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Group actions on fibered three-manifolds

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1. Introduction

In this paper we present several results about finite group actions on three-dimensional manifolds. The results are primarily directed toward a geometric understanding of periodic knots in the 3-sphere, that is, knots left invariant by periodic homeomorphism of S^3 which fix a simple closed curve in the complement of the knot.

One noteworthy application of the present techniques is that nontrivial periodic knots have "Property R," that is, surgery on such a knot cannot produce $S^1 \times S^2$.

Let G be a finite group acting effectively and smoothly on an orientable 3-manifold M, preserving orientation. The first result is that if the orbit manifold M^* contains an incompressible surface F^*, then (after suitably adjusting the embedding) the preimage of F^* in M is an incompressible surface. The proof of this makes use of the Equivariant Loop Theorem of Meeks and Yau [12, 13] and is given in Section 2.

An immediate corollary of this result is that a periodic knot has an invariant incompressible Seifert surface. If there is a bound on the possible genera of the incompressible Seifert surfaces of a given knot K (as is the case for fibered knots), then the Riemann-Hurwitz formula places nontrivial bounds on the possible periods of K. See Section 2. There is an extensive literature devoted to the problem of determining the possible periods of a given knot [1, 5, 6, 7, 11, 15, 16, 18, 22]. Most previous work on this problem has been heavily algebraic in nature, in contrast to the present more geometric approach.

In Section 3 the preceding work is applied to prove that periodic knots have Property R.

Now suppose that F is a compact, orientable surface and that G acts on $F \times [0, 1]$ preserving orientation and leaving both $F \times \{0\}$ and $F \times \{1\}$ invariant. We show that the action is equivalent to the level-preserving action which is the product of the action of $F \times \{0\}$ with the trivial action on the interval $[0, 1]$, except

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possibly when the action on $F\times\{0\}$ has fewer than four exceptional orbits and orbit space the 2-sphere. The proof is given in Section 4 and makes essential use of the Equivariant Dehn Lemma of Meeks and Yau [12, 13] and of the solution of the classical Smith Conjecture [19], which corresponds to the case here that F is a 2-disk. After the first version of this paper was written we learned that W. Meeks and G. P. Scott have proved the above result in the more difficult excluded case as well.

An easy application of the preceding result, given in Section 5, is that quite generally a group action on a surface bundle over the circle *with unique fibering* is equivalent to a fiber-preserving action. In particular, any periodic fibered knot in S^3 admits a fibering preserved by the corresponding group action.

The final section contains further applications of the preceding results to group actions preserving fibered knots. In particular one has a very short list of fibered knots of a given genus g which admit periods m which are maximal $(m=2g+1)$ or nearly maximal $(m=g+1)$. Several instructive examples are given as well. Finally we show that the present techniques give a partial answer to a question of D. Goldsmith in the Kirby problem set [9; 1.28].

The techniques used in this paper are similar to those used by Gordon and Litherland [4]. That work is primarily concerned with invariant incompressible surfaces in the complement of the exceptional set, while our main interest is in incompressible surfaces which intersect the exceptional set.

Quite similar and more complete results in the case $G=\mathbb{Z}_2$ are due to Tollefson [21] and Kim and Tollefson [8].

2. Lifting incompressible surfaces

Throughout this section let G be a finite group acting effectively and smoothly on an orientable, differentiable 3-manifold M preserving orientation. The *exceptional set* E for the action is the set of all points in M having nontrivial isotropy group. If nonempty, E is a one-dimensional CW complex with all vertices of valence one lying in ∂M. The exceptional set is a 1-manifold precisely if all isotropy groups are cyclic. Let M^* denote the orbit manifold and let $p: M\to M^*$ be the orbit map. The set $B=p(E)$ is called the *branch set*.

Two embedded surfaces F and F' in a 3-manifold N are said to be *disk equivalent* if there is a sequence $F_1=F, F_2, \ldots, F_n=F'$ of surfaces in N such that for each k, $1<k\leq n$, there exist disks $D_k\subset \operatorname{int} F_k$ and $D_k'\subset \operatorname{int} F_{k-1}$ such that $F_k-D_k=F_{k-1}-D_k'$. If N is irreducible, then standard techniques show that disk equivalent surfaces are isotopic.

Let F be a surface in a 3-manifold N such that $F\cap\partial N=\partial F$. A *compressing* *disk* for F is a 2-disk $D \subset \text{int } N$ such that $D \cap F = \partial D$ and ∂D is a homotopically nontrivial simple loop on F. If F has a compressing disk or is a nullhomotopic 2-sphere or is a 2-disk which can be deformed rel ∂F into ∂N, then F will be called *compressible*; otherwise F is said to be *incompressible*.

