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Braided surfaces and Seifert ribbons for closed braids

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Abstract. A *positive band* in the braid group B_n is a conjugate of one of the standard generators; a negative band is the inverse of a positive band. Using the geometry of the configuration space, a theory of bands and *braided surfaces* is developed. Each representation of a braid as a product of bands yields a handle decomposition of a *Seifert ribbon* bounded by the corresponding closed braid; and up to isotopy all Seifert ribbons occur in this manner. Thus, *band representations* provide a convenient calculus for the study of ribbon surfaces. For instance, from a band representation, a Wirtinger presentation of the fundamental group of the complement of the associated Seifert ribbon in D^4 can be immediately read off, and we recover a result of T. Yajima (and D. Johnson) that every Wirtinger-presentable group appears as such a fundamental group. In fact, we show that every such group is the fundamental group of a Stein manifold, and so that there are finite homotopy types among the Stein manifolds which cannot (by work of Morgan) be realized as smooth affine algebraic varieties.

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§0. Introduction

Stallings, reporting [S] on constructions of fibred knots and links, mentions (almost in passing) a construction which associates to any braid $\beta \in B_n$ a certain Seifert surface in S^3 bounded by the closed braid $\hat{\beta}$. Actually – and importantly – that construction begins *not* with a braid (an element of the group B_n) but with a

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braid word (an expression of the braid as a word in the standard generators $\sigma_1, \ldots, \sigma_{n-1}$ of B_n, and their inverses). Stallings describes the constructed Seifert surface as being plumbed together from $n-1$ simpler surfaces.[(2)] More naively, the surface is simply given as a handlebody: the union of n (2-dimensional) 0-handles connected by orientable 1-handles whose number and location are specified by the particular braid word.

The plumbing description, in Stallings's context of "homogeneous braids," is appropriate because it shows that the surface constructed from a homogeneous braid (word) is actually a fibre surface for the closed braid. In this paper I hope to show that the naive handlebody description, and a generalization of it which produces *Seifert ribbons*, can be appropriate in other contexts.

This work fits into a circle of ideas going back to Alexander, E. Artin, van Kampen, and Zariski. In 1923, without bringing the (then undiscovered) *braid groups* into it, Alexander [Al] showed that every (tame) link type contains representative *closed braids*. In other words: the construction that begins with a braid $\beta \in B_n$ and produces an oriented link $\hat{\beta} \subset S^3$ is perfectly general – every link type can be so produced. Artin introduced the braid groups B_n in 1925, giving algebraic structure to the geometric braids, and used that algebraic structure to describe (among other things) a class of group presentations which included presentations of precisely the link groups $\pi_1(S^3 - L)$. Meanwhile, Zariski [Z1, Z2] was investigating the groups $\pi_1(\mathbf{C}P^2 - \Gamma)$, where Γ was a (possibly singular) complex algebraic curve, and seems actually to have commissioned van Kampen to prove the now-famous "van Kampen's Theorem" [vK] precisely to get presentations of those groups – which are of course intimately related to the groups $\pi_1(\mathbf{C}^2 - \Gamma)$.

A *ribbon surface* in the 4-disk D^4 is a 2-manifold-with-boundary embedded in a certain restricted way (see §§1 and 2, below). A (non-singular) piece of algebraic curve is, as it turns out, always a ribbon surface (cf. [Mi]). In §2 I show how, from a braid β together with $\vec{b}$, an expression for β as a word in certain generators of B_n (the set of conjugates of the standard generators), one can construct a ribbon surface in D^4 bounded by (a link of the type of) $\hat{\beta}$; and in §3 I show that this construction is perfectly general, and produces representatives of each isotopy class of orientable ribbon surface. One may say that Alexander's theorem is the boundary of these results. (A modification of the construction produces, equally generally, "ribbon immersions" in S^3; and in particular all ordinary Seifert surfaces can be constructed from "embedded band representations" of braids.)

In §4 a presentation for the group $\pi_1(D^4 - S(\vec{b}))$, of the form called a *Wirtinger presentation*, is derived from $\vec{b}$. Every group that has a Wirtinger presentation at

[2] This kind of plumbing was first discovered by Murasugi [Mu].

all, has one of the sort that appears here (and from such a presentation $\vec{b}$ is immediately read off). Thus we recover Dennis Johnson's improvement [J] of T. Yajima's [Y] result that any group with a Wirtinger presentation can be realized as $\pi_1(S^4 - S)$ for some smooth orientable surface S (the improvement being in the ribbon-like nature of the surface, see below). Actually, I show somewhat more, and as an application show that each such group also appears as the fundamental group of a Stein manifold (in fact a complex surface in $\mathbf{C}^5$). John Morgan [Mo] has ruled out many groups, for instance $(x, y : [x, [x, [x, [x, y]]]] = 1) = G$ say, from being fundamental groups of (affine) smooth algebraic varieties; but G, as it happens, has a Wirtinger presentation.

Ribbon genus and related matters are attacked in §5. An appendix indicates how the work can be extended to (alternatively) ribbon surfaces with nodes in D^4, or surfaces immersed in S^3 with both ribbon and clasp singularities.

The long §1 lays the groundwork for the rest of the paper, relating information about the geometry of the *configuration space* (the space of which B_n is the fundamental group) to algebraic information about B_n and geometric information about surfaces in S^3 and D^4.

§1. Loops and disks in the configuration space: closed braids, braided surfaces, and band representations

Apparently it was only as late as 1962 that topologists first realized that "B_n may be considered as the fundamental group of the space ... of configurations of n undifferentiated points in the plane" (this "previously unnoted remark" being then made by Fox and Neuwirth [F–N, p. 119]).[(3)] In this section some further relations among the geometry of that space, the geometry of links and surfaces, and the algebra of the braid group, will be explored. Simply for convenience here, the plane $\mathbf{R}^2$ (in which the configurations of n points lie) will be identified with the complex line $\mathbf{C}$; for a further application of the theory, where the complex structure is really at the heart of things, see [Ru].

By identifying the complex n^{th} degree monic polynomial $\prod_{j=1}^{n} (w - w_j) = w^n + c_1 w^{n-1} + \cdots + c_{n-1} w + c_n$ with on the one hand the un-ordered n-tuple $\{w_1, \ldots, w_n\}$ of its roots, and on the other hand the ordered n-tuple $(c_1, \ldots, c_n)$ of its non-leading coefficients, we effect the well-known identification of $\mathbf{C}^n/\mathfrak{S}_n$ with $\mathbf{C}^n$. (The symmetric group $\mathfrak{S}_n$ on n letters acts on $\mathbf{C}^n$ by permuting the coordinates.) Now, $\mathbf{C}^n/\mathfrak{S}_n$ (being the quotient of $\mathbf{C}^n$ by a finite group of automorphisms) inherits from $\mathbf{C}^n$ a natural structure of (singular, affine) algebraic

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[3] Magnus [M], in a review of [Bi], indicates that Hurwitz, [Hu], studying monodromy in 1891, had in fact noted this definition.

variety; its singular locus $\mathcal{S}(\mathbf{C}^n/\mathfrak{S}_n)$ is the quotient by $\mathfrak{S}_n$ of the multi-diagonal in $\mathbf{C}^n$, that is, it contains exactly those n-tuples $\{w_1, \ldots, w_n\}$ in which for some $j \neq k$, $w_j = w_k$. But, via the identification of $\mathbf{C}^n/\mathfrak{S}_n$ (the space of roots) with $\mathbf{C}^n$ (the space of coefficients), we also give $\mathbf{C}^n/\mathfrak{S}_n$ a non-singular structure, which is the normalization and the minimal resolution of the quotient structure. Let us denote $\mathbf{C}^n/\mathfrak{S}_n$ with this non-singular structure by E_n, and let Δ denote its subset which "is" the old singular locus. Then Δ is a hypersurface of the affine space E_n; when $n \geqq 3$, Δ is singular. (Algebraic geometers know Δ as the *discriminant locus.*) Still, a smooth map of a manifold into E_n may be perturbed arbitrarily slightly to make it transverse to Δ, since Δ is the image of a smooth manifold (any one of the hyperplanes $w_j = w_k$ back in the multidiagonal of the space of roots) by a smooth map. (Incidentally, this resolution shows that Δ is irreducible, so that its regular set $\mathcal{R}(\Delta)$ is connected, a fact we need later.) In particular, all the transversality we will need in the sequel is collected in the following lemma.

LEMMA 1.1. *Let M be a compact, smooth manifold-with-boundary of dimension no greater than 3. Then any smooth map $f: M \to E_n$ may be perturbed by an arbitrarily small homotopy to a smooth map which misses the singular locus $\mathcal{S}(\Delta)$ entirely (since $\mathcal{S}(\Delta)$ has real codimension 4) and which intersects the smooth, codimension-2 manifold $\mathcal{R}(\Delta)$ of regular points of Δ transversely. If $f \mid \partial M$ is already transverse to Δ in this sense then the homotopy need not alter $f \mid \partial M$.* □

The (open, dense) set $E_n - \Delta \subset E_n$ is the *configuration space* (of n "undifferentiated points in the plane"). The fundamental group $\pi_1(E_n - \Delta)$ (we will suppress basepoints whenever it is decent to do so) is called the *braid group B_n*. (Its structure will be recalled later. General reference: [Bi].) Since E_n is contractible, every loop $f: \partial D^2 \to E_n - \Delta$ extends to a map $f: D^2 \to E_n$ – we can assume f is smooth, and by Lemma 1.1, transverse to Δ. Now, what is called a *geometric braid* is nothing more nor less than a loop in $E_n - \Delta$. What then is such an extension to a map of a disk?

DEFINITION 1.2. A (smooth) *singular braided surface* in a bidisk $D = D_1^2 \times D_2^2 = \{(z, w) \in \mathbf{C}^2 : |z| \leqq r_1, |w| \leqq r_2\}$ is a (smooth) map of pairs $i: (S, \partial S) \to (D, \partial_1 D)$ (here $\partial_1 D$ denotes the solid torus $\partial D_1^2 \times D_2^2$ which is half of the boundary of D), such that

(1) $pr_1 \circ i: (S, \partial S) \to (D_1^2, \partial D_1^2)$ is a branched covering map (and an honest covering on the boundaries),

(2) S is so oriented that $pr_1 \circ i$, away from its finite set of branch points, is orientation preserving (with respect to the complex orientation of $D_1^2 \subset \mathbf{C}$).

From (1) we see that S is orientable, so (2) makes sense.

The degree n of the branched covering $pr_1 \circ i$ is the *degree* of the braided surface; all but finitely many points $z \in D_1^2$ have n distinct preimages in S.

By an *embedded braided surface* in D let us mean a singular braided surface for which i is a smooth embedding, or, by abuse of language, also the image $i(S) \subset D$ of such an i.

EXAMPLE 1.3. If Γ is a complex-analytic curve in a neighborhood of D (possibly analytically reducible, but without multiple components), so situated that $\Gamma \cap \partial D$ is the transverse intersection of $\mathcal{R}(\Gamma)$ and $\partial_1 D$, then the normalization of $\Gamma \cap D$ mapping into D is a singular braided surface; and if there are no singularities of the curve inside D then it is an embedded braided surface. (Such analytic curves motivated these investigations, but by no means exhaust the examples.)

Here is the connection between braided surfaces and the configuration space.

On the one hand, given a smooth map $f:(D_1^2, \partial D_1^2) \to (E_n, E_n - \Delta)$ for which $f^{-1}(\Delta)$ is a finite subset of Int D_1^2, one can create a singular braided surface $f^{\#}$ in $D_1^2 \times D_2^2$, where the second radius r_2 is any strict upper bound for the absolute value of all elements w_j in all n-tuples $\{w_1, \ldots, w_n\} = f(z)$ for $z \in D_1^2$. (Begin by considering the set $S_f' = \{(z, w) \in D : w \in f(z)\}$. Then there is a finite subset $X \subset S_f'$ so that $S_f' - X$ is a genuine n-sheeted covering space of $D_1^2 - pr_1(X)$, embedded as a submanifold of D, with pr_1 as covering projection. Just from the continuity of f it is easy to resolve the singularities of S_f', yielding a surface-with-boundary S_f on which the map $f^{\#}$ is forced; and this is clearly the desired singular braided surface. Note that its degree is n.)

On the other hand, given a singular braided surface $i:(S, \partial S) \to (D, \partial_1 D)$, of degree n, there is a corresponding smooth map $i_{\#}:(D_1^2, \partial D_1^2) \to (E_n, E_n - \Delta)$: on the set of those $z \in D_1^2$ where $\{w : (z, w) \in i(S)\}$ has n distinct elements, one sets $i_{\#}(z) = \{w : (z, w) \in i(S)\}$; again the extension to all z in D_1 is forced.

Note also that if f, as above, is transverse to Δ, then $f^{\#}$ is an embedded braided surface, and is also "in general position" – meaning here that branch points of $pr_1 \circ f^{\#}$ are all "simple vertical tangents". And conversely, given i as above, $i_{\#}$ will be transverse to Δ only if S is in fact embedded and its vertical tangents are all simple. Of course, any embedded braided surface is arbitrarily close (isotopic through embedded braided surfaces) to an embedded braided surface in general position.

Recall that a surface embedded in $D^4 = \{(z, w) \in \mathbf{C}^2 : |z|^2 + |w|^2 \leqq 1\}$ is a *ribbon* *surface* if the restriction to the surface of $|z|^2+|w|^2$ is a Morse function, identically 1 on the boundary, which may have saddles as well as local minima, but which has no local maxima. Ribbon surfaces in D^4, and the related ribbon immersions in S^3 and $\mathbf{R}^3$, will be discussed in greater detail in §§2 and 3. Here we will make the connection to braided surfaces.

PROPOSITION 1.4. *If $S \subset D$ is an embedded braided surface then there is an isotopic deformation of D to D^4 (in $\mathbf{C}^2$) which carries S onto a ribbon surface.*

Of course D has corners and D^4 is smooth, but the isotopy will be smooth except near the corners of D, which without loss of generality are missed by S.

