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Quasiregular mappings and metrics on the n-sphere with punctures

SEPPO RICKMAN*

1. Introduction

Let D be a domain in the Euclidean n-space R^n and $f: D \to R^n$ continuous. We call f *quasiregular* if f belongs to the local Sobolev space $W^1_{n,\mathrm{loc}}(D)$, i.e. f has generalized first order partial derivatives which are locally L^n-integrable and there exists K, $1 \le K < \infty$, such that the distortion inequality

$$|f'(x)|^n \le K J_f(x) \quad \text{a.e.} \tag{1.1}$$

holds. Here $f'(x)$ is the formal derivative of f at x defined by the partial derivatives, $|f'(x)|$ its operator norm, and $J_f(x)$ the Jacobian determinant. The definition extends immediately to maps $f: M \to N$ where M and N are oriented connected Riemannian n-manifolds, see for example [6]. If here N is $\bar{R}^n = R^n \cup \{\infty\}$, equipped with the spherical metric

$$d\sigma^2 = \frac{dx^2}{(1+|x|^2)^2},$$

where dx^2 is the Euclidean metric, and M is a domain in $\bar{R}^n$, we also call f *quasimeromorphic.* A quasiregular homeomorphism is called *quasiconformal.* The smallest K in (1.1) is the outer dilatation $K_O(f)$ of f and the smallest K in

$$J_f(x) \le K \inf_{|h|=1} |f'(x)h|^n \quad \text{a.e.}$$

is the inner dilatation $K_I(f)$ of f. A quasiregular mapping f is called K-*quasiregular* if the dilatation $K(f) = \max(K_O(f), K_I(f))$ satisfies $K(f) \le K$.

Quasiregular mappings form a natural generalization of analytic functions in plane to the real n-dimensional space. For the basic properties we refer to [2],

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[12]. For some years ago a Picard type theorem on omitted values was proved in the following form:

1.2. THEOREM [9]. *For $n \geq 3$ and $K \geq 1$ there exists a constant $q = q(n, K)$ such that every K-quasiregular mapping $f: R^n \to R^n \setminus \{a_1, \ldots, a_q\}$, where $a_1, \ldots, a_q$ are distinct points in R^n, is constant.*

The proof of 1.2 in [9] is based on two basic tools in the theory of quasiregular mappings, namely, the method of moduli of path families and the theory of quasilinear partial differential equations. A proof which uses only the first of these methods is given in [11] by means of ideas from [10]. It was recently proved by the author that at least for $n = 3$ Theorem 1.2 is qualitatively best possible, in fact, any number of points can be omitted.

The purpose of this paper is to give some geometrical insight from a different point of view to Theorem 1.2. We shall study quasimeromorphic mappings of the unit ball $B = \{x \in R^n \mid |x| < 1\}$ into $Y = \bar{R}^n \setminus \{a_1, \ldots, a_q\}$ where q is sufficiently large. We consider B as the Poincare model of the hyperbolic n-space with the hyperbolic metric

$$d\rho^2 = \frac{4\, dx^2}{(1 - |x|^2)^2}.$$

Our main result is that if Y is equipped with a metric with a certain natural singularity behavior near the points a_j, then f is a Lipschitz mapping if small distances are ruled out (Theorem 2.4).

Let us first take a look at the classical case $n = 2$. If $q \geq 3$, the analytic universal covering surface of Y is conformally equivalent to B. Let $\pi: B \to Y$ be an analytic covering projection. The map π induces a complete metric $d\tau^2$ on Y, called the Poincare metric of Y. If $f: B \to Y$ is analytic, we can lift f to an analytic function $\tilde{f}: B \to B$ such that $\pi \circ \tilde{f} = f$. According to the Schwarz–Pick lemma $\tilde{f}$ is distance decreasing, and with the metric $d\tau^2$ on Y, so is f. For the case $q = 3$ one gets from estimates on the metric $d\tau^2$ the Picard–Schottky theorem (see [1, Theorem 1–13]).