A surface F in the orbit manifold M^* is said to be *transverse* to the branch set B provided that F only meets B in the subset B_0 of points where B is a 1-manifold and F is transverse to B_0 in the usual sense.

As indicated previously we shall need the following basic result of Meeks-Yau [12, 13]. We quote from [4].

EQUIVARIANT LOOP THEOREM. *Let N be a 3-manifold, and G a finite group acting on N. Suppose F is a compressible component of ∂N. Then there exists a compressing disk D for F such that for all $g \in G$, either $g(D) = D$ or $g(D) \cap D = \varnothing$.*

Although Meeks and Yau only assert the result when N is compact and orientable, the theorem is true in the generality given. Further one may assume G acts freely on $G(\partial D)$.

THEOREM 2.1. *Any two-sided incompressible surface in the orbit manifold M^* is disk equivalent to an incompressible surface F^* which meets the branch set B transversely and such that $F = p^{-1}(F^*)$ is a two-sided incompressible surface in M.*

Proof. It may be assumed that F^* is an incompressible surface in M^* which is transverse to B and meets the non-singular part of B in the minimum number of points possible for incompressible surfaces in its disk equivalence class. Transversality implies that $F = p^{-1}(F^*)$ is a surface in M.

Suppose that F is compressible in M. If a component F_1 of F is a nullhomotopic 2-sphere, then $F_1 = \partial \Delta$ where Δ is a homotopy 3-cell in M. Let H be the subgroup of G leaving Δ invariant. Then $F = \partial H(\Delta)$ and $F^* = \partial p(\Delta)$, a 2-sphere. Moreover $p(\Delta)$ is a simply connected 3-manifold with boundary F^*, and this implies that F^* is compressible.

Similarly, if some component F_1 of F is a disk homotopic into ∂M, it follows that F^* is a disk homotopic into ∂M^*, contradicting incompressibility of F^*.

Finally suppose F contains no spheres or disks. Let N be an invariant tubular neighborhood of F in M. Then the manifold $W = (M - \partial M) - \text{int } N$ has as boundary two copies of int F, and has compressible boundary. By the Equivariant Loop Theorem there is a compressing disk D_1 for ∂W in W such that for any $g \in G$ either $g(D_1) = D_1$ or $g(D_1) \cap D_1 = \varnothing$. By using an equivariant product structure in N one may expand D_1 to a compressing disk D for F such that for $g \in G$ either $g(D) = D$ or $g(D) \cap D = \varnothing$.

Let $D^* = p(D)$ in M^*. Then D^* is a disk transverse to the branch set B, $D^* \cap B$ consists of at most one point and $D^* \cap F^* = \partial D^*$. Let F_0^* be the component of F^* meeting D^*. First suppose F_0^* is neither a 2-sphere nor a 2-disk. Since F_0^* is incompressible there is a 2-disk $D_0^* \subset F_0^*$ with $\partial D_0^* = \partial D^*$. The minimality condition on F^* implies that D_0^* meets B in at most one point. Therefore each component of $p^{-1}(D_0^*)$ is a disk in F. Since the boundary of some component of $p^{-1}(D_0^*)$ is ∂D, it follows that D is not a compressing disk for F after all. If F_0^* is a 2-sphere, then ∂D^* divides F_0^* into two 2-disks F_1^* and F_2^*. Since F contains no 2-sphere and ∂D is nontrivial on F, each F_i^* must contain at least two branch points, while D^* contains at most one. One of the two 2-spheres $D^* \cup F_1^*$ or $D^* \cup F_2^*$ must be incompressible and meet B in fewer points than F_0^* did. This contradicts the choice of F. If F_0^* is a 2-disk, then ∂D^* divides F_0^* into a 2-disk F_1^* and an annulus F_2^*. Since no component of F is a 2-disk, and ∂D is nontrivial on F, $p^{-1}(F_1^*)$ does not consist of disks, and so F_1^* meets B at least twice. But D^* meets B at most once. Thus a disk move would reduce $F^* \cap B$, a contradiction. This completes the proof. □

Remark. It follows that if the orbit manifold M^* is sufficiently large, then so is M.