Proof. After a slight perturbation of S, perhaps, the function $L_0(z, w)=|z|^2$ will, when restricted to S, be a Morse function with n (the degree of S) minima, a saddle point for each branch point, and no local maxima, and it will be identically 1 on ∂S. Then for small $\varepsilon>0$, $L_\varepsilon=|z|^2+\varepsilon\,|w|^2$, when restricted to S, has the same properties, except that it is not quite constant on the boundary. A small isotopy of D, supported near its own boundary, will fix $L_\varepsilon \mid \partial S$. The rest is clear. □

Remark 1.5. A consequence of the construction in §3 is that a converse to this proposition holds – every (orientable!) ribbon surface is isotopic to an embedded braided surface. This is the exact analogue, for ribbon surfaces, of Alexander's theorem [Al] for links, that they all occur as closed braids. I don't know a more direct proof of this converse.

Next we will dip into the algebra of B_n for a while.

The *standard generators* of B_n are $\sigma_1, \ldots, \sigma_{n-1}$. (With respect to a basepoint $* \in E_n$, for instance $*=\{1, \ldots, n\}$, σ_j is represented by a loop which as a motion of the n points leaves all but j and $j+1$ fixed constantly, while exchanging j and $j+1$ by a *counterclockwise* 180° rotation [this is the East Coast convention!].) The *standard presentation* of B_n is $B_n=(\sigma_1, \ldots, \sigma_{n-1}: R_i \quad (i=1, \ldots, n-2),\ R_{ij}$ $(1 \leqq i<j-1 \leqq n-1))$, where $R_i: \sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}$ and $R_{ij}: \sigma_i\sigma_j=\sigma_j\sigma_i$ are the *standard relations*. All the standard generators belong to one conjugacy class: for R_i may be rewritten as $\sigma_{i+1}=\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}=(\sigma_i\sigma_{i+1})\sigma_i(\sigma_i\sigma_{i+1})^{-1}$ and so by induction each σ_i is conjugate to σ_1. Also, this class is not equal to its inverse, and in the infinite cyclic abelianization of B_n, each generator σ_i maps to 1.

For reasons which will be evident in the next section, I call any element of the conjugacy class of σ_1 a *positive band*. The inverse of a positive band (i.e., a conjugate of σ_1^{-1}) is a *negative band*. A *band* is a positive or negative band.

In any group, I will use the notation ${}^ab$ to denote the conjugate aba^{-1}, when convenient.

For $n>2$ there are infinitely many bands in B_n. (B_2 is infinite cyclic.) Intermediate between the set of $2(n-1)$ standard generators and their inverses, and the set of all bands, is a set of $(n-1)n$ *embedded bands*. The positive embedded bands are $\sigma_{i,j}={}^{A(i,j-1)}\sigma_j$, where (just here) $A(i,j-1)=\sigma_i\cdots\sigma_{j-1}$, and $1\leqq i\leqq j\leqq n-1$. (So $\sigma_{j,j}=\sigma_j$.)

NOTATION 1.6. An ordered k-tuple $\vec{b}=(b(1),\ldots,b(k))$ with each $b(i)$ a band in B_n (of either sign) is a *band representation* (*in* B_n) *of the braid* $\beta(\vec{b})=b(1)\cdots b(k)$, which we call *the braid of* $\vec{b}$. The *length* $l(\vec{b})$ is k. Conventionally, the braid of the unique 0-tuple is the identity of B_n.

If each $b(i)$ is an embedded band we call $\vec{b}$ an *embedded band representation.* If each $b(i)$ is a standard generator or the inverse of a standard generator, we identify $\vec{b}$ with a *braid word* in the usual sense; in that case length is more usually called *letter length.*

Since every braid is the braid of some braid word, it makes sense to define the *rank of* β in B_n, written $rk_n(\beta)$ or $rk(\beta)$, to be the least k such that some band representation of β has length k. Only the identity has rank 0. Rank is constant on conjugacy classes, and is less than or equal to "least letter length" (the analogue of rank when only braid words and not all band representations are used) and greater than or equal to the absolute value of the exponent sum (an invariant of words in the free group on $\sigma_1,\ldots,\sigma_{n-1}$ which clearly passes on to B_n). A band representation in which each band is positive is a *quasipositive band representation* and its braid is a *quasipositive braid* (cf. [Ru]); the length of a quasipositive band representation equals the exponent sum and the rank, and equals the least letter length if and only if the braid of the representation is actually a positive braid in the usual sense.

Remark 1.7. The notion of band is algebraic, geometric in E_n, *and* (as we shall see) geometric in S^3. The notion of embedded band is not algebraic, and seems to be geometric only in the latter context. Thus the idea of "embedded rank" seems to be unnatural and will be ignored.

There are some natural operations that relate different band representations of the same braid β. (Perhaps some natural incidence structure, of the "building" sort, awaits discovery in the set of such representations.) Let $\vec{b}=(b(1),\ldots,b(k))$, $k\geqq 2$. If for some j between 1 and $k-1$ we have $b(j)b(j+1)=1\in B_n$, then $(b(1),\ldots,b(j-1), b(j+2),\ldots,b(k))$ is another band representation of the same braid, gotten by *elementary contraction at the* j^{th} *place.*If j is between 1 and $k+1$

($k \geqq 0$), and a is any band, then the *elementary expansion of* $\vec{b} = (b(1), \ldots, b(k))$ *by* a *at the* j^{th} *place* is the band representation of the same braid $\vec{b}' = (b'(1), \ldots, b'(k+2))$ with $b'(i) = b(i)$ $(i<j)$, $b'(j) = a$, $b'(j+1) = a^{-1}$, $b'(i) = b(i-2)$ $(i > j+1)$.

Now let $1 \leqq j < k = l(b)$. The effect of S_j, the *forward slide at the* j^{th} *place*, is to replace $\vec{b}$ with $S_j\vec{b} = (b'(1), \ldots, b'(k)) : b'(i) = b(i)$ if $i \neq j, j+1$; $b'(j) = {}^{b(j)}b(j+1)$; and $b'(j+1) = b(j)$. The effect of S_j^{-1}, the *backward slide at the* j^{th} *place*, is to replace $\vec{b}$ with $S_j^{-1}\vec{b} = (b'(1), \ldots, b'(k)) : b'(i) = i$ if $i \neq j$, $j+1$; $b'(j) = b(j+1)$; $b'(j+1) = {}^{b(j+1)^{-1}}b(j)$. It is easy to check that $\beta(\vec{b}) = \beta(S_j\vec{b}) = \beta(S_j^{-1}\vec{b})$ and that S_j and S_j^{-1} are, indeed, inverse to each other.

(After preparing this paper, the author became aware of Moishezon's work [Moi] on "braid monodromies" of complex plane curves. My slides are Moishezon's "elementary transformations"; because he is dealing purely with what I have called quasipositive band representations, he does not introduce expansions and contractions.)

For a fixed $k \geqq 2$, the $k-1$ slides $S_1, \ldots, S_{k-1}$ generate a group which acts on the set of all band representations (of various braids) of length k. It is readily checked that these slides satisfy the standard relations $R_i(S_1, \ldots, S_{k-1})$ and $R_{ij}(S_1, \ldots, S_{k-1})$, and therefore mediate an action of the braid group B_k on this set of length-k band representations. Let two band representations (necessarily of the same braid) which are in the same B_k-orbit be called *slide-equivalent*. This will be elucidated in the next section, and in Prop. 1.11.

EXAMPLE 1.8. Let (a, b) be a band representation of length 2. It is easily checked that $S_1^{2m}(a, b) = ({}^{(ab)^m}a, {}^{(ab)^{m-1}a}b)$, $S_1^{2m-1}(a, b) = ({}^{(ab)^{m-1}a}b, {}^{(ab)^{m-1}}a)$ for any $m \in \mathbf{Z}$.

Remark 1.9. It is tempting to conjecture that a single slide-equivalence class should fill out the set of band representations of β of a given length, at least when that length is the rank of β. This fails to be true. For instance, in B_3, $(\sigma_1, \sigma_2^2\sigma_1\sigma_2^{-2})$ and $(\sigma_2^{-1}\sigma_1\sigma_2, \sigma_2\sigma_1\sigma_2^{-1})$ have the same braid and (being quasipositive) are of minimal length for that braid, but they are not slide-equivalent. (Sketch of proof: For typographical convenience, let σ_1 and σ_2 be abbreviated to 1, 2, respectively. Using Example 1.8 it suffices to show that ${}^{2^{-1}}1$ cannot be written as ${}^{(1\cdot{}^{22}1)^m}1$ or as ${}^{(1\cdot{}^{22}1)^m1}2 \equiv {}^{(1\cdot{}^{22}1)^m2^{-1}}1$ for any integer m. Now, in any group, three elements u, v, x satisfy ${}^ux = {}^vx$ if and only if uv^{-1} commutes with x. So we have to show that $(1 \cdot {}^{22}1)^m$ and $(1 \cdot {}^{22}1)^m2$ don't commute with 1 for any m. A straightforward but unilluminating computation in $SL(2, \mathbf{Z})$, using the well-known representation $\sigma_1 \mapsto \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $\sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, suffices to verify this.) It happens that these two band representations are "conjugate" in the obvious sense (by σ_2^{-1}) but not all examples of this phenomenon arise so simply.

Remark 1.10. In §4 we will see an example of a braid β of rank 2, and a band representation $\vec{b}$ of β of length 4, which is not slide-equivalent to any elementary expansion of any band representation of the minimal length 2.

Our next task is to relate band representations to disks in E_n. Now we fix a basepoint $* \in E_n - \Delta$, and identify B_n with $\pi_1(E_n - \Delta, *)$. For each $k \geqq 1$, moreover, we fix a set P_k of k distinct interior points of D^2 – let us be definite and say $P_k = \{1/m - 1 \in \mathbf{C} : m = 1, \ldots, k\} \subset D^2 = \{z \in \mathbf{C} : |z| \leqq 1\}$. Let the basepoint of D^2 be $* = -\sqrt{-1}$. Then $\pi_1(D^2 - P_k; *)$ is the free group of rank k on free generators x_j $(j = 1, \ldots, k)$, where x_j is the class of a loop consisting of a straight line segment from $*$ to a point on the circle of radius $1/2k(k-1)$ centered at $1/j - 1$, followed by the circle traversed once counterclockwise, followed by the segment back to $*$. If h is a diffeomorphism of D^2 to itself which is the identity on ∂D^2 and which preserves P_k as a set, then the automorphism $h_* : \pi_1(D^2 - P_k; *) \to \pi_1(D^2 - P_k; *)$ satisfies $h_*([\partial D^2]) = [\partial D^2]$, where $[\partial D^2]$ is the homotopy class of the (counterclockwise oriented) boundary of D^2, namely, $x_1 x_2 \cdots x_k$. It is a fact (cf. [Bi]) that the group of all such automorphisms h_* is naturally isomorphic to the braid group B_k; a diffeomorphism which is supported in a $1/2k(k-1)$-neighborhood of the interval

$$\left[\frac{1}{m+1} - 1, \frac{1}{m} - 1\right] \subset D^2$$

and rotates the interval 180° counterclockwise will induce the automorphism Σ_m corresponding to σ_m.

PROPOSITION 1.11. (i) *Let $f : (D^2, \partial D^2, *) \to (E_n, E_n - \Delta, *)$ be smooth and transverse to Δ, and suppose that $f^{-1}(\Delta)$ contains precisely k points. Let $h : D^2 \to D^2$ be a diffeomorphism, fixing ∂D pointwise, such that $h(P_k) = f^{-1}(\Delta)$. Then the k-tuple $((f \circ h)_* x_1, \ldots, (f \circ h)_* x_k)$ is a band representation in B_n, and its braid is $\beta = f_*([\partial D^2])$. The band representations which correspond to different choices of h are slide-equivalent, and vice versa.*

(ii) *Conversely, given a band representation $\vec{b}$ of β, and a smooth map $f : (\partial D^2, *) \to (E_n - \Delta, *)$ such that $f(\partial D^2)$ (oriented counterclockwise) represents β, then there is a smooth extension of f over the whole disk D^2, $f : (D^2, \partial D^2, *) \to (E_n, E_n - \Delta, *)$, which is transverse to Δ, with $f^{-1}(\Delta) = P_k$, and such that the band $b(j)$ equals $f_* x_j$ $(j = 1, \ldots, k)$. Such an extension is unique up to homotopy. If f is an embedding on ∂D^2 the extension may be taken to be an embedding also, unique up to isotopy.*

Proof. If $k=1$, then (i) says that a loop (through $*$) which bounds a disk that meets Δ transversely in exactly one point represents a band in B_n; and the existence half of (ii) says that every band arises like this. Both statements are true: for, indeed, an obvious explicit loop representing σ_1 (as in [Bi, p. 18]) bounds an equally obvious disk of the sort required in (i), and (up to orientation) all loops which bound such disks are conjugate in B_n because (by transversality) the map $\pi_1(E_n-\Delta, *)\to\pi_1(E_n-\mathcal{R}(\Delta), *)$ induced by inclusion is an isomorphism and (as remarked before Lemma 1.1) $\mathcal{R}(\Delta)$ is a *connected* submanifold of E_n of codimension 2.

Now, to prove (i) for any k, note that since $(f\circ h)_*:\pi_1(D^2-P_k; *)\to\pi_1(E_n-\Delta; *)$ is a homomorphism, certainly the product $(f\circ h)_*x_1\cdots(f\circ h)_*x_k$ equals $(f\circ h)_*(x_1\cdots x_k)$ which is $f_*([\partial D^2])$ since h is the identity on ∂D^2; and by the case $k=1$, each braid $(f\circ h)_*x_j$ is indeed a band; so we do have a band representation of β. A different choice of h corresponds to composing the original $f\circ h$ on the right with a diffeomorphism of D^2 which fixes the boundary pointwise and P_k as a set, and therefore to composing the original $(f\circ h)_*$ on the right by an automorphism in the group generated by the Σ_j's. But one quickly sees that, on the level of band representations, Σ_m corresponds to the forward slide S_m. So (i) is proved for all k.