Let then $n \geq 3$. To some extent the covering projection π in the 2-dimensional case can be replaced by a branched covering which is quasimeromorphic. In Section 3 we consider such maps $h: B \to Y = \bar{R}^n \setminus \{a_1, \ldots, a_q\}$ which are automorphic with respect to some discrete group G of Möbius transformations acting on B and which are injective in each fundamental set. Such a map h induces a distance $\tau(y, z)$ for points y, z in Y from the hyperbolic metric in B. The singular behavior of the metric τ is similar to the behavior in the classical case as is shown in Proposition 3.2. In dimension three we give explicitly an example of this type where the dilatation of h has an absolute bound and q is arbitrarily large. The possible sets $\{a_1, \ldots, a_q\}$ in these constructions depend on G and the dilatation of h.

On the other hand, if we take an arbitrary sufficiently large set $\{a_1, \ldots, a_q\}$ in $\bar{R}^n$ and a metric τ on $Y = R^n \setminus \{a_1, \ldots, a_q\}$ which has a singular behavior near each a_j like in Proposition 3.2, then we obtain a counterpart (Theorem 2.4) for the classical distance decreasing result mentioned above. As a corollary we get an analogue for the Picard–Schottky theorem and in this way also a new proof for Theorem 1.2.

1.3. *Notation.* The Euclidean (spherical) ball and the $(n-1)$-dimensional sphere with center x and radius r are denoted by $B(x, r)$ $(D(x, r))$ and $S(x, r)$ $(C(x, r))$ respectively. We write $B(r) = B(0, r)$, $S(r) = S(0, r)$, $B = B(1)$, $S = S(1)$. The hyperbolic metric in B is denoted by ρ and the spherical metric in $\bar{R}^n$ by σ.

2. The main result

Let $a_1, \ldots, a_q$, $q \geq 3$, be distinct points in $\bar{R}^n$. We fix $\beta > 0$ such that

$$\beta \leq \tfrac{1}{4} \min_{j \neq k} \sigma(a_j, a_k)$$

and write $Y = \bar{R}^n \setminus \{a_1, \ldots, a_q\}$, $U_j = D(a_j, \beta) \setminus \{a_j\}$, and

$$U = \bigcup_{j=1}^{q} U_j.$$

We shall consider metrics τ in Y which satisfy the conditions

$$\left| \tau(y_1, y_2) - \left| \log \frac{\log(1/\sigma(a_j, y_1))}{\log(1/\sigma(a_j, y_2))} \right| \right| \leq P \quad \text{if} \quad y_1, y_2 \in U_j, \tag{2.1}$$

$$\tau(y_1, y_2) \leq Q\sigma(y_1, y_2) \quad \text{if} \quad y_1, y_2 \in Y \setminus U, \tag{2.2}$$

for some positive constants P and Q.

Metrics τ satisfying (2.1) and (2.2) are for example obtained from conformal metrics

$$d\tau^2 = \gamma^2 \, d\sigma^2 \tag{2.3}$$

where γ is continuous in Y, constant in $Y \setminus U$, and

$$\gamma(y) = \frac{1}{\sigma(a_j, y) \log (1/\sigma(a_j, y))} \quad \text{if} \quad y \in U_j.$$

We formulate our main result as follows.

2.4. THEOREM. *For each $K \geq 1$ and for each integer $n \geq 3$ there exists a number $\delta = \delta(n, K) > 0$ and a positive integer $q_0 = q_0(n, K)$ such that the following holds. If $f: B \to \bar{R}^n \setminus \{a_1, \ldots, a_q\} = Y$ is a K-quasimeromorphic mapping where $a_1, \ldots, a_q$ are distinct and $q \geq q_0$, then*

$$\tau(f(x_1), f(x_2)) \leq C \max (\rho(x_1, x_2), \delta), \qquad x_1, x_2 \in B, \tag{2.5}$$

where τ is a metric in Y satisfying (2.1) *and* (2.2) *and C is a constant depending only on n, K, β, P, and Q.*

The proof of 2.4 includes some value distribution results which we shall first list below.

