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Closed forms on symplectic fibre bundles

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A *bundle of symplectic manifolds* is a differentiable fibre bundle $F \to E \xrightarrow{\pi} B$ whose structure group (not necessarily a Lie group) preserves a symplectic structure on F. The vertical subbundle $\mathcal{V} = \operatorname{Ker}(T\pi) \subseteq TE$ carries a field of bilinear forms which we call the *symplectic structure along the fibres* and denote by ω. Any 2-form Ω on E has a restriction to $\mathcal{V}$; if this restriction is ω, we call Ω an *extension* of ω.

In this note, we discuss the problem of finding a closed extension of the symplectic structure along the fibres. This is the first step toward finding a symplectic extension – a problem already considered in special cases in [Th] and [Wn].

The first theorem shows that the existence of a closed extension is a purely topological problem.

THEOREM 1. *Let $F \to E \to B$ be a differentiable fibre bundle carrying a field ω of p-forms on the vertical bundle $\mathcal{V}$, defining a closed form on each fibre. Then there is a closed p-form Ω on E extending ω if and only if there is a de Rham cohomology class c on E whose restriction to each fibre is the class determined by ω.*

Theorem 1 is tacitly assumed in the usual identification of the E_1 term in the Leray spectral sequence for de Rham cohomology as forms on the base with values in the cohomology of the fibres [Gr–Ha]. No proof is given in the cited reference, and indeed the only proof we have found in the literature [Ha] applies only when the fibres have finite-dimensional cohomology groups.

Using a partition of unity on B, it is easy to reduce Theorem 1 to the following lemma.

LEMMA. *Let $\{\omega_u\}$ be a family of p-forms on the manifold F, depending*

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smoothly on a parameter u ranging over an open set $\mathcal{U}$ in $\mathbb{R}^n$. If ω_u is exact for each u, then there is a smooth family $\{\eta_u\}$ of $p-1$ forms on F such that $d\eta_u = \omega_u$ for each u in $\mathcal{U}$.

For compact F, the lemma can be proven by Hodge theory. For general F, the first step in our proof is to pass from de Rham to Čech cohomology by the usual sheaf-theoretic argument [Gr–Ha] [W1] using smooth dependence on parameters in the Poincaré lemma. Now one applies the universal coefficient theorem for Čech cohomology to the case of coefficients in $C^\infty(\mathcal{U})$. (See Thm. 5.4.13(c) in [Hi–Wy].)

From now on, we confine our attention to the case $p=2$. The following theorem seems to be something of a surprise.

THEOREM 2. *Let $F \to E \to B$ be a fibre space with F and B 1-connected. If the restriction map $H^2(E;\mathbb{R}) \to H^2(F;\mathbb{R})$ is not surjective, then $H^{2k}(F;\mathbb{R})$ is non-trivial for all $k \geq 0$.*

Proof. 1. There is a map $\alpha: S^3 \to B$ such that $\partial_*[\alpha] \in \pi_2(F)$ has infinite order: Since F is 1-connected, $\pi_2(F) \cong H_2(F)$ and so $\pi_2(F) \otimes \mathbb{R} \cong H_2(F) \otimes \mathbb{R} \cong H_2(F;\mathbb{R})$. Also, $H^2(F;\mathbb{R}) \cong \mathrm{Hom}\,(H_2(F;\mathbb{R}),\mathbb{R})$. Since $\pi_1(B)$ is trivial, so is $\pi_1(E)$. Thus $H^2(E;\mathbb{R}) \to H^2(F;\mathbb{R})$ not surjective is equivalent to $H_2(F;\mathbb{R}) \to H_2(E;\mathbb{R})$ not injective, and hence equivalent to $\pi_2(F) \otimes \mathbb{R} \to \pi_2(E) \otimes \mathbb{R}$ not injective. Thus α exists.

2. There is a map $\psi: \Omega S^3 \to F$ such that $\psi^*: H^2(F;\mathbb{R}) \to H^2(\Omega S^3;\mathbb{R})$ is non-trivial: Consider the commutative diagram

$$\begin{array}{ccccc}
\Omega S^3 & \xrightarrow{\psi} & F & \xrightarrow{=} & F \\
\downarrow & & \downarrow & & \downarrow \\
PS^3 & \xrightarrow{\varphi} & \alpha^* E & \rightarrow & E \\
{\scriptstyle q}\downarrow & & \downarrow & & \downarrow \\
S^3 & \xrightarrow{=} & S^3 & \xrightarrow[\alpha]{} & B
\end{array}$$

where PS^3 is the space of paths beginning at the base point of S^3, q is the endpoint projection, φ is any lift of q, and ψ is the restriction of φ. From the homomorphism of exact homotopy sequences we see that $\psi_*: \pi_2(\Omega S^3) \to \pi_2(F)$ sends the generator onto $\partial_*[\alpha]$. Hence $\psi_*: H_2(\Omega S^3;\mathbb{R}) \to H_2(F;\mathbb{R})$ is non-trivial, and so ψ^* is non-trivial.