As to (ii), given b one readily constructs a map g from a bouquet of k disks $\bigvee_{j=1}^k(D_j^2, *)$, identified at a common boundary point $*$, into E_n so that each restriction $g\mid D_j^2$ is smooth and transverse to Δ, meeting it at a single point, and taking ∂D_j^2 to a loop in the class $b(j)$. Then there is a map $q:(\partial D^2, *)\to(\bigvee_{j=1}^k\partial D_j^2, *)$ with $(g\circ q)_*[\partial D^2]=\beta$; and $g\circ q$ is homotopic (rel. $*$) to the given map f *in the complement of* Δ. Using q to glue the annulus (which is the domain of the homotopy between f and $g\circ q$) to $\bigvee_{j=1}^k D_j^2$, one creates a disk D^2 and a continuous extension of f from ∂D^2 across D^2. This extension is smooth on the boundary and near the preimage of Δ, to which it is transverse; and a small perturbation will preserve those properties, while rendering the extension smooth everywhere. Two different extensions differ, up to homotopy, by an element of $\pi_2(E_n-\Delta)$ but according to [F–N] the space $E_n-\Delta$ is a $K(B_n, 1)$: so any two extensions of f are homotopic. Finally, if $n>2$ the assertions about embeddings and isotopies are easy by general position, the ambient dimension being then at least 6; while if $n=2$, B_2 is $\mathbf{Z}$ and what little there is to be said can be justified by *ad hoc* arguments. □

The following proposition shows how any two band representations of a braid are related. The proof given is geometric; the algebraically-minded reader may supply an algebraic proof.

PROPOSITION 1.12. *Two band representations of β in B_n may always be joined by a finite chain in which adjacent band representations differ either by an elementary expansion or contraction or by a forward or backwards slide.*

Proof. Let $f: D^2 \to E_n$ be smooth and transverse to Δ. Then the natural (complex) orientations of D^2 and $\mathcal{R}(\Delta)$ give the finite set $f^{-1}(\Delta)$ an orientation – the sign of a point equals the sign of a corresponding band. Let $F: D^2 \times I \to E_n$ be a homotopy between two such maps $f_i = F(\cdot, i)$, $i = 0, 1$, with $F \mid \partial D^2 \times \{t\}$ independent of t, and F smooth and transverse to Δ in the interior of the solid cylinder $D^2 \times I$. Then the set $F^{-1}(\Delta)$ is a smooth 1-manifold-with-boundary in $D^2 \times I$, with $\partial(F^{-1}(\Delta)) = f_0^{-1}(\Delta) \cup f_1^{-1}(\Delta)$; and in fact $F^{-1}(\Delta)$ has a natural orientation for which, as a relative cycle, $\partial F^{-1}(\Delta) = -f_0^{-1}(\Delta) + f_1^{-1}(\Delta)$. After possibly a small perturbation we can assume that $pr_2 \mid F^{-1}(\Delta): F^{-1}(\Delta) \to I$ is a Morse function. For all but critical values $t_1, \ldots, t_N$, $F(\cdot, t): D^2 \to E_n$ gives a band representation of the braid $[f_0(\partial D^2)]$. The band representations just below and above a local minimum (resp., maximum) differ by an elementary expansion (resp., an elementary contraction). In an interval without critical points, F is an isotopy rel. Δ and the band representations at the ends of such an interval differ by a sequence of slides (slides really appear: it may not be possible, as it were, to choose a fixed normal form for the disks $D^2 \times \{t\}$ over the whole interval). $\square$

Note that $F^{-1}(\Delta) \subset D^2 \times I$ may well be knotted and linked. There is a homomorphism $\pi_1(D^2 \times I - F^{-1}(\Delta)) \to B_n$ which takes meridians to bands. In a picture, it can be helpful to label arcs of the diagram of $F^{-1}(\Delta)$ with names of bands.

EXAMPLE 1.13. Figure 1.1 shows the geometric equivalent of the following chain of band representations (as before, we simplify typography by writing i for σ_i):

$$\begin{aligned}(1, {}^{22}1) &\to (1, {}^{22}1, 2, 2^{-1}) \to (1, 2, {}^{2}1, 2^{-1}) \\ &\to (2, {}^{2^{-1}}1, {}^{2}1, 2^{-1}) \to (2, {}^{2^{-1}}1, {}^{21}2^{-1}, {}^{2}1) \\ &\to (2, {}^{2^{-1}1221}2^{-1}, {}^{2^{-1}}1, {}^{2}1) \to ({}^{2^{-1}}1, {}^{2}1).\end{aligned}$$

The last link in the chain depends on the calculation $2 \cdot {}^{2^{-1}1221}2^{-1} = \text{identity in } B_3$.

The reader may like to check that if a and b are any two bands satisfying $aba = bab$ (for instance, σ_i and σ_{i+1}) then the following chain corresponds to an arc knotted in a trefoil: $(a) \to (a, b^{-1}, b) \to ({}^{a}b^{-1}, a, b) \to ({}^{a}b^{-1}, b, {}^{b^{-1}}a) \to ({}^{ab^{-1}a^{-1}}b, {}^{a}b^{-1}, {}^{b^{-1}}a) \to ({}^{ab^{-1}a^{-1}}b) = (a)$ again.

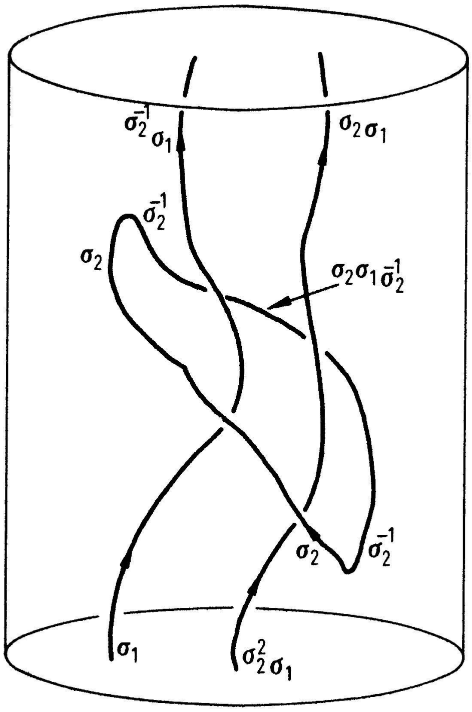

Figure 1.1

Because the relationship between different band representations of the same braid is of interest, the further study of the configurations $F^{-1}(\Delta)$ may be worth undertaking. In this regard one further construction may be mentioned here. For $n \geqq 3$ there is room in E_n to alter a homotopy F by surgeries, as follows. First, one may assume that $F(D^2 \times I)$ is an embedded 3-disk (i.e., F identifies only along intervals $\{z\} \times I$, $z \in \partial D^2$). Let L be any link in the interior of this 3-disk, disjoint from Δ. Then in E_n, L is the boundary of a collection of 2-disks which are pairwise disjoint and disjoint, except along L, from $F(D^2 \times I)$, and which are smoothly embedded transverse to Δ. Corresponding to any framing of any component L_i of L in the 3-disk there is an embedding of a bidisk $D_i^2 \times D^2$ in E_n so that $D_i^2 \times \{0\}$ is mapped to the 2-disk bounded by L_i and $\partial D_i^2 \times D^2$ with its product structure induces the given framing of L_i in the 3-disk, while $D_i^2 \times \partial D^2$ is transverse to Δ. Make a 3-manifold in E_n, with boundary equal to the 2-sphere $F(\partial(D^2 \times I))$, by removing the solid tori $\partial D_i^2 \times D^2$ from the 3-disk and replacing them with the solid tori $D_i^2 \times \partial D^2$; this 3-manifold is transverse to Δ and easily smoothed at its corners. In case L is a split link of trivial knots, each framed with ± 1, the new 3-manifold is again a 3-disk and a new homotopy has been created between the original pair of band representations f_0, f_1. It may be hoped that such surgeries, properly chosen, can replace general configurations $F^{-1}(\Delta)$ with ones that are special enough in some way to be more easily understood. For instance, crossings in a diagram of the link $F^{-1}(\Delta)$ can be reversed, at the expense (in general) of introducing new components (each component L_i of L will contribute a $(\pm 2, 2)$ torus link binding together the arcs whose crossing has switched sign). Remark 1.9 shows that what might be conceived to be the ultimate simplification is not always possible: we cannot assume that $F^{-1}(\Delta)$ is simply a braid (with respect to projection on Γ).

§2. Constructions of surfaces from band representations

The real content of this section, and the next, is in the pictures.

Figure 2.1 shows a surface of the type described in [S] (there named T_β) for the "homogeneous" braid word $\sigma_1\sigma_2^{-2}\sigma_1^3\sigma_2^{-1} \in B_3$. (Although the notation T_β would seem to suggest that the surface depends only on the braid, in fact the particular

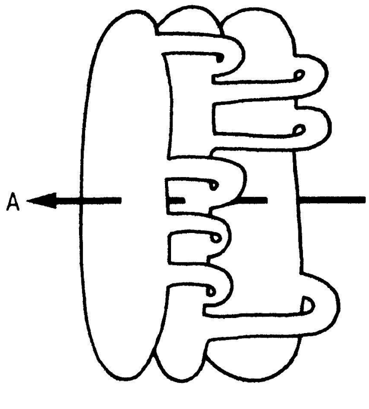

Figure 2.1

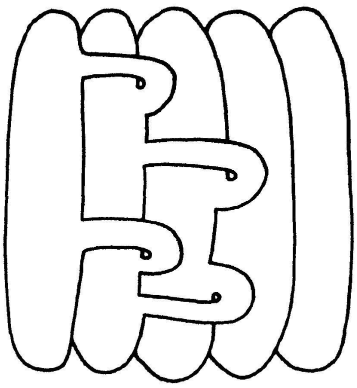

Figure 2.2

FOUR SURFACES $S(\vec{b})$

Figure 2.1. $\vec{b} = (\sigma_1, \sigma_2^{-1}, \sigma_2^{-1}, \sigma_1, \sigma_1, \sigma_1, \sigma_2^{-1})$ in B_3.
Figure 2.2. $\vec{b} = (\sigma_1, \sigma_2, \sigma_1, \sigma_2^{-1})$ in B_5.
Figure 2.3. $\vec{b} = (\sigma_{2,4}, \sigma_{1,2}, \sigma_{2,3}, \sigma_{3,4}, \sigma_{1,3})$ in B_5.
Figure 2.4. $\vec{b} = ({}^{\sigma_1\sigma_3}\sigma_2, {}^{\sigma_2^3}\sigma_1^{-1})$ in B_4.

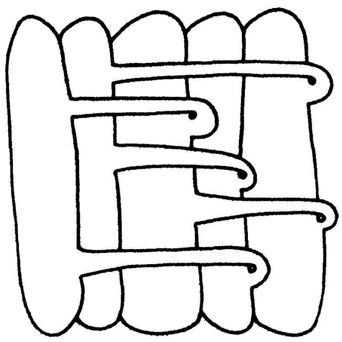

Figure 2.3

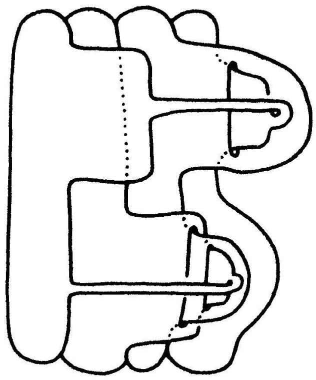

Figure 2.4

word is used to make the surface.) Instead of drawing the surface just as [S] would have it, with a twist (positive or negative according to the exponent of the corresponding letter in the braid word) to each band, I have preferred to give the bands half-curls: then, in Fox's expressive words [F, p. 151], "the resulting surface ... may be laid down flat on the table so that only one side of it is visible," whereas twists expose a bit of the back side.

Here is the procedure for making a surface according to a braid word (homogeneous or not): if the word represents an element of B_n and is of letter length k, the surface has an ordered handlebody decomposition $h_1^0 \cup \cdots \cup h_n^0 \cup h_1^1 \cup \cdots \cup h_k^1$; the 0-handles are embedded in $\mathbf{R}^3$ as planar cells, stacked in order in parallel planes; the 1-handles are attached (orientably) along the front edges of the 0-handles, in order; if the jth letter in the word is $\sigma_{i(j)}^{\varepsilon(j)}$, $\varepsilon(j) = \pm 1$, then the jth 1-handle connects $h_{i(j)}^0$ to $h_{i(j)+1}^0$; the half-curl is downwards (i.e., towards the next 1-handle) if $\varepsilon(j) = +1$ and upwards if $\varepsilon(j) = -1$. (The referee observes that this is really just Seifert's method of "Seifert circles" [F], applied to a natural oriented link diagram for the closure of the given braid word.)

Figure 2.2 illustrates the surface corresponding to the braid word $\sigma_1\sigma_2\sigma_1\sigma_2^{-1}$ considered as an element of B_5; just as including a braid in B_n (here, B_3) into the group B_{n+m} adds m trivial components to the link which is its closure, so does such an inclusion add disks to the constructed surface.

Somewhat more generally, if $\vec{b} = (b(1), \ldots, b(k))$ is an embedded band representation of $\beta = \beta(\vec{b})$ in B_n, then there is a Seifert surface for $\hat{\beta}$ made of n 0-handles connected by k 1-handles, where now the 1-handles may have to stretch across several intervening disks between their two ends.

It should be noted that while the surfaces constructed from braid words are all unknotted (that is, the fundamental group of the complement of the surface is free – as the referee remarks, this is always true for surfaces constructed by Seifert's procedure), this is not true of all surfaces constructed from embedded band representations; see Fig. 2.3, an annulus knotted in a trefoil, corresponding to the embedded band representation $(\sigma_{2,4}, \sigma_{1,2}, \sigma_{2,3}, \sigma_{3,4}, \sigma_{1,4})$ in B_5.

Now consider a general band representation $\vec{b}$. Make a choice, for each $j = 1, \ldots, k$, of a particular braid word $w(j)$ such that $b(j) = {}^{w(j)}\sigma_{i(j)}^{\pm 1}$. (One is actually also choosing $i(j)$.) Then, as in Fig. 2.4, where the process is applied to $({}^{\sigma_1\sigma_3}\sigma_2, {}^{\sigma_2^2}\sigma_1^{-1})$ with $w(1)$, $w(2)$ as written, a surface $h_1^0 \cup \cdots \cup h_n^0 \cup h_1^1 \cup \cdots \cup h_1^k$ whose boundary is the closure of $\beta(\vec{b})$ can be constructed; but now it is not embedded in $\mathbf{R}^3$, but rather immersed. The 0-handles have interpenetrated each other according to the braid words $w(j)w(j)^{-1}$. Each component of the singular set of the immersed surface is of the same type: an arc of transverse double-points, of which the preimage on the abstract surface consists of two arcs, one entirely interior to the surface and one with both its endpoints on the boundary (briefly, a *proper arc*). A surface with only such singularities is called a *ribbon surface* (in $\mathbf{R}^3$ or S^3), or a *ribbon immersion*. We have constructed a *Seifert ribbon* for the closed braid.