2.6. *Averages of the counting function over spheres.* Let V be a ball $B(x_0, r_0)$ and $g: V \to \bar{R}^n$ a nonconstant K-quasimeromorphic mapping. For $y \in \bar{R}^n$ and for a Borel set E such that $\bar{E} \subset V$ we define

$$n(E, y) = \sum_{x \in g^{-1}(y) \cap E} i(x, g)$$

where $i(x, g)$ is the local topological index of g at x; see [2, p. 6]. If E is as above and X is an $(n-1)$-dimensional sphere in $\bar{R}^n$, we let $\nu(E, X)$ be the average of $n(E, y)$ over X with respect to the $(n-1)$-dimensional spherical metric. Especially, if $E = \bar{B}(r)$ and $X = S(t)$, we call $n(r, y) = n(E, y)$ the counting function and write $\nu(r, t) = \nu(\bar{B}(r), S(t))$, in which case we also have that

$$\nu(r, t) = \frac{1}{\omega_{n-1}} \int_S n(r, ty) \, d\mathcal{H}^{n-1} y$$

where $\mathcal{H}^{n-1}$ is the normalized $(n-1)$-dimensional Hausdorff measure and $\omega_{n-1} = \mathcal{H}^{n-1}(S)$.

2.7. LEMMA. *For $r, s, t > 0$ and $\theta > 1$ such that $\bar{B}(\theta r) \subset V$ we have*

$$\nu(\theta r, t) \geq \nu(r, s) - \frac{K \left| \log \frac{t}{s} \right|^{n-1}}{(\log \theta)^{n-1}}.$$

This lemma is in a slightly weaker form in [8, 4.1]. The above form is due to M. Pesonen and A. Hinkkanen (independently) and the proof can be found in [7] and [11].

Let $A(r)$ be the average of $n(r, y)$ over $\bar{R}^n$ with respect to the n-dimensional spherical measure. From 2.7 we obtain (see [8, p. 456]).

$$\nu(r/\theta, t)-\frac{K(a+a'\,|\log t|^{n-1})}{(\log\theta)^{n-1}}\leq A(r)\leq\nu(\theta r, t)+\frac{K(a+a'\,|\log t|^{n-1})}{(\log\theta)^{n-1}} \tag{2.8}$$

where $a, a'>0$ depend only on n. Since $A(r)$ remains invariant if g is followed by a rotation in $\bar{R}^n$, we get from (2.8) the following lemma formulated with spherical radii.

2.9. LEMMA. *For $y, z\in\bar{R}^n$, for $0<s, t\leq\pi/2$, and $r>0$ and $\theta>1$ such that $\bar{B}(\theta r)\subset V$ we have*

$$\nu(\theta r, C(y, s))\geq\nu(r, C(z, t))-\frac{K[b+b'(|\log s|^{n-1}+|\log t|^{n-1})]}{(\log\theta)^{n-1}}.$$

where $b, b'>0$ depend only on n.

The next result is a variant of [9, 3.2] for spherical distances:

2.10. LEMMA. *There exists $\theta_0=\theta_0(n, K)>1$ such that the following holds. Let $r>0$ and $\theta>\theta_0$ be such that $\bar{B}(\theta^2 r)\subset V$, let $u, v\in\bar{B}(r)$ and $y\in\bar{R}^n$ be points such that $s=\sigma(g(u), y)<t=\sigma(g(v), y)$. If y and some z in $\bar{R}^n\setminus D(y, t)$ are not in gV, then for some $d_n>0$ depending only on n*

$$\nu(\theta^2 r, C(y, t))\geq\frac{d_n\log\theta}{K}\left(\log\frac{t}{s}\right)^{n-1}.$$