COROLLARY 2.2. *Let K be a knot in an integral homology 3-sphere Σ invariant under a semifree orientation-preserving action of the cyclic group C_m of order m, with fixed set A (the "axis") disjoint from K. Then K bounds an incompressible Seifert surface invariant under C_m.*

Proof. Since C_m has fixed points the quotient map $p : \Sigma \to \Sigma^*$ induces a surjection $H_1(\Sigma; \mathbb{Z}) \to H_1(\Sigma^*; \mathbb{Z})$. It follows that Σ^* is also an integral homology 3-sphere containing the knot $K^* = p(K)$. Let F^* be an incompressible Seifert surface for K^* meeting the branch set transversely a minimum number of times. Then by Theorem 2.1 $p^{-1}(F^*)$ is the required incompressible Seifert surface for K. □

Note that in the situation of Corollary 2.2 the linking number $lk(K, A) = lk(K^*, A^*)$ is relatively prime to m. Otherwise $p^{-1}(K^*) = \partial p^{-1}(F^*)$ would be disconnected.

The following well known lemma is the basis for further restrictions on the periods of knots.

LEMMA 2.3. *Let F be a compact, connected, orientable surface of genus $g > 0$ with one boundary component. If F admits a semifree, orientation-preserving action of the cyclic group C_m with nonempty fixed set, then $m \leq 2g + 1$.*

Proof. The Riemann-Hurwitz formula for the regular branched cyclic covering $p: F \to F^*$ says that

$$1-2g = m(1-2g^*) - k(m-1)$$

where g^* is the genus of the orbit surface F^* and k is the number of branch points. If $k=1$, then $mg^* = g$; so $m = g/g^* \leq g$. If $k>1$, then

$$\begin{aligned} m &= (2g-1+k)/(2g^*-1+k) \\ &\leq (2g-1+k)/(k-1) \\ &\leq 2g+1. \quad \square \end{aligned}$$

Remark. The maximum value $m = 2g+1$ occurs only when $k=2$. Otherwise $m \leq g+1$. If $m = g+1$, then $k \leq 3$. For all other values of $k \geq 4$, $m \leq g$.

COROLLARY 2.4. (cf. [15]). *If K is a fibered knot in S^3 of genus g and period m, then $m \leq 2g+1$.*

Proof. It is well known that a fibered knot has a unique incompressible Seifert surface, up to isotopy. See Lemma 5.1. By Corollary 2.2 K has an invariant Seifert surface of genus g. By Lemma 2.3 $m \leq 2g+1$. $\square$

See Section 6 for more precise results in this direction, where it will be shown that *very* few fibered knots of genus g admit periods greater than g.

3. Periodic knots have Property *R*

Let K be a knot in S^3 which is invariant under a semifree, orientation-preserving action of a cyclic group C_m, with nonempty fixed point set A disjoint from K. The manifold obtained by 0-surgery on K is defined to be $M(K, 0) = (S^3 - \operatorname{int} N(K)) \cup B^2 \times S^1$ where $N(K)$ is a tubular neighborhood of K and the image of the meridian $\partial B^2 \times pt$ is a longitude of K (nullhomologous in $S^3 - K$).

THEOREM 3.1. *If K is a periodic knot in S^3 and $\pi_1(M(K, 0)) \approx \mathbb{Z}$, then K is unknotted (and $M(K, 0) \cong S^1 \times S^2$).*

Proof. One may assume that the tubular neighborhood $N(K)$ is invariant under a given C_m action. As in the proof of Theorem 2.1 K has an invariant

Seifert surface. Therefore the group action leaves a preferred longitude invariant, and it is possible to extend the action on $S^3 - \operatorname{int} N(K)$ to $M(K, 0)$. The fixed point set of the extended action is $A \cup (0 \times S^1)$. The quotient of $M(K, 0)$ by the induced C_m action may be described as the result of 0-surgery on the quotient of K under the original action. Put succinctly, $M(K, 0)^* = M(K^*, 0)$. In particular then $H_1(M(K, 0)^*; \mathbb{Z}) \approx \mathbb{Z}$, generated by the image of the core $0 \times S^1$. Therefore $M(K, 0)^*$ contains a nonseparating incompressible surface S^* (in fact a 2-sphere). It may be assumed that among all incompressible surfaces disk equivalent to S^*, S^* meets the branch set transversely in $M(K, 0)^*$ a minimum number of points.

By Theorem 2.1, and its proof, the inverse image S of S^* in $M(K, 0)$ is an invariant, incompressible surface in $M(K, 0)$. Because $\pi_1(M(K, 0)) \approx \mathbb{Z}$, S must be a 2-sphere or a collection of m 2-spheres.

If S is not connected, then $M(K, 0) - S$ consists of m components cyclically permuted by the action. In this case the action would have empty fixed point set. This contradiction implies that S is a single invariant 2-sphere, which must therefore meet the fixed point set in exactly two points.