A band representation $\vec{b}$ of β in B_n thus gives various different Seifert ribbons for $\hat{\beta}$ (each one a ribbon immersion, conceivably an embedding, of the same abstract surface), differing according to specific ways of writing the bands. For our purposes there seems to be no need to distinguish these various ribbons, any one of which will therefore be denoted by $S(\vec{b})$.

Ribbon immersions in S^3 are related to the previously introduced ribbon surfaces in D^4 as follows (a detailed exposition has been written up by Joel Hass, [H]). Let $i: S \to S^3 = \partial D^4$ be a ribbon immersion. Then without changing i on ∂S, one may isotopically push i into D^4 so as to separate the double-arcs and produce an embedding $(S, \partial S) \subset (D^4, \partial D^4)$ which is a ribbon surface in the sense of §1; and every ribbon surface in D^4 arises in this way (from any one of many different ribbon immersions i).

We will also use the symbol $S(\vec{b})$ to denote such a pushed-in version of (any one of) the Seifert ribbons $S(\vec{b})$. On this interpretation, $S(\vec{b})$ is uniquely defined (up to isotopy), perhaps justifying the ambiguity in the other interpretation; we can see this by explicitly using the data of $\vec{b}$ alone (no choices of conjugators $w(j)$) to construct $S(\vec{b})$ in D^4. Figure 2.5 illustrates stages in such a construction of $S(\sigma_1, {}^{\sigma^3}\sigma_1)$. Figure 2.6 shows a Seifert ribbon $S(\vec{b})$ in S^3 adorned with representative level sets showing how to push the ribbon immersion into D^4.

This is the general construction: if $\vec{b} = (b(1), \ldots, b(k))$ is in B_n, of length k, think of an n-string (open) braid which changes in time, from the (constant) trivial braid at $t = 0$ to $\beta(\vec{b})$ at $t = 1$. In between there are k singular times, $0 < t_1 < \cdots < t_k < 1$; the interval $[0, 2\pi]$ which parametrizes the changing braid is also divided into subintervals, by values $0 = \theta_1 < \cdots < \theta_k < 2\pi$. Between $t = 0$ and $t = \frac{1}{2}(t_1 + t_2)$, the braid changes only in the θ-interval $\theta_1 < \theta < \theta_2$, in which before and after the singular time t_1 it moves by isotopies, passing at t_1 through a stage where a simple crossing (a point of order 4, like the center of an **X**) appears. Similarly, between $t = \frac{1}{2}(t_1 + t_2)$ and $t = \frac{1}{2}(t_2 + t_3)$, the braid changes only in the θ-interval $\theta_2 < \theta < \theta_3$, where it has a simple crossing when $t = t_2$; and so on. When this movie of a changing open braid is used to create a surface in the bidisk $D = \{(z, w) : |z| \leqq 1, |w| \leqq R\}$, by letting $z = t \exp i\theta$, the surface evidently is a braided surface, isotopic (*vide* Prop. 1.4) to a ribbon surface in D^4; and the boundary is (of the link type of) $\beta(\vec{b})$. We will use $S(\vec{b})$ also for the braided surface just constructed.

Remark 2.1. Of course, according to §1, $\vec{b}$ dictates an embedding $(D^2, \partial D^2) \to (E_n, E_n - \Delta)$ transverse to Δ, and this embedding in turn gives a

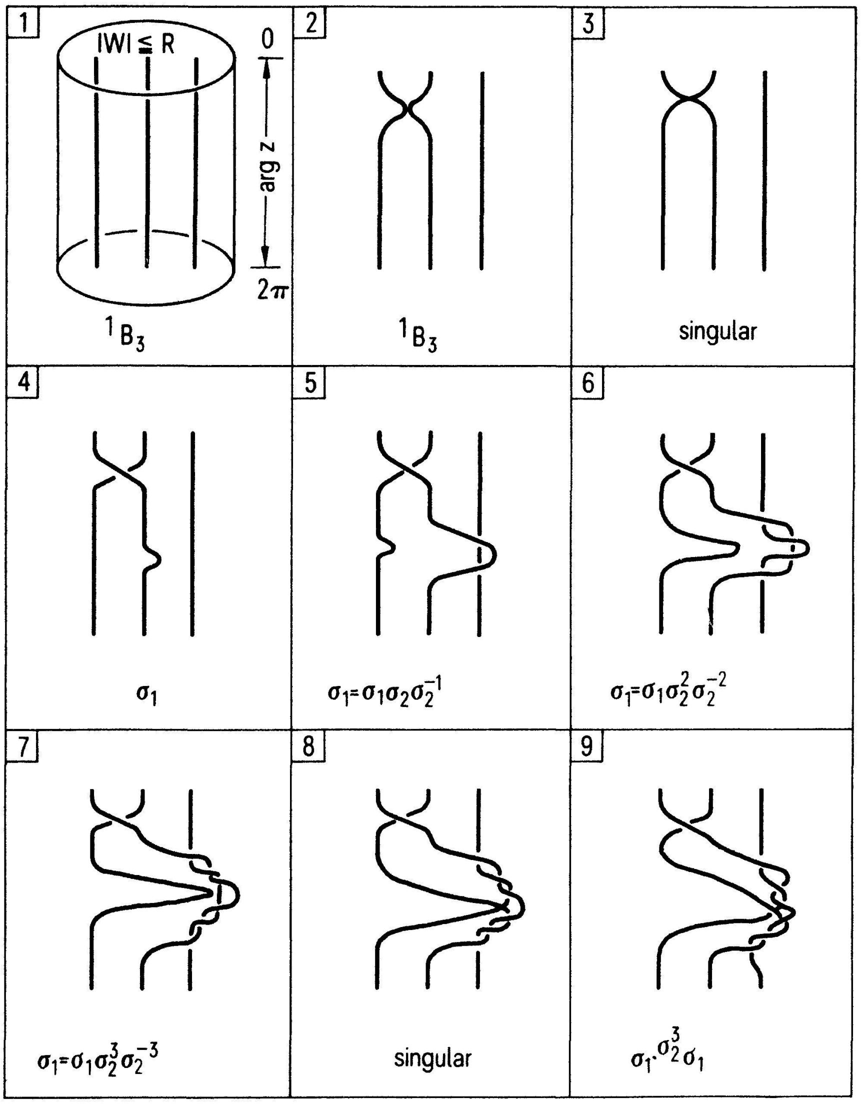

Figure 2.5. A movie of the construction of $S(\sigma_1, {}^{\sigma_2^3}\sigma_1)$

braided surface in D: it should be no surprise that this surface is none other than $S(\vec{b})$. It is hoped, however, that the pictorial approach taken has been of some help in understanding this situation.

Remark 2.2. The algebraic moves of §1 can now be interpreted geometrically. "Slides" on the level of band representations correspond to handle-slides of the surfaces $S(\vec{b}) \subset D^4$ with their ordered handlebody decompositions; thus, slide-equivalent band representations of β in B_n produce surfaces $S(\vec{b})$, $S(\vec{b}')$ which are isotopic in D^4 (but generally not through a level-preserving isotopy). An elementary expansion of $\vec{b}$ corresponds to adding a (hollow) handle to $S(\vec{b})$, either joining two components by a trivial tube $S^1 \times I$ or taking the connected sum of one component with a trivial torus $S^1 \times S^1$, in D^4 (the cases corresponding to whether or not the pair of inverse bands in question have a permutation that links

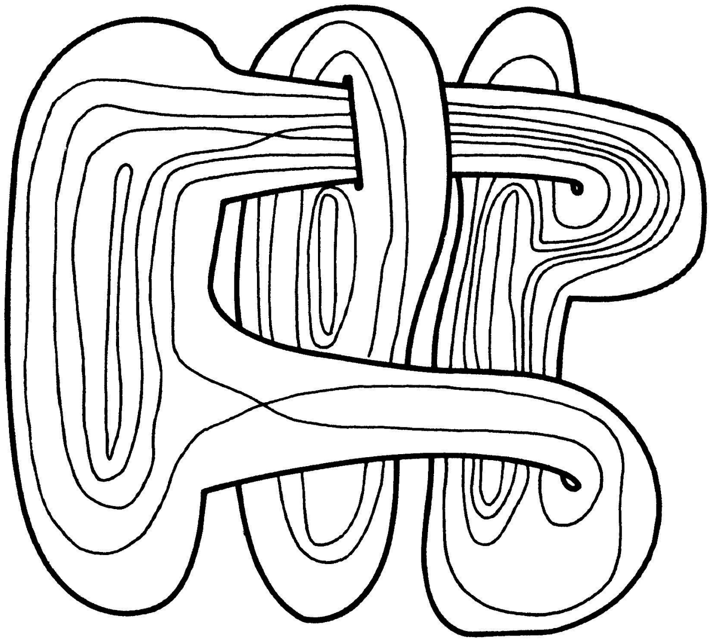

Figure 2.6.

previously disjoint cycles); an elementary contraction, when possible, corresponds to removing such a trivial tube or torus.

§3. The construction is general

We will show that every orientable ribbon surface is (isotopic to) some $S(\vec{b})$, $\vec{b}$ a band representation. We will do this on the interpretation of "ribbon surface" as "ribbon immersion in $\mathbf{R}^3 \subset S^3$"; the isotopy will be ambient isotopy. The proof is in two steps. First we show that every ribbon immersion "may be laid down flat on the table." Then we show how to move any such tabled surface around until it is an $S(\vec{b})$. As before, the text is secondary to the pictures.

Let us say that a surface immersed in $\mathbf{R}^3$ is *tabled* if it is oriented, and we have an oriented 2-plane (the table) so that orthogonal projection from the surface to the plane is an orientation-preserving immersion. In [F], Fox attributes to Seifert essentially the following procedure for finding a tabled surface in the isotopy class of a given embedded surface (oriented and without closed components, necessarily), S. There is a handlebody decomposition of S with n 0-handles h_i^0, k 1-handles h_j^1, and no 2-handles; and the 1-handles are attached orientably. (We might of course require that n be the number of components of S, but we don't have to; and when we come to the case of immersions this won't be possible.) Let $T \subset \mathbf{R}^3$ be an oriented 2-plane. By isotopy of S in $\mathbf{R}^3$, we may make each h_i^0 a 2-cell lying in a translate T_i of T, bearing the proper orientation there; and we can assume that the projections of these 0-handles into T are pairwise disjoint. Now by isotopy arrange that the core arcs of the h_j^1 project into T in general position, and with no points except their endpoints in the images of the h_i^0. Shrink each h_j^1 down to a narrow band around its core arc; then, without loss of generality, the projection of h_j^1 identifies some number of transverse arcs to points, and is otherwise an immersion, alternately preserving and reversing orientation in the regions between the transverse arcs – that is, h_j^1 is twisted (as seen from T). Also, of course, h_j^1 may be knotted, and the various 1-handles may link each other, too. As far as twisting goes, however, since the 0-handles already projected orientably and S is oriented, each h_j^1 has an even number of twists; and by further isotopy "these twists can be replaced by curls (just half as many curls as twists)" ([F, p. 151]). When the twists are all out, the surface is tabled.

Figure 3.1 illustrates this procedure as applied to a particular Seifert surface for the figure-8 knot, without regard to economy in the number of handles.

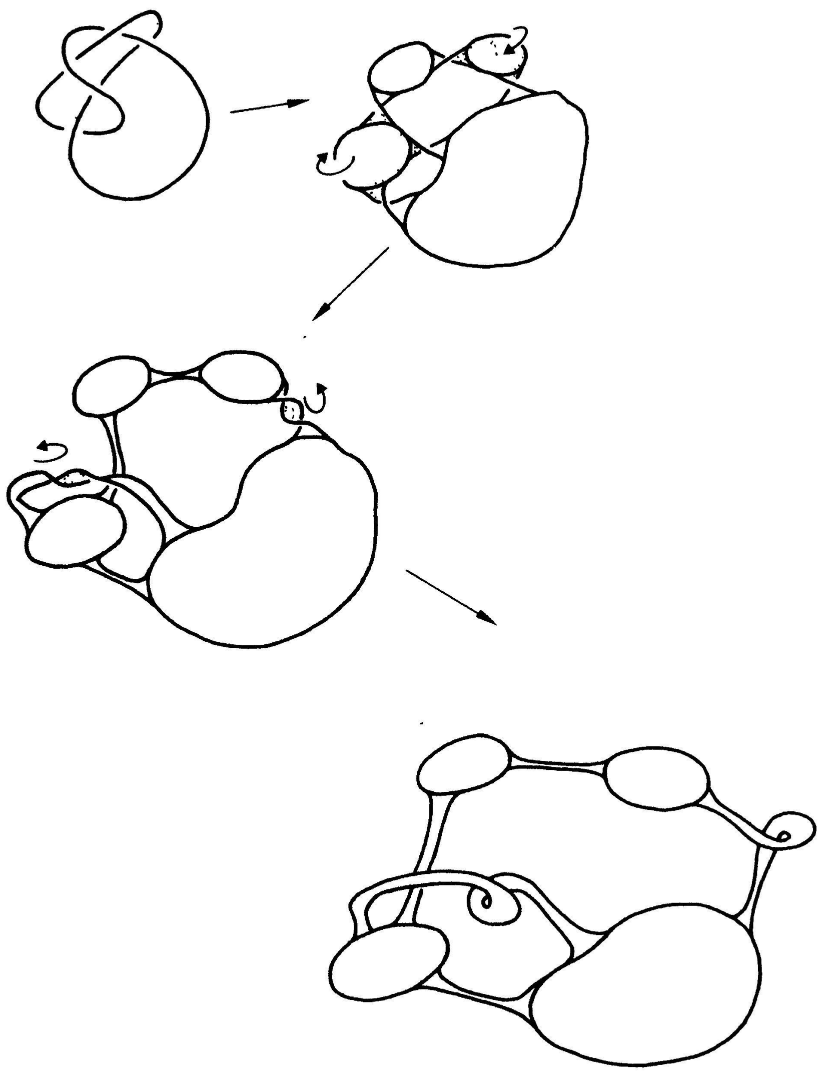

Figure 3.1. Tabling an embedded surface by twisting handles.