2.11. *Proof of Theorem* 2.4. We may assume that f is nonconstant. We write

$$c_1=\frac{(b+2b')K}{(\log 2)^{n-1}},\qquad \theta_1=\max(\theta_0, \exp(3c_1Kd_n^{-1})),$$

where b, b', θ_0 and d_n are the constants appearing in 2.9 and 2.10. Let q_0 be the smallest integer such that

$$q_0\geq\omega_{n-1}\Omega_{n-1}^{-1}2^{3n-3}\theta_1^{2n-2} \tag{2.12}$$

and let

$$\delta = 2^{-5}\theta_1^{-2}. \tag{2.13}$$

Here Ω_{n-1} is the $(n-1)$-measure of the unit ball in R^{n-1}. Because $\beta < \frac{1}{3}$, it is possible to choose $p \geq 3$ such that

$$(\log p)^{n-1} = \frac{1}{2}\left(\log \frac{p}{\beta}\right)^{n-1}. \tag{2.14}$$

Let $x_1, x_2 \in B$ be such that $\rho(x_1, x_2) = \delta$ and write $y_i = f(x_i)$, $i = 1, 2$. Because f is open, it suffices to find a suitable estimate for $\tau(y_1, y_2)$. We consider different cases according to the location of y_1 and y_2.

Case 1. $y_1, y_2 \in D(a_k, \beta/p)$ for some k.

Set $s_i = \sigma(a_k, y_i)$, $i = 1, 2$, and assume $s_2 \leq s_1$. By (2.1) we have

$$\tau(y_1, y_2) \leq \log \frac{\log s_2^{-1}}{\log s_1^{-1}} + P. \tag{2.15}$$

Write $r_1 = |x_1 - x_2|\,\theta_1^2$. By (2.13) and by simple estimation of the hyperbolic distance we get $r_1 \leq 2^{-4}(1 - |x_1|)$. Lemma 2.10 gives

$$\nu(\bar{B}(x_1, r_1), C(a_k, s_1)) \geq \frac{d_n \log \theta_1}{K}\left(\log \frac{s_1}{s_2}\right)^{n-1}. \tag{2.16}$$

By Lemma 2.9 we obtain

$$\nu(\bar{B}(x_1, 2r_1), C(a_j, \beta/p)) \geq \nu(\bar{B}(x_1, r_1), C(a_k, s_1)) - c_1\left(\log \frac{1}{s_1}\right)^{n-1} \tag{2.17}$$

for all j. The left hand side of (2.17) is positive if

$$\nu(\bar{B}(x_1, r_1), C(a_k, s_1)) > c_1\left(\log \frac{1}{s_1}\right)^{n-1}.$$

By (2.16) this in turn is true if

$$\left(\frac{\log s_2^{-1}}{\log s_1^{-1}} - 1\right)^{n-1} = \left(\frac{\log (s_1/s_2)}{\log s_1^{-1}}\right)^{n-1} > \frac{c_1 K}{d_n \log \theta_1}.$$

Suppose now that $\tau(y_1, y_2) > c_2$ where

$$c_2 = P + \log\left[\left(\frac{c_1 K}{d_n \log \theta_1}\right)^{1/(n-1)} + 1\right]. \tag{2.18}$$

Then the left hand side of (2.17) is positive by (2.15).

Since a_j is omitted and $\nu(\bar{B}(x_1, 2r_1), C(a_j, \beta/p)) > 0$, we have $E_j = S(x_1, 2r_1) \cap f^{-1}C(a_j, \beta/p) \neq \emptyset$ for all j. Let b be the smallest of the Euclidean distances $d(E_j, E_i)$, $j \neq i$, and let $b = d(E_l, E_m)$. Then $q\Omega_{n-1}(b/2)^{n-1} \leq \omega_{n-1}(2r_1)^{n-1}$. By (2.12) $b \leq |x_1 - x_2|/2$. Let $x_1^2 \in E_l$ and $x_2^2 \in E_m$ be such that $b = |x_1^2 - x_2^2|$ and write $r_2 = |x_1^2 - x_2^2|\, \theta_1^2$. Since $f(x_1^2)$ and $f(x_2^2)$ are separated by the ring $D(a_l, \beta) \backslash \bar{D}(a_l, \beta/p)$, Lemma 2.10 implies

$$\nu(\bar{B}(x_1^2, r_2), C(a_l, \beta)) \geq \frac{d_n \log \theta_1}{K} (\log p)^{n-1}. \tag{2.19}$$