Since the core $0 \times S^1$ of $B^2 \times S^1$ represents a generator of $H_1(M(K, 0))$, the intersection number of $0 \times S^1$ with S must be ± 1. But S meets $0 \times S^1$ at most twice. Therefore S meets $0 \times S^1$ once and the original axis A once.

Finally, the manifold obtained by removing from $M(K, 0)$ a small tubular neighborhood of $0 \times S^1$ is just $S^3 - \operatorname{int} N(K)$. But in this space S becomes a disk with boundary a longitude of K. Hence K is unknotted. □

4. Actions on $F^2 \times [0, 1]$

As the second step toward standardizing actions on fibered knots and other fibered 3-manifolds we show how to straighten actions on the product of a surface with the unit interval.

Let F be a compact surface and G be a finite group of diffeomorphisms of $F \times [0, 1]$, leaving $F \times \{0\}$ and $F \times \{1\}$ invariant. The restriction of G to $F \times \{0\}$ induces an action of G on $F: g(x) = \pi_F g(x, 0)$ where $\pi_F : F \times [0, 1] \to F$ is the projection. The *associated straight action* of G on $F \times [0, 1]$ is given by $g(x, t) = (g(x), t)$.

THEOREM 4.1. *Let F be a compact, orientable surface and let G be a finite group acting smoothly and effectively on $F \times [0, 1]$, preserving orientation and leaving $F \times \{0\}$ and $F \times \{1\}$ invariant. Assume that the orbit space F/G is not a 2-sphere with less than 4 branch points. Then the given action is equivalent to its associated straight action, by a diffeomorphism of $F \times [0, 1]$ which is the identity on $F \times \{0\}$.*

Proof. It is easy to see that one may assume F is connected: Separate orbits of components may clearly be handled separately; if F_1 is component of F such that $G(F_1)=F$, let H be the subgroup of G leaving $F_1\times[0, 1]$ invariant and suppose the action of H can be straightened, so there is a diffeomorphism $\phi: F_1\times[0, 1]\to F_1\times[0, 1]$ such that $\phi \mid F_1\times\{0\}$ is the identity and $\phi(h(x), t)=h\phi(x, t)$ for $h\in H$, $x\in F_1$, $t\in[0, 1]$; define $\theta: F\times[0, 1]\to F\times[0, 1]$ by $\theta(g(x), t)=g\phi(x, t)$ for $g\in G$, $x\in F_1$, $t\in[0, 1]$; then θ is the required equivariant diffeomorphism satisfying $\theta(g(x), t)=g\theta(x, t)$ for all $g\in G$, $x\in F$, $t\in[0, 1]$.

So suppose F is connected. We proceed by induction on the ordered pairs $(r, b)=$ (genus of F, number of components of ∂F), ordered lexicographically.

If $r=0$, then $b\geq 1$ since $(0, 0)$ is an excluded case. If $b=1$, then F is a disk. In this case G must be cyclic. The solution of the Smith Conjecture [19] then shows that the action on $F\times[0, 1]$ can be straightened. Note that if one assumes the action is already straight on $\partial F\times[0, 1]$, then the action can be straightened rel $\partial F\times[0, 1]$.

Inductively, first consider the cases where F has nonempty boundary $(b\geq 1)$. Covering space theory shows that we may assume the action is already straight on $\partial F\times[0, 1]$. We show inductively that the action on $F\times[0, 1]$ can be straightened by a diffeomorphism which is the identity on $F\times\{0\}\cup\partial F\times[0, 1]$. Choose an invariant family of disjoint, properly embedded arcs $A_1, \ldots, A_m\subset F$, where m is the order of G, such that if F is cut open along $A_1, \ldots, A_m$, then the complexity (r, b) is reduced. Such a family is easily constructed as the preimage of an arc A^* in the orbit surface F/G which, if homotopic into $\partial F/G$, does not cut off a disk containing less than two branch points.

Let $B_i=A_i\times\{1\}$ in $F\times\{1\}, 1\leq i\leq m$. This family of arcs is probably not G-invariant. We may isotope $\{B_i\}$, rel end points, to an invariant collection as follows: Impose a G-invariant hyperbolic metric on $F\times\{1\}$ with totally geodesic boundary curves. (If S is an annulus, one must use a euclidean metric instead.) Each B_i is then homotopic, rel end points, to a unique geodesic arc C_i with the same end points as B_i. By the uniqueness of the choice of these geodesics the collection $\{C_i\}$ is G-invariant.