Now suppose that we begin with a surface which is not embedded, but is ribbon immersed by $i:S\to\mathbf{R}^3$, where on $i(S)$ the double-arcs are A_m ($m=1,\ldots,s$), and $i^{-1}(A_m)$ is the disjoint union of the proper arc A'_m and the arc $A''_m\subset \operatorname{Int} S$. It is easy to find a set A^*_m ($m=1,\ldots,s$) of proper arcs on S, pairwise disjoint and disjoint from all the A'_p, such that for each m, $A''_m\subset A^*_m$. Then there is a handlebody decomposition of S which includes among its 0-handles a neighborhood on S of each proper arc A'_m and A^*_m, and which has no 2-handles. (As always, S is oriented and without closed components.) It is now possible practically to mimic Seifert's procedure with $i(S)$, except of course that the 0-handles containing A^*_m and A'_m will not have disjoint images in T, and cannot both lie in planes parallel to T. Let us always take $A_m\subset i(S)$ to be actually a straight line segment, parallel to T; then of the two immersed 0-handles containing it, one can be taken to lie parallel to T, and the other to lie in another plane parallel to T except for a narrow tab which passes through A_m. (Note that to have both the 0-handles project orientably to T, one may have to "pivot" one of them about A_m.) Figure 3.2 illustrates this, for a particular ribbon immersion of a disk – the boundary being a stevedore's knot.

Returning to surfaces $S(\vec{b})$ for a moment, we see that they are of course tabled (as pictured in §2) – both from the point of view of the plane of the paper, and from the tilted plane perpendicular to the axis of the closed braid $\partial S(\vec{b})=\hat{\beta}$, in which perspective the 0-handles greatly overlap each other. So our second task is to take our ribbon surface, already assumed tabled, and isotope it until it has become an $S(\vec{b})$. First, skewer all the 0-handles; that is, pick an axis A perpendicular to T, and by isotopy of S through tabled surfaces arrange the 0-handles so that each one intersects A in its own plane (in the case of the 0-handles with tabs, let us make the intersection fall in the planar part, not in the tab). Now pick rectangular coordinates in T, and for reference a rectangle-with-rounded-corners R in T, its sides parallel to the axes, which we will call horizontal and vertical. Let one of the vertical sides of R be called its front edge. By further isotopy of S, we may so arrange the 0-handles so that each one projects either onto exactly R (if there is no double-arc in that 0-handle) or onto R suitably enlarged along the front edge (by a larger or smaller tab), and so that the double-arcs are vertical segments projecting outside R (past its front edge). Next we may arrange the 1-handles (if necessary, sliding their attaching maps along the boundaries of the 0-handles) so that: they attach only along the front edges of the projections (including front edges of tabs); so that their projections are neighborhoods of polygonal arcs composed solely of horizontal and vertical segments; and so that in the resulting "link diagram" of core arcs, each over-arc is a horizontal segment. (We always assume general position, so it also is assured that the $2k$ endpoints of the k 1-handles' core arcs have $2k$ distinct vertical coordinates; let also the

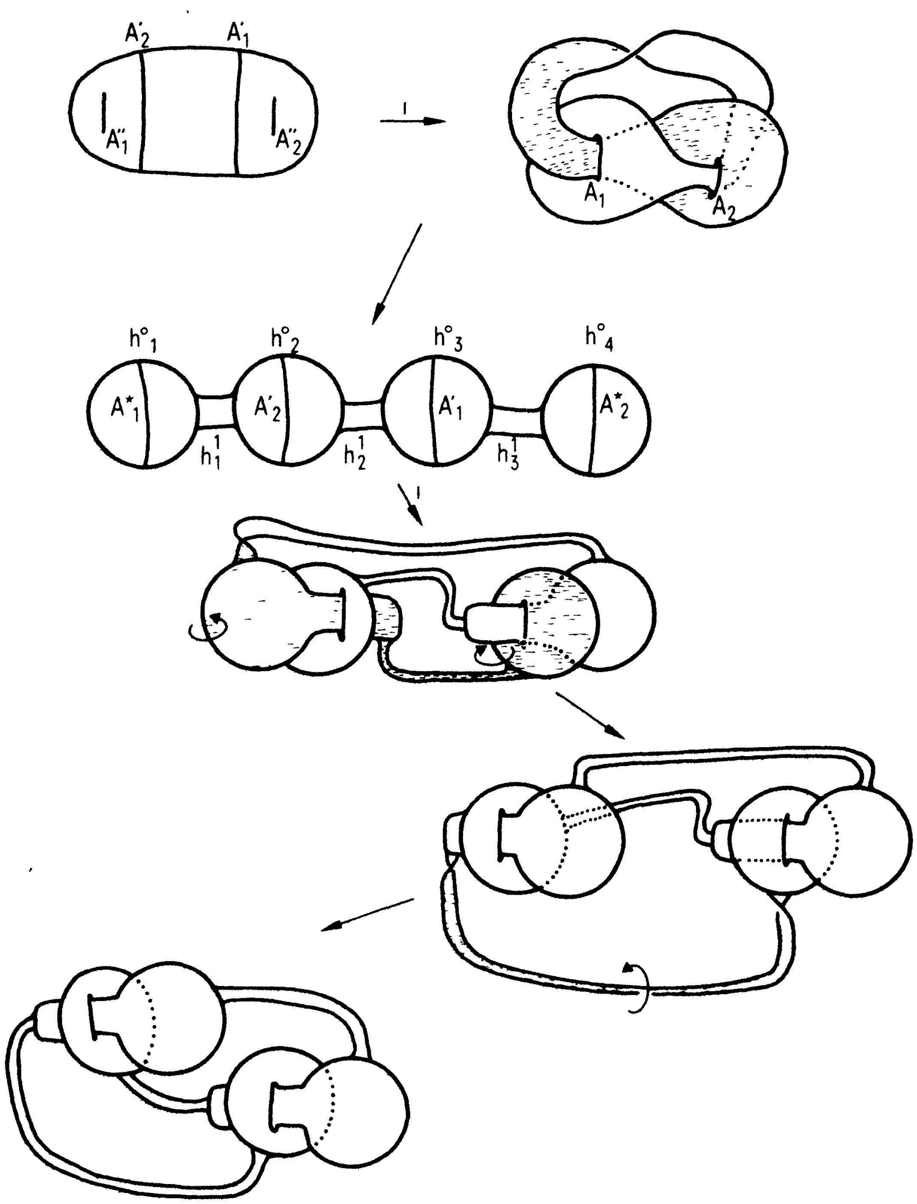

Figure 3.2. Tabling a ribbon immersed surface by twisting handles.

1-handles be sufficiently narrow that all attaching takes place inside $2k$ disjoint intervals.) This is illustrated in Fig. 3.3, continuing the example of the stevedore's knot.

We are nearly done now. One by one, vertical parts of the bands may be expanded into full-fledged 0-handles and these 0-handles slipped into the stack impaled by A – the adjacent horizontal segments, if they approach from the left, being given half-curls to allow the attachment to stay within the realm of tabled surfaces with all bands attached along front edges. When no vertical parts are left, the resulting surface is of the form $S(\vec{b})$, where $\beta(\vec{b})$ is a braid on some large number of strings (n plus the number of vertical segments).

Remark 3.1. Each band in such a band representation $\vec{b}$ is actually of the form ${}^{w}\sigma_j^{\pm 1}$ where for some $i \leqq j$ either $w = \sigma_i\sigma_{i+1} \cdots \sigma_{j-1}$ or $w = \sigma_i\sigma_{i+1} \cdots \sigma_{j-1}^{-1}$ – it is either embedded or has a single double-arc but goes directly from h_i^0 to h_{j+1}^0.

Figure 3.4 shows how this last part of the construction was used to make the surface in Fig. 2.3, beginning with an annulus knotted in a trefoil (already tabled); and Fig. 3.5 finishes the stevedore's knot with its ribbon disk.

Acknowledgement. The conviction that every ribbon surface should arise as $S(\vec{b})$ for some $\vec{b}$ came upon the author in 1978, after experimentation with cardboard models. It was some time before the idea of using, essentially, link diagrams with only vertical and horizontal segments in them, and every over-crossing horizontal, was incorporated into a proof. And it was only much later that the author remembered having first heard of such a construction at the October, 1977, topology conference in Blacksburg, Va. (at VPI&SU) from Herbert Lyon, in whose hands the construction was used to show that every (embedded, orientable) surface in S^3, without closed components, is a subsurface

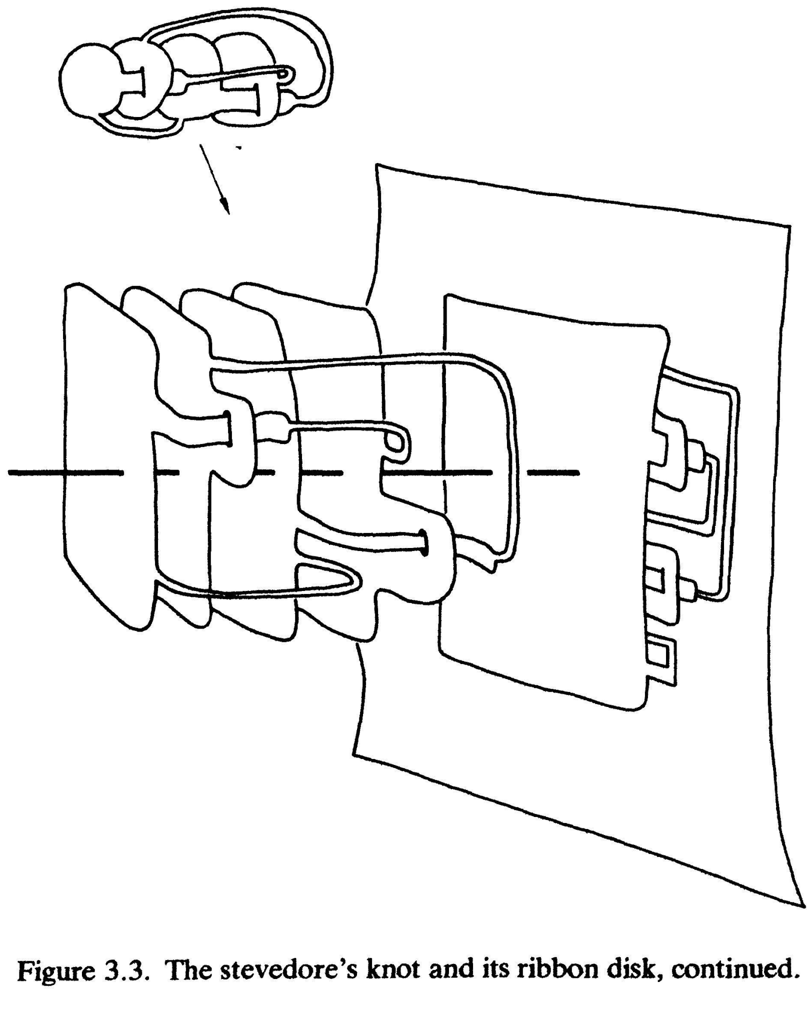

Figure 3.3. The stevedore's knot and its ribbon disk, continued.

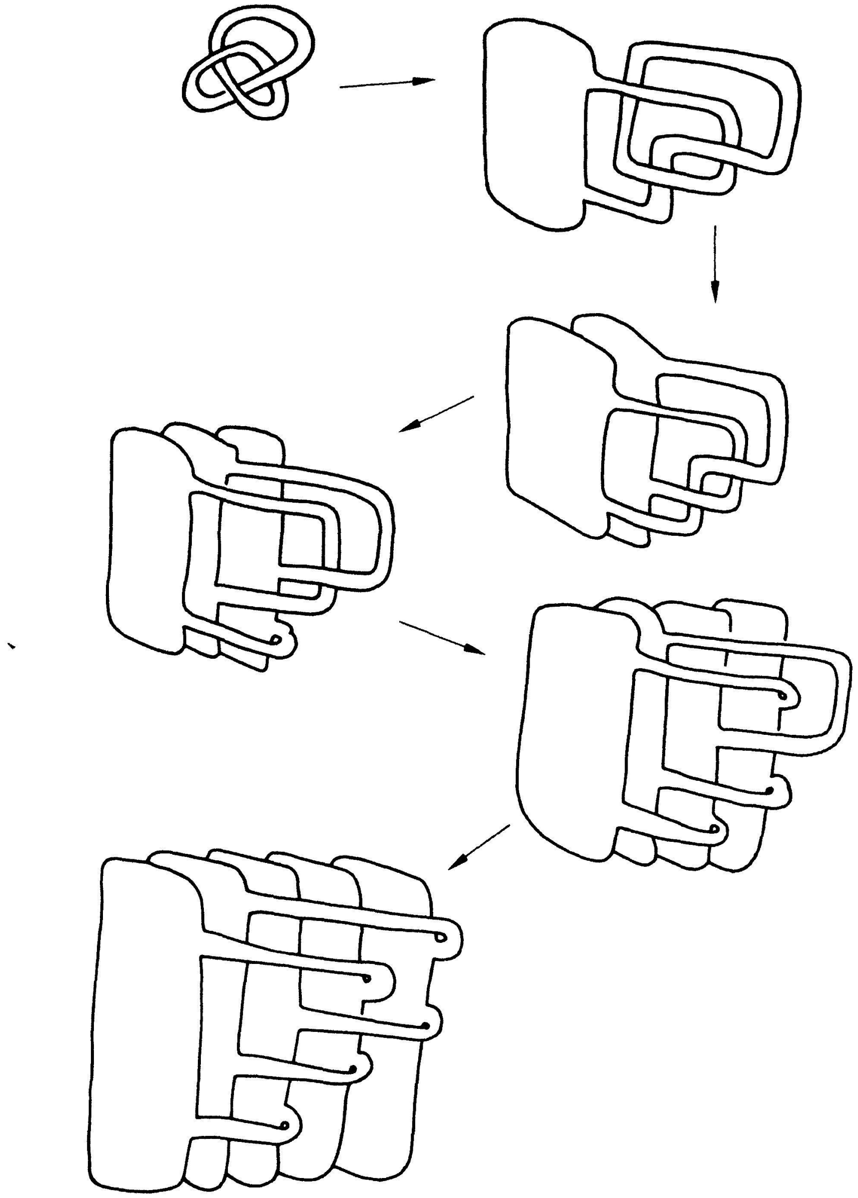

Figure 3.4. Thickening vertical parts of 1-handles into 0-handles.