Lemma 2.9 gives then for all j

$$\nu(\bar{B}(x_1^2, 2r_2), C(a_j, \beta/p)) \geq \nu(\bar{B}(x_1^2, r_2), C(a_l, \beta)) - c_1\left(\log\frac{p}{\beta}\right)^{n-1}. \tag{2.20}$$

The left hand side of (2.20) is positive because

$$\frac{d_n \log \theta_1}{K} (\log p)^{n-1} > c_1\left(\log\frac{p}{\beta}\right)^{n-1}$$

according to the choices of θ_1 and p.

Continuing similarly we get a sequence $(x_1, x_2) = (x_1^1, x_2^1), (x_1^2, x_2^2), (x_1^3, x_2^3), \ldots$ of pairs in B such that $x_1^{m+1}, x_2^{m+1} \in \bar{B}(x_1^m, 2r_m)$ and $r_m = |x_1^m - x_2^m|\, \theta_1^2 \leq r_{m-1}/2$. Then $|x_1^m - x_1| < 4r_1 \leq 2^{-2}(1 - |x_1|)$ which implies that $x_1^m, x_2^m \to x_0 \in B$. But $\sigma(f(x_1^m), f(x_2^m)) > \beta$ for all m which contradicts the continuity of f at x_0.

We have thus proved that

$$\tau(y_1, y_2) \leq c_2 \tag{2.21}$$

where c_2 is defined in (2.18).

Case 2. $y_1 \in D(a_k, \beta/p)$, $y_2 \notin D(a_k, \beta/p)$ for some k.

Assume first that $y_1 \in D(a_k, \beta/p^2)$ or $y_2 \notin D(a_k, \beta)$. Then y_1 and y_2 are separated by the ring $D(a_k, \beta/p) \backslash \bar{D}(a_k, \beta/p^2)$ or $D(a_k, \beta) \backslash \bar{D}(a_k, \beta/p)$. Starting as in

Case 1 from the inequality (2.19) we get a contradiction with continuity if $\tau(y_1, y_2) > c_2$.

If $y_1 \notin D(a_k, \beta/p^2)$, $y_2 \in D(a_k, \beta)$, we get

$$\tau(y_1, y_2) \le P + \log \frac{\log(p^2/\beta)}{-\log \beta} = c_3. \tag{2.22}$$

Case 3. $y_1, y_2 \notin \bigcup_j D(a_j, \beta/p) = U'$.

From (2.1) and (2.2) we obtain

$$\tau(y_1, y_2) = P + 2\log \frac{\log(p/\beta)}{-\log \beta} + \frac{\pi}{2} Q = c_4. \tag{2.23}$$

Our conclusion from the inequalities (2.21), (2.22), and (2.23) is that in any case

$$\tau(y_1, y_2) \le \max(c_2, c_3, c_4) = C_1.$$

For the constant C in the theorem we can by (2.13) take

$$C = 2^6 \theta_1^2 C_1.$$

The theorem is proved.

As a corollary of Theorem 2.4 we obtain a substitute for the Picard–Schottky theorem in the following form.