Moreover $\{C_i\}$ consists of embedded, pairwise disjoint arcs. The essential facts here are that the minimum number of self-intersections in an arc representing a given relative homotopy class is realized by its geodesic representative and that the minimum number of intersections between two arcs representing two given relative homotopy classes is also realized by their geodesic representatives. A proof of these assertions follows the lines of the proof of the analogue of the second statement in the context of closed curves as given in [3; Exposé 3, Proposition 10]. Since the B_i are disjoint and embedded, it follows that the same is true of the C_i.

The closed curves $\alpha_i = A_i \times \{0\} \cup \partial A_i \times [0, 1] \cup C_i$ form a G-invariant family of pairwise disjoint simple loops in $\partial(F \times [0, 1])$. Each α_i is nullhomotopic in $F \times [0, 1]$, since $\alpha_i \simeq A_i \times \{0\} \cup \partial A_i \times [0, 1] \cup A_i \times \{1\}$. By the Equivariant Dehn Lemma [13], there is a G-invariant family of disjoint, embedded disks $D_i \subset F \times [0, 1]$ with $\partial D_i = \alpha_i$. Since $F \times [0, 1]$ is irreducible, $\bigcup D_i$ is isotopic to $\bigcup A_i \times [0, 1]$. Therefore, cutting $F \times [0, 1]$ open along $\bigcup D_i$ results in a new G-manifold of the form $F' \times [0, 1]$ where the product structure imposed on the copies of D_i extends that on $\partial F \times [0, 1]$. One may apply covering space theory to straighten the action on the copies of D_i prior to cutting open along $\bigcup D_i$. Now the inductive hypothesis says that the action on $F' \times [0, 1]$ is equivalent rel $\partial F' \times [0, 1]$ to the associated straight action. Gluing $F' \times [0, 1]$ back together along the cuts provides an equivalence of the given action on $F \times [0, 1]$ with its associated straight action.

Finally suppose $\partial F = \emptyset$ and either the orbit space F^* has positive genus or is a sphere with at least four branch points. Choose an invariant family of disjoint embedded simple loops $A_1, \ldots, A_m \subset F$, $m \leq |G|$, which are homotopically nontrivial in F. Such a family is constructed as the preimage of a suitable closed loop in F^* which does not bound a disk containing less than two branch points.

Let $B_i = A_i \times \{1\}$, $1 \leq i \leq m$. This family of loops is not in general G-invariant. But by replacing them with a corresponding set of geodesics for a G-invariant hyperbolic metric on $F \times \{1\}$ we obtain an invariant family of simple, closed loops $\{C_i\}$ such that $C_i \simeq B_i$ for each i.

Now the Equivariant Dehn Lemma (for planar domains) of Meeks-Yau [13] implies that there is a G-invariant family of disjoint embedded annuli $V_i \subset F \times [0, 1]$ with $\partial V_i = A_i \times \{0\} \cup C_i$. Since $C_i \simeq A_i \times \{1\}$ and $F \times [0, 1]$ is irreducible, V_i is isotopic rel $A_i \times 0$ to $A_i \times [0, 1]$. Thus the given action is equivalent to one which preserves the family $\{A_i \times [0, 1]\}$ and is straight there. Now cut open along $\bigcup A_i \times [0, 1]$, apply the inductive hypothesis, and glue back together to complete the argument. □

Remark. We have been informed that W. Meeks and G. P. Scott have proved Theorem 4.1 without the hypothesis that F/G is not a 2-sphere with fewer than four branch points. The proofs involve singular incompressible surfaces.

5. Actions on bundles over the circle

Let M denote a compact, connected, orientable 3-manifold which fibers over the circle S^1, in such a way that ∂M, if nonempty, is a torus which inherits an induced fibering by restriction, and satisfies $H_1(M; \mathbb{Q}) \approx \mathbb{Q}$. For example M might be the complement of an open tubular neighborhood of a fibered knot in a homology sphere. The condition on homology is used to insure that M admits an essentially unique fibering. The following lemma is well known.

LEMMA 5.1. *Any two connected, properly embedded, non-separating, two-sided, incompressible surfaces* F_1, $F_2 \subset M$ *are isotopic.*

Proof sketch. One may suppose that F_1 is the fiber of a fibering $M \to S^1$. Let $\bar{M} \cong F_1 \times \mathbb{R}$ be the corresponding infinite cyclic covering. Because F_2 in non-separating, connected, and two-sided, intersection with F_2 defines a surjective homomorphism $\pi_1(M) \to \mathbb{Z}$, which must coincide, up to sign, with the corresponding homomorphism associated with the fibration since $H_1(M; \mathbb{Z})/\text{torsion} \approx \mathbb{Z}$. Thus F_2 lifts homeomorphically to an incompressible surface $\bar{F}_2 \subset \bar{M}$. If $\pi_1(\bar{F}_2) \to \pi_1(\bar{M})$ were not also surjective, then an argument with Van Kampen's theorem would imply that $\pi_1(\bar{M})$ is not finitely generated. (Compare [20; p. 97].)