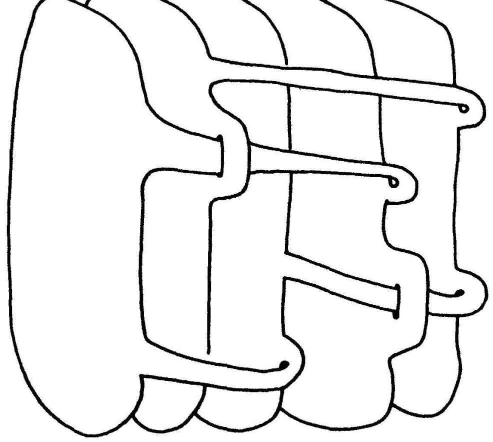

Figure 3.5. The ribbon disk bounded by a stevedore's knot, concluded.

of a fibre surface of some fibred knot, [L]. The unconscious memory of Professor Lyon's talk was undoubtedly an important ingredient in the genesis of the author's proof.

The fact that every ribbon surface appears as $S(\vec{b})$ for some $\vec{b}$ has some immediate consequences which may be noted here.

PROPOSITION 3.2. *Every (orientable) ribbon surface in the 4-disk is isotopic to a braided surface in the bidisk.* □

As stated in Remark 1.5, I don't know a more direct proof of this.

PROPOSITION 3.3 (Alexander). *Every link can be represented as a closed braid.* □

We might say that Proposition 3.3 is the boundary of Proposition 3.2; of course it relies on the existence of Seifert surfaces for every link.

Further consequences will be reserved to the next section.

§4. The fundamental group $\pi_1(D - S(\vec{b}))$

A *Wirtinger presentation* of a group G is a presentation $G = (x_1, \ldots, x_n : x_{i(r)} = {}^{w(r)}x_{j(r)},\ r = 1, \ldots, k)$, in which each $w(r)$ is a word in $x_1, \ldots, x_n$; a group with a Wirtinger presentation is a *Wirtinger group.* A *special* Wirtinger presentation is one in which each conjugator $w(r)$ is actually one of the generators, $x_{m(r)}$. It is clear that any Wirtinger group has a special Wirtinger presentation.

That any link group $\pi_1(S^3 - L)$ (for L tame) is Wirtinger is classical (presumably due to Wirtinger); and indeed that Wirtinger presentation which is written down in the usual way from inspection of a link diagram is special. Then Fox's method of cross-sections (for instance), or, indeed, Morse theory relative to the submanifold X, shows that any group $\pi_1(S^N - X)$, X a smooth orientable submanifold of codimension 2, is Wirtinger. Not every Wirtinger group appears as a link group in S^3. But the following is true.

PROPOSITION 4.1. (Yajima [Y], Johnson [J]). *If G is a Wirtinger group, then there is a smooth, orientable surface $S \subset S^4$ with $\pi_1(S^4 - S) \cong G$; and S may be taken to be the double of a ribbon surface in the 4-disk D^4.*

(The papers of Yajima and Johnson were pointed out to me by Professor Jonathan Simon; Johnson's proof, obtained independently of Yajima, introduces the ribbon refinement; the proof to be given here uses the formalism of band representations to render Johnson's construction by "band moves" even more perspicuous.)

Proof. First, let us derive a (Wirtinger) presentation for $\pi_1(D-S(\vec{b}))$ from $\vec{b}$. If $\vec{b}$ is in B_n, of length k, there will be n generators and k relations. Thinking of $S(\vec{b})$ as a closed braid changing in time from the (constant) trivial braid to $\hat{\beta}(\vec{b})$, one can identify the generators $x_1, \ldots, x_n$ as standard meridians at any one of the stages; the relations appear at the singular stages, and as in [F, p. 133] each relation takes the form "two meridians are equal" (not, of course, necessarily standard meridians). More explicitly: recall that there is a (faithful) representation of B_n as a group of left automorphisms of the free group $F_n = (x_1, \ldots, x_n :)$, given on generators by $\sigma_i x_i = {}^{x_i}x_{i+1}$, $\sigma_i x_{i+1} = x_i$, $\sigma_i x_j = x_j$ $(j \neq i,\ i+1)$; and that, if $\gamma \in B_n$ and the open geometric braid $K \subset D^2 \times I$ represents γ, then in terms of the standard meridians $x_1, \ldots, x_n$ of K in $D^2 \times \{0\}$, the standard meridians of K in $D^2 \times \{1\}$ (taken in the same order) are $\gamma x_1, \ldots, \gamma x_n$. (This is readily checked graphically; or see [Bi] where, however, right automorphisms are used.) Let $\vec{b} = (b(1), \ldots, b(k))$, $b(j) = {}^{w(j)}\sigma_{i(j)}^{\pm 1}$; then the j^{th} stage contributes the relation $w(j)x_{i(j)} = w(j)x_{i(j)+1}$. Noting that for any braid w and any x_i, wx_i is a conjugate of some x_j (true by inspection for w a generator, then generally true by induction), we see that this relation can be rewritten in Wirtinger form.

Consider two particular types of bands. An embedded band $\sigma_{i,j}^{\pm 1}$ contributes the relation $x_i = x_{j+1}$. A band of the form ${}^{w}\sigma_j^{\pm 1}$, with $w = (\prod_{m=i}^{j-2} \sigma_m)\sigma_{j-1}^{-1}$, contributes the relation $x_i = {}^{x_i}x_{j+1}$. According to Remark 3.1, every orientable ribbon surface $S \subset D^4$ can be constructed as $S(\vec{b})$ for some band representation with bands only of those two types; the corresponding presentation is special Wirtinger. Conversely, given any special Wirtinger presentation of a group G, after possibly adding new generators set equal to old ones, we can assume that each relation is of one of the two forms $x_i = x_{j+1}$, $x_i = {}^{x_i}x_{j+1}$; and it is easy to find a band representation with that as the corresponding presentation.

So every Wirtinger group appears as $\pi_1(D^4 - S(\vec{b}))$ for some $\vec{b}$. By Morse theory, because $S(\vec{b})$ is ribbon, the homomorphism $\pi_1(S^3 - \hat{\beta}(\vec{b})) \to \pi_1(D^4 - S(\vec{b}))$ is onto. Now, by van Kampen's theorem, the groups $\pi_1(D^4 - S(\vec{b}))$ and $\pi_1(S^4 - 2S(\vec{b}))$, where $2S(\vec{b}))$ is the double of $S(\vec{b})$ in S^4 (the double of D^4), are isomorphic. □

EXAMPLE 4.2. Let $\vec{b} = (\sigma_1, {}^{\sigma_2^3}\sigma_1^{-1})$ in B_3; then $\hat{\beta}(\vec{b})$ is a square knot, $S(\vec{b})$ is a ribbon disk, and it is readily checked that $\pi_1(D - S(\vec{b})) = (x_1, x_2, x_3:$

$x_1 = x_2,\ x_1 = {}^{x_2 x_3 x_2}x_3) = (x, y : xyx = yxy)$, the group of the trefoil knot. In fact, in this case the double of the disk pair $(S(\vec{b}), D)$ is the spun trefoil in S^4.

EXAMPLE 4.3. Here is the example, using a knotted 2-sphere, promised in Remark 1.10 to show the subtle structure of the set of band representations of a given braid. As before, let 1, 2, 3 abbreviate σ_1, σ_2, σ_3 respectively; and let $\bar{x}$ abbreviate x^{-1}. Then $\beta = \beta(3, {}^{\overline{222}}\bar{1}) \in B_4$ closes to a split link with two unknotted components, and evidently $rk(\beta) = 2$ since β is not the trivial braid. One calculates $\pi_1(D - S(3, {}^{\overline{222}}\bar{1})) = (x_2, x_3 : \ )$. If b is any band in B_4, the elementary expansion $(3, {}^{\overline{222}}\bar{1}, b, \bar{b})$ gives a presentation $\pi_1(D - S(3, {}^{\overline{222}}\bar{1}, b, \bar{b})) = (x_2, x_3 : r(x_2, x_3))$ with (at most) one new relation, since b and $\bar{b}$ give rise to the same relation according to Theorem 4.1. Then any band representation of β of length 4, which is slide equivalent to an elementary expansion of $(3, {}^{\overline{222}}\bar{1})$, gives a presentation of the same form.

On the other hand, consider the band representation $\vec{b} = ({}^{2}3, {}^{\overline{22}}\bar{3}, {}^{3\bar{2}}1, {}^{1\bar{2}}\bar{1})$. Its conjugate ${}^{w}\vec{b}$, $w = 3\bar{2}12333$, has braid equal to β (we use $\vec{b}$ for ease of computation). We calculate $\pi_1(D - S(\vec{b})) = (x_1, x_2, x_3, x_4 : x_2 = x_4, {}^{\overline{x_3 x_2}}x_3 = x_4, x_1 = {}^{x_3}x_4, {}^{x_1}x_2 = x_3) = (x_2, x_3 : x_2 x_3 x_2 = x_3 x_2 x_3, [x_2^2, x_3] = 1)$. This is the group of the 2-twist spun trefoil, and in fact when we cap off the two trivial components of the closed braid the 2-twist spun trefoil is the knotted 2-sphere we get. In any case, this is not a one-relator group, so the length-4 band representation ${}^{w}\vec{b}$ of $\beta(3, {}^{\overline{222}}\bar{1})$ cannot be slide equivalent to an elementary expansion of $(3, {}^{\overline{222}}\bar{1})$. (Presumably two elementary expansions, some sliding, and one elementary contradiction suffice to connect the two slide-equivalence classes, but I have not checked this.)

Remark 4.4. The Wirtinger presentation of $\pi_1(D - S(\vec{b}))$ derived in Proposition 4.1 evidently takes no notice of the signs of the bands in $\vec{b}$, so every Wirtinger group arises as $\pi_1(D - S(\vec{b}))$ for some quasipositive band representation $\vec{b}$. As shown in [Ru], if $\Gamma \subset D \subset \mathbf{C}^2$ is a piece of complex-analytic curve with $\partial\Gamma = \Gamma \cap \partial_1 D$ (transverse intersection), then $\partial\Gamma = \hat{\beta}$ is a closed quasipositive braid, and conversely every quasipositive braid arises in this manner. Examining the proof given there in the light of this paper, one sees that in fact (up to isotopy) such pieces of (non-singular) complex-analytic curves are precisely the surfaces $S(\vec{b})$ for $\vec{b}$ quasipositive.

We can use this to produce a Stein manifold $M \subset \mathbf{C}^N$ with fundamental group G, for any Wirtinger group G. In fact, find a quasipositive band representation $\vec{b}$ with $\pi_1(D - S(\vec{b})) = G$; realize $S(\vec{b})$ as a piece of complex-analytic curve (non-singular, and extendible to a slightly larger bidisk) in D. By the solvability of the Cousin problem for the bidisk (cf. [G−R]), there is a holomorphic function $f(z, w)$ in (a neighborhood of) D, of which the zero-set in D is precisely $S(\vec{b})$. Also, the 2-disk $\mathring{D}^2 \subset \mathbf{C}$ may be embedded as a Stein submanifold of some $\mathbf{C}^N$

(and actually $N=2$ will do), by a proper analytic embedding $g:\mathring{D}^2 \to \mathbf{C}^N$. Then $(z, w) \mapsto (g(z), g(w), 1/f(z, w))$ is a proper analytic embedding of $\mathring{D} - S(\vec{b})$ onto a Stein submanifold $M \subset \mathbf{C}^{2N+1}$. (In fact, by Forster [Fr], if $N = 2$ or 3, M must be an analytic complete intersection, since it is parallelizable.)

SCHOLIUM. *There are finite homotopy types which can be realized as Stein manifolds but not as non-singular affine algebraic varieties.*

For John Morgan, using Hodge theory, has shown that, for instance, the group $G=(x, y:1=[x,[x,[x,[x,y]]]])$ is not the fundamental group of any non-singular algebraic variety (affine or not), [Mo]. Yet G has the Wirtinger presentation $G=(x, y, s, t, u, v, w: s={}^yx, x={}^st,\ v={}^xt, x={}^vu, w={}^xu, x={}^wx)$. (To see this, one uses repeatedly that in any group ${}^c[a, b]=[{}^ca, {}^cb]$, and $[a, b]=1$ iff $[a, b^{-1}]=1$.)

Of course, there are infinite homotopy types among the Stein manifolds; for instance, any open subset of $\mathbf{C}$ (e.g., the complement of the integers) is a Stein manifold, [G−R]. (The analogue in $\mathbf{C}^n$, $n>1$, is naturally quite false.)

The abelianization of a Wirtinger group is free abelian, so there are certainly finitely presented non-Wirtinger groups, and some of these appear as fundamental groups of Stein manifolds, indeed of algebraic varieties. Is it possible that every finitely presented group appears as the fundamental group of a Stein manifold? Given a finite presentation, it is easy to construct various (open) complex manifolds of complex dimension 2 with G as the fundamental group, but it is not at all clear how to make such a construction yield Stein manifolds.

§5. Rank and ribbon genus

Recall that $rk_n(\beta)$, for $\beta \in B_n$, is the least k such that some band representation of β in B_n has length k. Call such a shortest band representation *minimal in* B_n.

Recall also the definition of the *ribbon genus* of a knot or link, $L \subset S^3 = \partial D^4$. Every such L is, of course, the boundary of various connected orientable smooth surfaces $S \subset \partial D^4$. The ribbon genus $g_r(L)$ is the least integer that appears as the genus of such a surface which is ribbon embedded in D^4; clearly we have $g(L) \geqq g_r(L) \geqq g_s(L)$, where the (classical) *genus* $g(L)$ restricts the surfaces over which the minimum is taken to those actually in S^3, and the *slice genus* $g_s(L)$ makes no restrictions.

The genus is quite a classical invariant; ribbon genus and slice genus are of more recent interest; both have been under study by some quite high-powered methods, cf. Gilmer [G1, G2]. As band representations and ribbon surfaces are so closely related, there might be some hope that the more naive methods of this paper would be relevant to the study of g_r. This section presents some observations on the beginnings of such a program.

It should be noted that (because of the requirement that the surfaces involved be connected) g, g_r, and g_s all are most satisfactory when applied to links which can bound connected surfaces (without closed components) only: for instance, knots, or more generally links in which any two distinct components have non-zero linking number.

