\xi\eta\xi^{-1}) \cdot (1-\xi) \qquad -(\xi-[\xi, \eta]) \cdot \eta = 1-\xi$$
$$(1-\xi\eta\xi^{-1}) \cdot (\xi\eta - \eta + 1) + (\xi - [\xi, \eta]) \cdot (\eta^2 - \eta) = 1 - \eta,$$

and thus $I = IF$, as contended.

Assume next L is of the form (31). We assert that $\sum_j (\partial/\partial\xi_j(\rho)) \cdot \mathbb{Z}L = \sum_j (1+x_j) \cdot \mathbb{Z}L$; it will suffice to establish the corresponding statement for $\mathbb{Z}F$. Set $\rho' := \xi_1^2 \cdot \cdots \cdot \xi_{g-1}^2$. Then $\partial/\partial\xi_j(\rho') = \partial/\partial\xi_j(\rho)$ for $j = 1, \ldots, g-1$ and thus

$$J:=\sum_{1 \leq j < g} (\partial/\partial\xi_j(\rho)) \cdot \mathbb{Z}F = \sum_{1 \leq j < g} (1+\xi_j) \cdot \mathbb{Z}F.$$

Since $1-\rho' = \sum_{1 \leq j < g} (\partial/\partial\xi_j(\rho))\ (1-\xi_j)$ is in J and as $(\partial/\partial\xi_g(\rho)) = \rho' \cdot (1+\xi_g)$, the assertion is established. Finally the quotients $\mathbb{Z}F/\sum_j (1+\xi_j) \cdot \mathbb{Z}F$ and $\mathbb{Z}L/\sum_j (1+x_j) \cdot \mathbb{Z}L$ are infinite cyclic, as can easily be verified and so (ii) holds also for the groups of the form (31).

Conversely, assume (ii) is true. Then $I := \sum_j D_{\alpha_j}\rho \cdot \mathbb{Z}L$ contains the element $1-r$ and I is a two-sided ideal. Therefore

$$H^2(\bar{\bar{L}}, \mathbb{Z}\bar{\bar{L}}) \cong \mathbb{Z}\bar{\bar{L}} \Big/ \sum_j (\bar{\bar{D}}_{\alpha_j}\rho) \cdot \bar{\bar{\mathbb{Z}}}L \cong \mathbb{Z}L/(I + \mathbb{Z}L(1-r)\mathbb{Z}L) \cong \mathbb{Z}L/I$$

is infinite cyclic. As in [1, p. 155, Remark] one sees next that $\bar{\bar{L}}$ is actually a Poincaré-duality group of dimension 2, and hence a surface group by a result of B. Eckmann and H. Müller [8, Thm. 1]. Work of H. Zieschang and N. Peczynski ([21], [12], cf. [22, p. 58, **2.11.9**]) now guarantees the existence of a basis of $F(\{\alpha_1, \ldots, \alpha_n\})$ with either $\rho = \prod_j [\xi_j, \eta_j]$ or $\rho = \prod_j \xi_j^2$.

Finally, we establish that (ii) implies (iii), the converse being evident. It suffices to show that (ii) implies that $H^1(L, \mathbb{Z}L) = 0 = H^2(L, \mathbb{Z}L)$. From Lemma 8 one sees that $H^1(L, \mathbb{Z}L) = 0$, and that

$$H^2(\bar{L}, \mathbb{Z}\bar{L}) \cong H^3(L, \mathbb{Z}L) \cong \mathbb{Z} \quad \text{and that} \quad H^2(L, \mathbb{Z}L) \cong H^1(\bar{L}, \mathbb{Z}\bar{L}).$$

Much as in [1, p. 155, Remark] one deduces from Stallings' theory of groups with infinitely many ends and from $H^2(\bar{L}, \mathbb{Z}\bar{L}) \cong \mathbb{Z}$ that $H^1(\bar{L}, \mathbb{Z}\bar{L}) = 0$, whence $H^2(L, \mathbb{Z}L) = 0$. □

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REFERENCES

[1] R. Bieri, *Homological Dimension of Discrete Groups*, Queen Mary College Mathematics Notes; Queen Mary College, London 1976.

[2] K. S. Brown, *Cohomology of Groups*, Graduate Texts in Math. 87; Springer-Verlag, Berlin Heidelberg New York 1982.

[3] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton 1956.

[4] H. S. M. Coxeter, *The binary polyhedral groups, and other generalizations of the quaternion groups*, Duke Math. J. 7 (1940), 367–379.

[5] H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups*. Erg. Math. Bd. 14 (3rd ed.); Springer-Verlag, Berlin Heidelberg New York 1972.

[6] P. M. Curran, *Cohomology of finitely presented groups*, Pacific J. Math. *42* (1972), 615–620.

[7] M. J. Dunwoody, Answer to a conjecture of J. M. Cohen; J. Pure Applied Algebra *16* (1980), 249.

[8] B. Eckmann and H. Müller, *Poincaré-duality groups of dimension two*, Comment. Math. Helv. *55* (1980), 510–520.

[9] R. C. Lyndon and P. Schupp, *Combinatorial Group Theory*, Erg. Math. 89, Springer-Verlag, Berlin Heidelberg New York 1977.

[10] W. Magnus, *Noneuclidean Tesselations and Their Groups*; Academic Press, New York and London 1974.

[11] W. Magnus, A. Karrass and D. Solitar, *Combinatorial Group Theory*, Interscience Publishers, John Wiley & Sons, New York London Sidney 1966.

[12] N. Peczynski, *Eine Kennzeichnung der Relation der Fundamentalgruppe einer nicht-orientierbaren geschlossenen Fläche*, Diplomarbeit Ruhr-Universität Bochum 1972.

[13] P. Orlik, E. Vogt und H. Zieschang, *Zur Topologie gefaserter dreidimensionaler Mannigfaltigkeiten*, Topology *6* (1967), 49–64.

[14] H. Seifert, *Lösung der Aufgabe 84*, Jahresber. Deutschen Math. Vereinigung *41* (1932), 7–8.

[15] H. SEIFERT, *Topologie dreidimensionaler gefaserter Räume*, Acta Math. *60* (1933), 147–238.
[16] J-P. SERRE, *Arbres, amalgames, SL_2; astérisque* 46; Soc. Math. de France, Paris 1977.
[17] J. STALLINGS, *Group Theory and three-dimensional Manifolds*, Yale Math. Monographs 4; Yale University Press, New Haven and London 1971.
[18] R. STREBEL, *A remark on subgroups of infinite index in Poincaré-duality groups*, Comment. Math. Helv. *52* (1977), 317–324.
[19] W. THRELFALL, *Lösung der Aufgabe 84*, Jahresber. Deutschen Math. Vereinigung *41* (1932), 6–7.
[20] W. THRELFALL und H. SEIFERT, *Topologische Untersuchungen der Diskontinuitätsbereiche des dreidimensionalen sphärischen Raumes I & II*, Math. Annalen *104* (1931), 1–70 und *107* (1933), 543–586.
[21] H. ZIESCHANG, *Über die Nielsensche Kürzungsmethode in freien Produkten mit Amalgam*, Invent. math. *10* (1970), 4–37.
[22] H. ZIESCHANG, E. VOGT and H-D. COLDEWEY, *Surfaces and Planar Discontinuous Groups*, Lecture Notes Math. 835, Springer-Verlag, Berlin Heidelberg New York 1980.

Mathematisches Institut
Im Neuenheimer Feld 288
D-6900 Heidelberg
Fed. Rep. Germany

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