THEOREM 5.2. *Let G be a finite group acting by orientation-preserving diffeomorphisms on the fibered 3-manifold M as above, with exceptional set E disjoint from ∂M. Assume* (i) *that the orbit manifold M^* is not $S^2 \times S^1$ where $S^2 \times$ point meets the branch set in less than four points, and* (ii) *that* $H_1(M; \mathbb{Q})^G \approx \mathbb{Q}$. *Then the given fibering of M is isotopic to a fibering in which G maps fibers into fibers.*

Proof. The quotient M^* is a manifold with ∂M^*, if nonempty, a torus. If $\partial M \neq \varnothing$, this immediately implies $H_1(M^*; \mathbb{Q}) \neq 0$. In general $H_1(M^*; \mathbb{Q}) \approx H_1(M; \mathbb{Q})^G$, nonzero by assumption. Therefore M^* contains a nonseparating, connected, two-sided incompressible surface F^*. We may assume that F^* meets the branch set B transversely in a minimal number of points. Then by Theorem 2.1 the preimage F of F^* in M is an incompressible surface. Let $F_1, \ldots, F_r$ be the components of F. By Lemma 5.1 each component F_i is isotopic to a fiber of M. Cutting M open along F one obtains a group action on $F \times I$. By Theorem 4.1 this action is equivalent to a straight action on $F \times I$, each component of $F \times \{t\}$ being a fiber. Reidentifying $F \times \{0\}$ with $F \times \{1\}$ one obtains the required equivariant fibering. □

Remarks. If the action has fixed points then F is connected, and each fiber is G-invariant. The recent result of Meeks-Scott mentioned earlier shows that the hypothesis (i) above can be dropped.

COROLLARY 5.3. *The quotient M^* fibers over S^1.* □

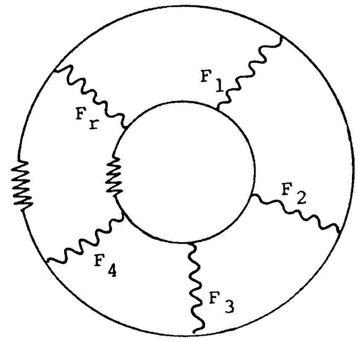

Figure 1

An alternative elementary proof of the corollary invokes Stallings' fibering theorem [20]. Compare [16].

6. Applications to fibered knots

Let $K \subset S^3$ be a fibered knot invariant under a smooth, orientation-preserving, semifree action of the cyclic group C_m of order m, having fixed axis A, a knot disjoint from K. The solution of the Smith Conjecture [19] implies that A is unknotted, and hence that the orbit space is again S^3 with branch set $B = A^*$ also an unknotted circle. The following simply interprets Theorem 5.2 in this setting.

PROPOSITION 6.1. *The fibering of K is isotopic to a fibering preserved by the action of C_m so that the axis A is transverse to the fibers; thus the quotient knot K^* inherits a fibering with all fibers transverse to the branch set B.* □

Let $F \subset S^3$ denote a typical fiber for the knot K transverse to A and let F^* denote its image in the orbit space. Note that since the axis is connected the local two-dimensional representations of C_m in the tangent space of F at the points of $F \cap A$ are all equivalent to rotation by $\pm 2\pi k/m$ for some fixed integer k.

Our first application of Proposition 6.1 is based on combining Corollary 2.4 with the fact that a genus 0 fibered knot is unknotted.

PROPOSITION 6.2. *For $g \geq 1$ the only fibered knot of genus g which is invariant under an action of C_m as above, with $m = 2g+1$, is the $(2, 2g+1)$ torus knot, up to orientation.*

Proof. One can check using the Riemann-Hurwitz formula that C_m, $m = 2g+1$, acts semifreely on a surface F of genus g with one boundary component,