The following Proposition is an immediate consequence of the construction in §3, applied to any Seifert ribbon for L which happens to have genus $g_r(L)$.

PROPOSITION 5.1. *Let L be a link for which every Seifert ribbon is connected. Then for some n and some braid $\beta \in B_n$ with $\hat{\beta} = L$, we have $g_r(L) = \frac{1}{2}(2 - n + rk_n(\beta) - c(\beta))$ (where $c(\beta)$ is the number of cycles in the permutation of β; or equivalently the number of components of L).* □

It would be nice if the quantity $\frac{1}{2}(2 - n + rk_n(\beta) - c(\beta))$ always computed $g_r(\hat{\beta})$. This, alas, is not the case, as Professor Andrew Casson pointed out to me. Example 5.3. is due to him. (Below, i abbreviates σ_i, and $\bar{i}$ abbreviates σ_i^{-1}.)

EXAMPLE 5.2. In B_4, consider $\beta = \overline{33}2\overline{3}211\overline{2}1\overline{2}$. Then $\hat{\beta}$ is a split link of two unknotted circles. (The reader familiar with Markov moves may verify this by first increasing the string index to five by the move $\beta \to \overline{343}2\overline{3}211\overline{2}1\overline{2}$; conjugating this braid to get $\overline{3}2\overline{343}211\overline{2}1\overline{2}$; reducing the string index to four by moving back to $\overline{3}2\overline{33}211\overline{2}1\overline{2}$; and then by a fairly straightforward series of conjugations and reductions in string index, proceeding to the identity in B_2, which certainly has the closure advertised. Alternatively, experiments with string, or pencil and eraser, may give the result more quickly.) So $\hat{\beta}$ bounds a pair of disjoint disks. If there were a band representation of β in B_4 of which the associated ribbon surface was a pair of disks – even ribbon disks – then it would have to have length 2, and (since β has exponent sum 0) it would have one positive and one negative band.

But in fact β is not the product of two bands in B_4. We check this as follows. There is a homomorphism ϕ of B_4 onto $SL(2, \mathbf{Z})$, given by

$$\phi(\sigma_1) = \phi(\sigma_3) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \phi(\sigma_2) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

The image of β is found to be $\begin{bmatrix} 49 & 30 \\ -18 & -11 \end{bmatrix}$. One also finds that the general form of the image of a band is $\begin{bmatrix} 1-ac & \pm a^2 \\ \mp c^2 & 1+ac \end{bmatrix}$, where a and c are coprime integers; the upper (lower) sign corresponding to a positive (negative) band. Then, up to conjugation in $SL(2, \mathbf{Z})$, a product of one positive and one negative band takes the form

$$\begin{bmatrix} 1-ac+c^2 & -1-ac+a^2 \\ -c^2 & 1+ac \end{bmatrix}.$$

This has trace $2+c^2$, so if it is conjugate to $\phi(\beta)$ we must have $c^2=36$. But let a unimodular integral matrix $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ conjugate $\begin{bmatrix} 49 & 30 \\ -18 & -11 \end{bmatrix}$; the lower left hand corner of the conjugate is $-18w^2+60wz-30z^2$. Yet $-18w^2+60wz-30z^2=-36$ can have no integral solutions (z, w), for dividing it by 6 and taking both sides modulo 5 yields $2w^2 \equiv -1 \pmod 5$, which is impossible.

Although β is not a product of two bands in B_4, the product ${}^{\bar{3}}\bar{4} \cdot \beta \in B_5$ is a product of three bands in B_5, namely ${}^{\bar{3}}\bar{4} \cdot \beta = {}^{\overline{43}2\bar{3}}\bar{4} \cdot {}^{\overline{4332}}1 \cdot {}^{\bar{4}32}\bar{1}$. It would be interesting to know whether β *considered as an element of* B_5 is a product *of two bands in* B_5. If it were, we would have here an example in which the rank of a braid decreases when the braid is considered to lie in a braid group of larger string index.

Now, β when considered as an element of B_5 has closure a split link of three unknotted components. If $rk_5(\hat{\beta})=2$, it is at least reasonable to suppose that among the minimal band representations of β in B_5, some at least correspond to the Seifert ribbon for $\hat{\beta}$ which consists of three unknotted disks. But if such a band representation of length 2 does exist, it still will not be possible to get from it to the band representation of length 4 given by $\beta = {}^{\bar{3}}4 \cdot {}^{\overline{43}2\bar{3}}\bar{4} \cdot {}^{\overline{4332}}1 \cdot {}^{\bar{4}32}\bar{1}$ simply by inserting a pair of cancelling bands and sliding: this can be shown by an argument like that in Example 4.3, comparing the fundamental groups of the complements of the surfaces corresponding to the different band representations.

EXAMPLE 5.3. (Casson). In B_3, let $\gamma=(1\bar{2})^5$. Then $\hat{\gamma}$ is a ribbon knot, but $rk_3(\gamma) \geqq 4$; so that $\frac{1}{2}(2-n+rk_n(\gamma)-c(\gamma)) = \frac{1}{2}(-2+rk_3(\gamma)) \geqq 1 > 0 = g_r(\hat{\gamma})$.

To see that $\hat{\gamma}$ is ribbon, we consider $\gamma_1 = {}^{1\bar{2}1\bar{2}}\bar{3} \cdot \gamma \in B_4$, which has the same closure, and observe that the equation $\gamma_1 = {}^{12}\bar{3} \cdot \beta$ (where β is as in Example 5.2) displays $\hat{\gamma}_1$ as the boundary of a ribbon disk made from the two disks bounded by $\hat{\beta}$ and a single band joining them.

To see $rk_3(\gamma) > 2$ (whence it must be at least 4), we represent B_3 in $SL(2, \mathbf{Z})$ and make an argument similar to that above; details are left to the reader.

In an earlier draft of this paper, the following hypotheses were put forth as conjectures.

Hypothesis I. *$rk_{n+1}(\beta)=rk_n(\beta)$ for all $\beta\in B_n\subset B_{n+1}$.*

Hypothesis II. *$rk_{n+1}(\beta\sigma_n^{\pm 1})=rk_n(\beta)+1$ for all $\beta\in B_n$.*

Hypothesis III. *If $\vec{b}=(b(1),\ldots,b(k))$ is a minimal band representation in B_{n+1} of $\beta\in B_n$, then $b(k)\neq\sigma_n^{\pm 1}$.*

Hypothesis IV. *If b is a minimal band representation in B_{n+1} of $\beta\in B_n$, then actually each band $b(j)$ belongs to B_n.*

The logical relationships of these hypotheses are as follows.

PROPOSITION 5.4. *Hypothesis* II $\Rightarrow$ *Hypothesis* I; *Hypothesis* IV $\Rightarrow$ *Hypothesis* III; (*Hypothesis* III *&* *Hypothesis* I) $\Rightarrow$ *Hypothesis* II.

Proof. Certainly IV implies III.

Observe that all band representations of a given braid have lengths of the same parity; that $rk_{n+1}(\beta)\leqq rk_n(\beta)$ for any $\beta\in B_n$; and that $rk_n(\beta\gamma)\leqq rk_n(\beta)+rk_n(\gamma)$ for any $\beta,\gamma\in B_n$.

Suppose β falsifies I. Then $rk_{n+1}(\beta)\leqq rk_n(\beta)-2$. Then if $\vec{b}$ is a minimal band representation of β in B_{n+1}, $(\vec{b},\sigma_n^{\pm 1})$ is a band representation of $\beta\sigma_n^{\pm 1}$ in B_{n+1}, so $rk_{n+1}(\beta\sigma_n^{\pm 1})\leqq rk_n(\beta)-1$, and β also falsifies II. Thus II implies I.

Now suppose III and I are both true. Let $\vec{b}$ be a minimal band representation of $\beta\sigma_n^{\varepsilon}$ in B_{n+1} ($\varepsilon=\pm 1$). Then $(\vec{b},\sigma_n^{-\varepsilon})$ is a band representation of β in B_{n+1}; by III it is not minimal, so $rk_{n+1}(\beta)\leqq rk_{n+1}(\beta\sigma_n^{\varepsilon})-1$. But in any case $rk_{n+1}(\beta)\geqq rk_{n+1}(\beta\sigma_n^{\varepsilon})-1$; so in fact $rk_{n+1}(\beta\sigma_n^{\varepsilon})-1=rk_{n+1}(\beta)$, and by I this equals $rk_n(\beta)$; so II is true. □

However, Hypothesis II is not true.

To see this, recall Markov's Theorem (alluded to in Example 5.2), as proved in [Bi]: Let $\beta\in B_n$ and $\beta'\in B_{n'}$ be braids with closures of the same (oriented) link type. Then there is a finite sequence $\beta_1,\ldots,\beta_s$ of braids $\beta_i\in B_{n(i)}$, with $\beta_1=\beta$, $\beta_s=\beta'$, such that for each $i=2,\ldots,s$, one of the following holds – either

(M1) $n(i)=n(i-1)$ and β_i is conjugate to β_{i-1} in $B_{n(i)}=B_{n(i-1)}$, or

(M2) $n(i)=n(i-1)+1$ and $\beta_i=\beta_{i-1}\sigma_{n(i-1)}^{\pm 1}$, or

(M2^{-1}) $n(i)=n(i-1)-1$ and $\beta_{i-1}=\beta_i\sigma_{n(i)}^{\pm 1}$.

LEMMA 5.5. *Suppose Hypothesis* II *is true. Then if two braids* β, β' *differ by a Markov move* $(M1)$, $(M2)$, *or* $(M2^{-1})$, *they have the same difference between string index and rank.*

Proof. In (M1) string index is constant, and so is rank since it is a conjugacy-class invariant. In (M2), let $\beta \in B_n$, $\beta' = \beta\sigma_n^{\pm 1}$; then by Hypothesis II, the rank of β' is one greater than the rank of β, but so is its string index, and the difference is constant. Similarly for $(M2^{-1})$. □

By Lemma 5.5 and Markov's Theorem, if Hypothesis II is true, then the difference of string index and rank would be an invariant of oriented link type; but Examples 5.2 and 5.3 show that it is not, so Hypothesis II is false.

Then, by Proposition 5.4, not both of Hypothesis III (or the stronger Hypothesis IV) and Hypothesis I can be true. I will still conjecture (weakly) that Hypothesis I is true.

I will conclude by asking various questions.

Is there an algorithm for calculating the rank of an arbitrary braid? Such an algorithm, with Proposition 5.1, would at least estimate the ribbon genus of a knot.

The rank of a quasipositive braid is, of course, its exponent sum. But is there an algorithm for determining whether a braid (which has not been given as the braid of a quasipositive band representation) is quasipositive? Is there an algorithm for determining whether a given knot or link has, among its various expressions as a closed braid, one which is quasipositive? Is there even a criterion which can rule out certain knots as possibly quasipositive? (No criterion based on a Seifert form for some Seifert surface can work – not, e.g., signatures or Alexander polynomials; [Ru2].)

Is there a way of determining (perhaps for a limited class of braids) whether the stable situation of Proposition 5.1 has been achieved? In particular, if $\beta \in B_n$ is quasipositive, is $g_r(\hat{\beta}) = \frac{1}{2}(2 - n + rk_n(\beta) - c(\beta))$? For the particular case that $\hat{\beta}$ is one of the quasipositive iterated torus links associated to singular points of complex plane curves, that this equality holds has been conjectured by Milnor. Also, if equality fails for some quasipositive braid, then (using results of [Ru]) one could represent some positive homology class in $H_2(\mathbf{C}P^2; \mathbf{Z})$ by a smooth manifold of genus strictly less than that of the homologous smooth algebraic curve – a situation which Thom has conjectured cannot occur.

Finally, note that if $g_s(L) = g$, then for some m (the number of local maxima of the radius-squared on some surface in D^4, with boundary L and genus $g_s(L)$) the link $L \cup mo$ consisting of L and m (split) unknots has $g_r(L \cup mo) = g$. If $L = \hat{\beta}$, $\beta \in B_n$, then $L \cup mo$ is the closure of β considered as an element of B_{n+m}.

Particularly in case Hypothesis I is true, can the method of band representations give any information about the slice genus?

Appendix. Clasps and nodes, überschneidungszahl, etc.

Here we sketch briefly how the various sections of the paper proper can be extended to a broader class of surfaces.

A.1. A *nodal braided surface* is a singular braided surface $i: S \to D$ for which i is an immersion in general position (that is, each singularity is a transverse doublepoint, briefly, a *node*). A nodal braided surface is itself "in general position" if no two nodes lie over any one point of D^2_1, and no node lies over a branch point (whence also neither tangent plane at a node is vertical). We will tacitly take all nodal braided surfaces to be in general position.

Though the disk $i_\#: D^2_1 \to E_n$ which corresponds to a nodal braided surface is not transverse to Δ (if there really are nodes), its intersections with Δ are either transverse or simply tangent.

An immersion $f: S \to D^4$ of an orientable surface is a ribbon immersion in D^4 provided that $L \circ f$ (where $L(z, w) = |z|^2 + |w|^2$) is Morse without local maxima. Such an immersion, if it is in general position, has only nodes as singularities, and no node is a critical point of $L \circ f$.

The analogue of Proposition 1.4 holds: any nodal braided surface in the bi-disk is isotopic to a ribbon immersed surface in the disk.

Let a *node* in B_n be the square of a band. A *nodal band representation* $\vec{\nu}$ is a k-tuple $(\nu(1), \dots, \nu(k))$ in which each $\nu(i)$ is either a band or a node; as before, $l(\vec{\nu}) = k$ is the *length of* ν, $\beta(\vec{\nu}) = \prod_{i=1}^{k} \nu(i)$ is its *braid*. Let $\kappa(\vec{\nu})$ be the number of nodes in $\vec{\nu}$.

In analogy to Proposition 1.11, we see that to a nodal band representation $\vec{\nu}$ and a smooth map $f: \partial D^2 \to E_n - \Delta$ representing $\beta(\vec{\nu})$, there corresponds a smooth extension of f over D^2 with $l(\vec{\nu})$ intersections with Δ, of which $\kappa(\vec{\nu})$ are "node-like". A suitable converse holds.

Slides (as well as various species of expansion and contraction) can be defined as before. In particular one sees that any nodal band representation is slide-equivalent to (and thus has the same braid as) a nodal band representation with all the nodes at the end.