3. $H^{2k}(;\mathbb{R})$ is non-trivial for all $k \geq 0$: $H^*(\Omega S^3;\mathbb{R}) = \mathbb{R}[x]$, where $x \in H^2(\Omega S^3;\mathbb{R})$ is the generator. By step 2, $H^*(F;\mathbb{R}) \to H^*(\Omega S^3;\mathbb{R}) = \mathbb{R}[x]$ is surjective, and the theorem follows. ■

Remark. If $F \to E \to B$ has as its structure group a Lie group G, then $\pi_2(G)=0$ implies $\alpha^*E = S^3 \times F$. Hence we may choose φ so that ψ is the constant map and ψ^* is trivial. Thus $H^2(E;\mathbb{R}) \to H^2(F;\mathbb{R})$ must be surjective. Also note that $\partial_*: \pi_3(B) \to \pi_2(F)$ is trivial since it factors through $\pi_2(G)$.

More generally, there are smooth finite dimensional manifolds F with $\pi_2(\mathrm{Diff}\, F) \neq 0$, and hence smooth non-trivial bundles E over S^3 with fibre F [Hr] [La]. But we must have $\partial_*: \pi_3(S^3) \to \pi_2(F)$ of finite order since the conclusions of the theorem cannot hold for finite dimensional F. Such a bundle looks like a product when viewed with real (or rational) coefficients; i.e., $\pi_i(E) \otimes \mathbb{R} = [\pi_i(S^3) \otimes \mathbb{R}] \oplus [\pi_i(F) \otimes \mathbb{R}]$, and in some sense the action of Diff F on F resembles the action of a Lie group.

Combining Theorems 1 and 2 leads to the following conclusion:

THEOREM 3. *Let $F \to E \to B$ be a bundle of symplectic manifolds with 1-connected fibre and base. Unless $H^{2k}(F;\mathbb{R}) \neq 0$ for all $k \geq 0$, the symplectic structure along the fibres has a closed extension.*

Theorem 3 was originally proven for compact and 1-connected F in [Wn]. The following two examples show the necessity of the hypotheses on the fibre in Theorem 3. It is possible that the assumption of simple connectivity of B could be dropped or weakened (compare [B1]).

EXAMPLE 1. Begin with the Hopf bundle $S^1 \to S^3 \to S^2$ and cross the fibre with S^1 to make a torus bundle $S^1 \times S^1 \to S^3 \times S^1 \to S^2$. The structure group consists of translations of the torus, which preserve the standard area element; thus this is a bundle of symplectic manifolds, but $H^2(S^3 \times S^1;\mathbb{R}) = 0$.

EXAMPLE 2. The loop space fibration $K(\mathbb{Z},2) \to K(\mathbb{Z},3)$ can be pulled back to S^3 to give a bundle $\mathbb{C}P^\infty \to E \to S^3$ with $H^2(E;\mathbb{R}) = 0$. This bundle can be constructed with a gluing map from S^2 to the projective unitary group $PU(\mathbb{H}) = U(\mathbb{H})/S^1$ acting on the Hilbert space $\mathbb{H}$ for which $\mathbb{C}P^\infty$ is the space of lines. Since $PU(\mathbb{H})$ acts symplectically on $\mathbb{C}P^\infty$, we have a bundle of symplectic manifolds. It would be interesting to have an explicit geometric or analytic construction of this bundle.

Theorem 3 has a nice interpretation involving connections and reductions of the structure group for bundles of symplectic manifolds. If $F \to E \to B$ is any bundle of symplectic manifolds, there always exists an extension Ω (not necessarily closed) of the symplectic structure along the fibres. Since the restriction of Ω to the vertical bundle $\mathcal{V}$ is nondegenerate, the horizontal sub-bundle $\mathcal{H} = \{v \in TE \mid v \rfloor \Omega$ annihilates vertical vectors$\}$ is a complement to $\mathcal{V}$ and so defines a connection on E in the sense of [Eh]. The form Ω is determined by this connection together with the restriction of Ω to $\mathcal{H}$.

We shall call Ω *r-closed* if $(v_1 \wedge \cdots \wedge v_r) \rfloor \, d\Omega = 0$ whenever $v_1, \ldots, v_r$ belong to $\mathcal{V}$. Since Ω is closed on fibres, it is 3-closed. Ω is 0-closed if and only if it is closed. The intermediate steps in this (Leray) filtration have the following interpretations.

THEOREM 4. (i) *Ω is 2-closed if and only if parallel translations preserve the symplectic structure.*

(ii) *If Ω is 2-closed, the curvature of the connection takes values in the Lie algebra of locally hamiltonian vector fields along the fibres. The curvature takes values in the globally hamiltonian vector fields if and only if Ω is 1-closed. Thus there exists a 1-closed extension if and only if the structure group can be reduced to one which admits a momentum mapping.* (*Compare Example* 1).

(iii) *There exists a closed extension if and only if the structure group can be lifted to one which admits an* Ad*-*equivariant momentum mapping.* (*Compare Example* 2.)

Our results can also be interpreted in the language of geometric quantization. The bundle $F \to E \to B$ of symplectic manifolds may be considered as a family of classical mechanical systems depending on a parameter in B. If the symplectic cohomology class of F is integral, then it is the Chern class of a complex line bundle over F, i.e. a prequantization. One might wish to prequantize the whole family at once by finding a line bundle over E which has the right restriction to each fibre. Theorem 2 shows that, if we multiply the symplectic form on F by a suitably chosen integer (in fact, this might not be necessary), then the line bundle over E can be found if F is simply connected and finite dimensional. Examples 1 and 2 show that these conditions on the fibre cannot be omitted.

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