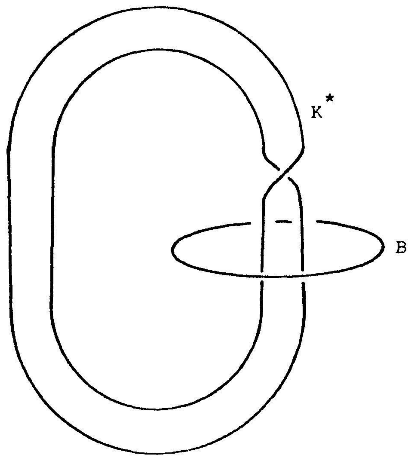

Figure 2

only so that the quotient F^* is disk with two branch points. Therefore the quotient of an m-periodic knot K is an unknotted circle K^* in S^3, and the branch set B is transverse to each disk fiber for K^*, meeting each fiber in exactly two points. The only such link $\{K^*, B\}$ up to orientation is shown in Figure 2. One easily checks that the preimage of K^* must be the $(2, \pm m)$ torus knot. □

PROPOSITION 6.3. *For $g \geq 1$ the only fibered knots of genus g which are invariant under an action of C_m as above, with $m = g+1$, are the $(3, \pm m)$ torus knot and the $(3, \pm m)$ Turk's head knot.*

Proof. The group C_m must act on the typical fiber F of genus $g = m-1$ with orbit surface a disk and exactly three branch points. According to [10] the only three-stranded braids (up to conjugation in the braid group, and reversal of orientation) closing to unknotted circles, are the two depicted in Figure 3. The preimages of K^* in the two cases are precisely, the $(3, \pm m)$ torus knot and the $(3, \pm m)$ Turk's head knot. □

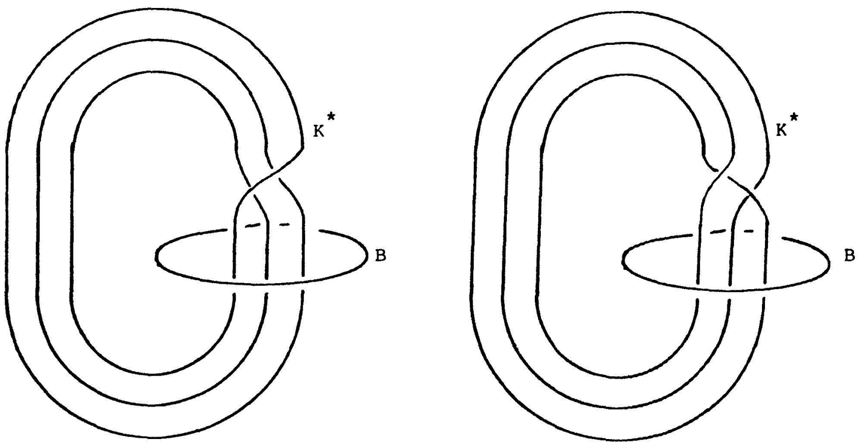

Figure 3

Remark. In working with specific examples the following observation, based on a fundamental result of Murasugi [15], is often helpful. If a knot K in S^3 is invariant under an action of the cyclic group C_m and the linking number of the axis with K is λ, then λ is completely determined by m. In the case of fibered knots this readily implies that the genus of the quotient fibered knot K^* is determined by m as well.

To see this assertion about λ, let p be any prime divisor of m, of order r in m. According to Murasugi

$$\Delta_K(t) \equiv [\delta_\lambda(t)]^{p^r-1}[f(t)]^{p^r} \bmod p,$$

where $\delta_\lambda(t) = (t^\lambda - 1)/(t-1)$. Notice that $\delta_\lambda(t)$ has no repeated factors as a polynomial with coefficients in $\mathbb{Z}_p$. This is because the derivative of $t^\lambda - 1$ has no nontrivial roots, since λ is relatively prime to p.

If there were two C_m actions on (S^3, K) with axes having different linking numbers with K, say λ_1 and λ_2, then we would have

$$[\delta_{\lambda_1}(t)]^{p^r-1}[f_1(t)]^{p^r} \equiv [\delta_{\lambda_2}(t)]^{p^r-1}[f_2(t)]^{p^r} \bmod p.$$

Now suppose $g(t)$ is an irreducible factor of this product, of order n. Then one has

$$n = \varepsilon_1(p^r - 1) + \delta_1 p^r = \varepsilon_2(p^r - 1) + \delta_2 p^r,$$

where ε_i is the order of g in δ_{λ_i} and δ_i is the order of g in f_i. Since δ_{λ_i} has no repeated roots, ε_i is 0 or 1. One concludes that $\varepsilon_1 = \varepsilon_2$, and hence that the factorization of δ_{λ_1} is the same as that of δ_{λ_2}. In particular they have the same degree, so $\lambda_1 = \lambda_2$.

The preceding uniqueness results do not hold in the presence of larger numbers of branch points, as the following example (inspired by a similar construction of Morton [14]) shows.