A.2. From a nodal band representation $\vec{\nu}$ a nodal braided surface $i: S \to D$, and hence a ribbon immersion $S \to D^4$, each with boundary (of the link type of) $\hat{\beta}(\vec{\nu})$, can be constructed; likewise, after a choice of conjugators $w(i)$ with

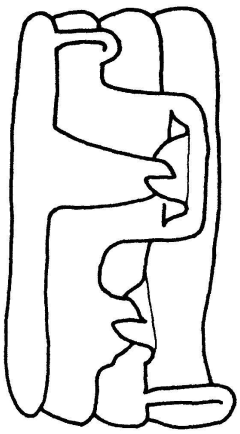

Figure A.1. A clasp/node surface derived from a nodal band representation.

$\nu(i) = {}^{w(i)}\sigma^{\pm s}_{j(i)}$, an immersion $S \to S^3$ with boundary $\hat{\beta}(\vec{\nu})$. All of these, as before, will be indiscriminately denoted by $S(\vec{\nu})$. The model in S^3 is no longer a ribbon immersion if $\kappa(\vec{\nu}) \neq 0$. It will have, besides ribbon singularities, so-called *clasp singularities.*

A component of the set of clasp singularities on the immersed surface is an arc of double-points A; the two inverse images A' and A'' each have one endpoint on the boundary of the abstract surface, and one in its interior; and the two sheets of the immersion are transverse along A. Figure A.1 shows how each node in the nodal band representation contributes one clasp (and, of course, possibly some ribbon singularities). Observe that the immersion isn't quite tabled – again, from each node there is a contribution of a single flap of the backside of the surface exposed to view.

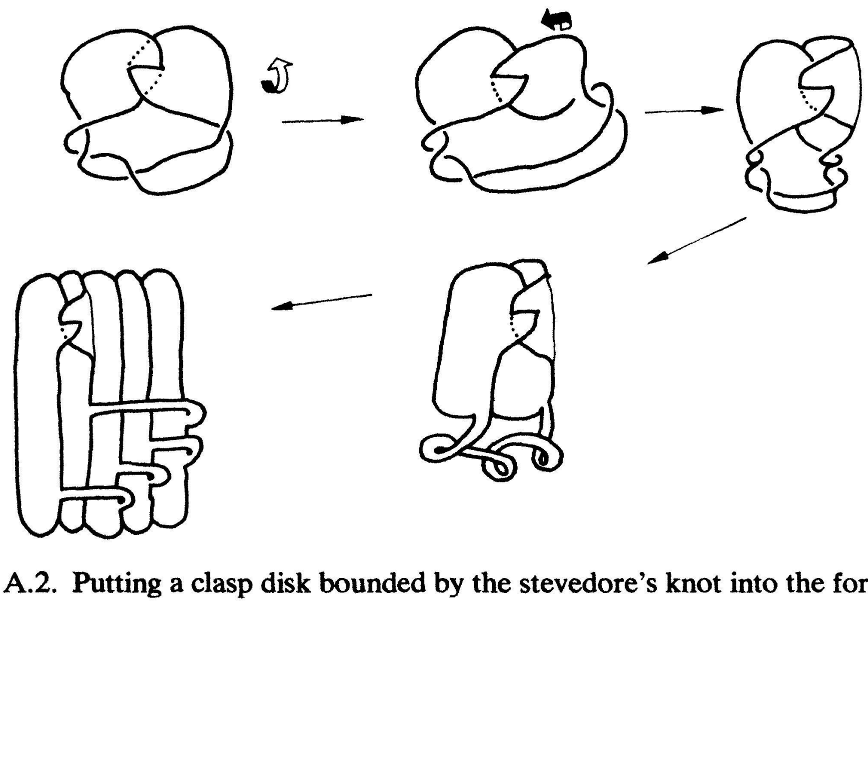

Figure A.2. Putting a clasp disk bounded by the stevedore's knot into the form $S(\vec{\nu})$.

A.3. Again, the construction is general; again, this is most easily seen in $\mathbf{R}^3$. One finds, in the isotopy class of the given clasp/ribbon surface, a surface which is "almost tabled" – tabled except for flaps such as those mentioned above.* As before, all the double-arcs of ribbon singularities can be made "vertical" segments in planes parallel to the table; and now all the double-arcs of clasp singularities are taken to be "horizontal." In the neighborhood of a clasp double-arc, two flaps (one tabled, the other not) interpenetrate each other, each attached to the front edge of one of the stacked 0-handles. Then one proceeds just as before. Figure A.2 illustrates this for a clasp disk bounded by the stevedore's knot.

A.4. If $\vec{\nu}$ is a nodal band representation in B_n of length k, the fundamental group $\pi_1(D-S(\vec{\nu}))$ can be presented with n generators and k relations; $\kappa(\vec{\nu})$ among the relations will be of the form "two (not necessarily standard) meridians commute," while the rest as before set two meridians equal. Analogues of all the results in §4 hold here.

A.5. If $K\subset S^3$ is a knot, define $\ddot{u}(K)$ [resp., $\ddot{u}_r(K)$; $\ddot{u}_s(K)$] to be the least integer k such that there is a ribbon immersion of a disk in D^4, bounded by K, with only one local minimum [resp., a ribbon immersion of a disk in D^4, bounded by K; an immersion of a disk in D^4, bounded by K] with exactly k singular points, each one a node. Then $\ddot{u}(K)$ is the ordinary *überschneidungszahl* of K, and may also be defined as the least number of self-crossings in a generic regular homotopy of K to an unknot; while $\ddot{u}_r$ and $\ddot{u}_s$ may be called the "ribbon überschneidungszahl" and "slice überschneidungszahl" respectively, for obvious reasons.

Now, if $S\subset M^4$ is any generically immersed surface in a 4-manifold, a surgery may be done on S inside M, replacing two 2-disks on S with a node as their intersection by an annulus with the same boundary, thereby increasing the genus of S (if it is connected) while decreasing the number of nodes by the same amount; and if S is ribbon-immersed in the 4-disk, such a surgery can be done within the class of ribbons, each annulus introducing two new saddles and no local extrema. Thus we have inequalities $\ddot{u}(K)\geqq\ddot{u}_r(K)\geqq g_r(K)$, $\ddot{u}_r(K)\geqq\ddot{u}_s(K)\geqq g_s(K)$, for any knot K.

PROPOSITION. *For any knot K, $\ddot{u}_r(K)$ is the least number k of self-crossings in a (generic) regular homotopy of K to a ribbon knot.*

Proof. The trace of a regular homotopy of K to K' is an annulus with singularities (generically, only nodes) in $S^3\times I$. So $k\geqq\ddot{u}_r(K)$. To see that $k\leqq\ddot{u}_r(K)$, let S be a disk, ribbon immersed in D^4 with boundary K, with exactly

* Added in proof: using vertical double arcs, clasps too may be tabled completely.

$\ddot{u}_r(K)$ nodes. Then there is a nodal band representation $\vec{\nu}$ such that S is ambient isotopic to $S(\vec{\nu})$; if $\vec{\nu}$ is in B_n, then an Euler characteristic argument shows that $l(\vec{\nu}) = \kappa(\vec{\nu}) + n - 1 = \ddot{u}_r(K) + n - 1$. After slides, if necessary, we may assume that all the nodes in $\vec{\nu}$ are collected at the end, $\vec{\nu} = (\vec{\nu}', \vec{\nu}'')$ where $\vec{\nu}'$ is an ordinary band representation of length $n-1$ in B_n, and $\vec{\nu}''$ is all nodes. The permutation of a node is trivial, so $\beta(\vec{\nu}')$ has the same permutation as $\beta(\vec{\nu})$, and $\hat{\beta}(\vec{\nu}')$ is a knot K', bounding a ribbon disk $S(\vec{\nu}')$. Now, evidently, $\vec{\nu}''$ may be understood as defining a regular homotopy of K to K' with $\ddot{u}_r(K)$ self-crossings. □

PROPOSITION. *If the knot K is the closure of a strictly positive braid $\beta \in B_n$, then $\ddot{u}(K) \leqq \frac{1}{2}(e(\beta) - n + 1)$.*

Proof. Recall that the diagram $\mathbf{D}(\beta)$ of a positive braid β is the (finite) set of positive braid words with that braid. Call β *square-free* if no word in its diagram has two consecutive letters equal.

Suppose that for $k < e(\beta)$, $m \leqq n$, if $\gamma \in B_m$ is a strictly positive braid, $e(\gamma) = k$, then there is a regular homotopy of $\hat{\gamma}$ to an unknot with $\frac{1}{2}(e(\gamma) - m + 1)$ self-crossings. If β_l is not square-free, let $\beta = \beta(\vec{b})$ where some two consecutive letters in $\vec{b}$ are equal, and let $\vec{b}_0$ be $\vec{b}$ with those two letters omitted; then there is a regular homotopy of K to $\hat{\beta}(\vec{b}_0)$ with one self-crossing, and this may be followed by the inductively-assumed homotopy of $\beta(\vec{b}_0)$ to the unknot, to produce the desired homotopy of K.

So let β be square-free. Each word in its diagram has at least one σ_{n-1} in it, for β is strictly positive. If some word in $\mathbf{D}(\beta)$ has σ_{n-1} in it exactly once, then that letter may be omitted to obtain a braid of smaller exponent sum (in one lower string index) with closure K, and the homotopy we seek exists by the inductive hypothesis. So we may assume each word in $\mathbf{D}(\beta)$ contains σ_{n-1} at least twice. Find $\vec{b}$ in $\mathbf{D}(\beta)$ with the fewest possible uses of σ_{n-1}, and, among those, with some two uses of σ_{n-1} separated by as few letters as possible, say $\vec{b} = \alpha\sigma_{n-1}\gamma_0\sigma_{n-1}\delta$, $\gamma_0 \in B_{n-1}$. Then γ_0 is not empty (since σ_{n-1}^2 cannot appear in $\vec{b}$), and it certainly begins with σ_{n-2} (for a letter with a smaller subscript could be commuted forwards past σ_{n-1}, shortening γ_0), and likewise ends with σ_{n-2}. Write $\gamma_0 = \sigma_{n-2}\gamma_1\rho_0$, where $\gamma_1 \in B_{n-2}$ and ρ_0 is either empty (in which case so is γ_1) or begins and ends in σ_{n-2}. Continue this process iteratively as long as possible, writing $\gamma_i = \sigma_{n-2-i}\gamma_{i+1}\rho_i$, where $\gamma_{i+1} \in B_{n-2-i}$ and ρ_i is either empty (in which case so is γ_{i+1}) or begins and ends in σ_{n-2-i}. The process must stop eventually, and at that point we have $\vec{b} = \alpha\sigma_{n-1}\sigma_{n-2} \cdots \sigma_{n-2-i}\rho_{i+1} \cdots \rho_0\sigma_{n-1}\delta$. But ρ_{i+1} isn't empty (the process stops at the first empty remainder), so it begins with σ_{n-1-i}, and we have $\vec{b} = \alpha \cdots \sigma_{n-1-i}\sigma_{n-2-i}\sigma_{n-1-i} \cdots \delta$. Now apply the standard relation to rewrite $\sigma_{n-1-i}\sigma_{n-2-i}\sigma_{n-1-i}$ as $\sigma_{n-2-i}\sigma_{n-1-i}\sigma_{n-2-i}$, and commute the first of the new letters forward past everything till it passes σ_{n-1}. Now the two σ_{n-1}'s are separated by fewer letters than in $\vec{b}$, contrary to assumption; so no square-free word has σ_{n-1} in it twice.

We are done, once we start the induction. But the only strictly positive braid of exponent sum 1 is σ_1 in B_2, for which $\ddot{u}(\hat{\beta}) = \frac{1}{2}(1-2+1)$. □

Remark. Milnor [Mi] conjectured the proposition, with an equality, in the particular case of links of singularities; and Henry Pinkham has given an inductive argument (based on the structure of such links as iterated torus knots) proving the proposition, again in that case.

CONJECTURE. *If β is a strictly positive braid (coming, for instance, from the link of an irreducible singular point of a complex plane curve), then there are equalities $\ddot{u}(\hat{\beta}) = \ddot{u}_r(\hat{\beta}) = g(\hat{\beta}) = g_r(\hat{\beta})$.*

This would follow from the discredited Hypothesis II of §5, and still seems a good bet.

Index of notation

Notations introduced in the paper (other than ephemera, used briefly in proof or exposition and then discarded) are listed with their page of definition. Standard symbols appear on the list if their use is somewhat idiosyncratic, or if they are very basic, or occasionally if their domain of standard use is remote from topology.

b; $b(i)$	a band; the i^{th} band in a band representation (6, 7)
$\vec{b}$	a band representation (7)
$\beta(\vec{b})$; $\hat{\beta}(\vec{b})$	the braid of a band representation (7); its closure
B_n	the braid group on n strings ($n-1$ generators) (4)
$c(\beta)$	the number of cycles in the permutation of β (27)
D; D^4	a bidisk $D^2 \times D^2$ (4); a round 4-disk (5)
Δ	the "discriminant locus" (4)
$e(\beta)$	the exponent sum of β (7) (the image of β in $\mathbf{Z} = H^1(B_n)$)
E_n	the ambient space of Δ (4)
$E_n - \Delta$	the "configuration space" of n points in $\mathbf{R}^2$ (4)
g, g_r, g_s	genus, ribbon genus, slice genus of a link (26)
$rk_n(\beta)$	the rank of β in B_n (7, 26)
R_i, R_{ij}	"standard relations" in the braid group (6)

$\mathscr{R}$	regular locus of an algebraic set (4)
S_j, S_j^{-1}	forward and backwards slides of band representations at the jth place (8)
$\mathfrak{S}_n$	the symmetric group on n letters
$\mathscr{S}$	singular locus of an algebraic set (4)
$S(\vec{b})$	the Seifert ribbon corresponding to a band representation $\vec{b}$ (15)
σ_i	a "standard generator" of B_n (6)
$\sigma_{i,j}$	an "embedded band" (7)
T_β	Stallings's notation for a particular kind of $S(\vec{b})$ (13)
$\ddot{u}$, $\ddot{u}_r$, $\ddot{u}_s$	the überschneidungszahl of a knot, and a ribbon and slice analogue thereof (33)

In any group, $[x, y]$ denotes $xyx^{-1}y^{-1}$, and ${}^xy$ denotes xyx^{-1}.

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