EXAMPLE. There exist infinitely many distinct genus two fibered knots with C_2 actions. Consider the link $\{K^*, B_n\}$ in Figure 4, where n denotes the number of half twists. For all n, B_n is unknotted. Let K_n be the preimage of K^* in S^3 under the two-fold cover of S^3 branched along B_n. Then K_n is a genus two fibered knot with a C_2 action.

The Alexander polynomial of K_n can be readily computed to be

$$\Delta_{K_n}(t) = t^4 + (n^2 + n - 1)t^3 + (-2n^2 - 2n + 1)t^2 + (n^2 + n - 1)t + 1.$$

Thus the knots K_n are all distinct, for $n > 0$.

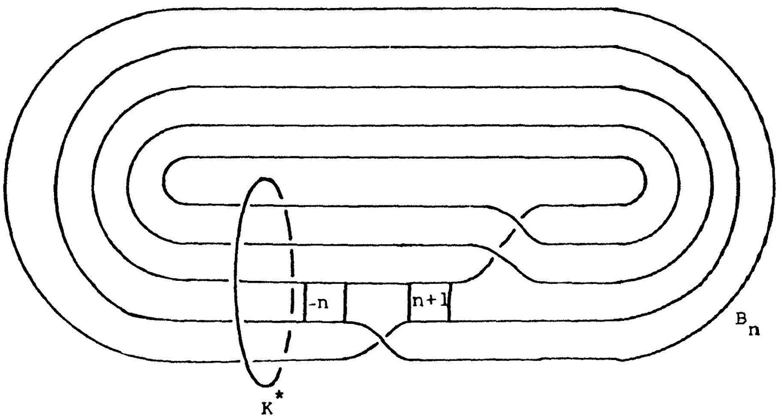

Figure 4

To compute $\Delta_{K_n}(t)$ proceed as follows: $S^3 - K^*$ is fibered by disks and hence $S^3 - (K^* \cup B_n)$ is fibered by 5 times punctured disks. The monodromy for that fibration is the product of twists σ_i with each σ_i corresponding to one half twist in the braid B_n. The monodromy for the genus 2 fibration of K_n is the product of the lifts of the σ_i to the branched covering space. The Alexander polynomial of K_n is the characteristic polynomial of the monodromy. Details of the calculation are left to the reader.

In cases where the quotient knot has positive genus further complexities arise.

EXAMPLE. Consider the two unknotted curves B_1 and B_2 in the complement of the trefoil knot K^* as depicted in Figure 5. In each case it is easy to check that B_i can be made transverse to the standard fibration of the trefoil complement, meeting each fiber exactly once. The preimage K_1 of K^* under the two-fold cover of S^3 branched along B_1 is the granny knot. The preimage K_2 of K^* under the two-fold cover of S^3 branched along B_2 is the knot 8_{21} in the standard knot tables [17].

In [9; 1.28] D. Goldsmith posed the following question. Suppose $p: M \to S^3$ is a cyclic cover of degree m branched over a link B; suppose K^* is an unknotted

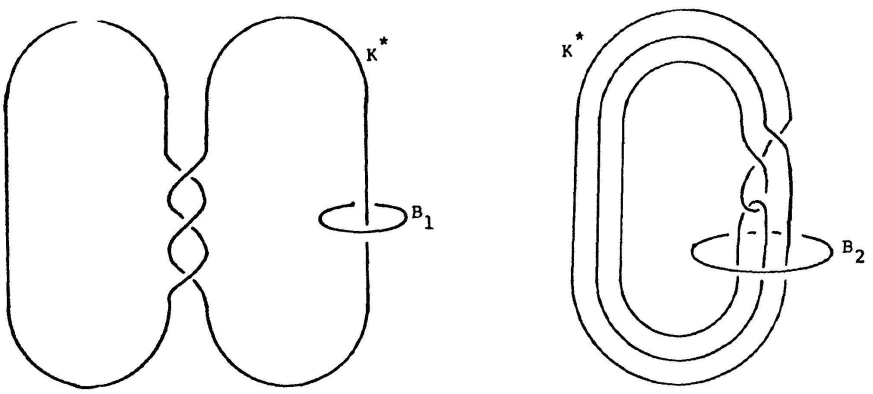

Figure 5

simple closed curve such that $K = p^{-1}(D^*)$ is a fibered knot in M. Is B isotopic to a braid about K^*? (The same question makes sense if K^* is just a fibered knot.) When M is a rational homology sphere, this is an immediate consequence of Theorem 5.2. If the degree m is a prime power and B is connected, then M is necessarily a rational homology sphere (see [2; §4]), so the answer is also yes in this case.

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