2.24. COROLLARY. *Let* $f: B \to R^n \setminus \{a_1, \ldots, a_{q-1}\}$, $n \ge 3$, *be* K*-quasiregular and* $q \ge q_0$ *where* q_0 *is as in* 2.4. *Then*

$$\log |f(x)| \le C_0(-\log s_0 + \overset{+}{\log} |f(0)|)(1 - |x|)^{-C} \tag{2.25}$$

where

$$s_0 = \tfrac{1}{4} \min_{j \ne k} \sigma(a_j, a_k)$$

and C_0 *and* C *are constants which depend only on* n, K, *and* s_0.

Proof. We choose a metric τ in $Y = R^n \setminus \{a_1, \ldots, a_{q-1}\}$ given by (2.3) with $a_q = \infty$ and $\beta = s_0$. Since $|f(x)| \le \pi/(2\sigma(f(x), \infty))$, we may assume that $f(x) \in$

$D(\infty, s_0)$. If $f(0) \in \bar{D}(\infty, s_0)$,

$$\frac{\log|f(x)|}{\log|f(0)|} \le \frac{4\log(1/\sigma(\infty, f(x)))}{\log(1/\sigma(\infty, f(0)))} \le 4\exp\tau(f(0), f(x))$$
$$\le 4\exp(C(\rho(0,x)+\delta)) \le C_0(1-|x|)^{-C}$$

and (2.25) holds. If $f(0) \notin \bar{D}(\infty, s_0)$, we choose a point $z \in C(\infty, s_0)$ with $\tau(f(0), f(x)) > \tau(z, f(x))$ and obtain

$$\frac{\log|f(x)|}{\log(1/s_0)} \le 4\exp\tau(z, f(x)) < 4\exp\tau(f(0), f(x)) \le C_0(1-|x|)^{-C}$$

and (2.25) holds also in this case.

2.26. *Remark.* Similarly as in the classical case we use Corollary 2.24 to give a new proof of Theorem 1.2 as follows. Let q be as in 2.24 and let $f: R^n \to R^n \setminus \{a_1, \ldots, a_{q-1}\}$ be K-quasiregular. Let $z \in R^n$ and h be the map $x \mapsto 2|z|x$ of the unit ball. Then 2.24 applied to $f \circ h$ gives

$$\log|f(z)| \le C_0(-\log s_0 + \overset{+}{\log}|f(0)|)2^C.$$

It follows that f is bounded and thus constant by [3, 3.7].

3. Branched coverings of sphere with punctures

Let M and N be oriented connected n-manifolds. A continuous map $f: M \to N$ is called a *branched covering* if

(a) f is discrete, open, and surjective,

(b) for each $y \in N$ there exists a neighborhood V of y such that each component of $f^{-1}V$ is relatively compact.

If $f: M \to N$ is a branched covering and V is as in (b) and connected, then every component D of $f^{-1}V$ is a normal domain, i.e. $f\,\partial D = \partial fD$, f maps D surjectively onto V, and the index (see 2.6)

$$\mu(y, f, D) = \sum_{x \in f^{-1}(y) \cap D} i(x, f)$$

is constant for all $y \in V$.

We shall consider special branched coverings from B onto some $Y = \bar{R}^n \setminus \{a_1, \ldots, a_q\}$. These will be quasimeromorphic and automorphic with respect to certain discrete Möbius groups G acting on B.

Let P be a convex (open) hyperbolic polyhedron in B which satisfies the following conditions:

(1) P has a finite number of faces and finite volume.

(2) Each dihedral angle in P is π/k for some integer $k > 1$.

(3) The set of vertices of P in ∂B is nonempty.

Let Γ be the group generated by reflections in the faces of P. Then Γ is a discrete group acting on B and P is a fundamental polyhedron for Γ [13]. Let G be the subgroup of Γ generated by an even number of reflections in the faces of P. Then G is a Möbius group. If T is the reflection in some (open) face A of P, $Q = \operatorname{int}(\bar{P} \cup T\bar{P})$ is a fundamental polyhedron for G.

3.1. LEMMA. *There exists a homeomorphism $\varphi : \bar{P} \to \bar{B}$ such that $\varphi \mid P$ is quasiconformal.*

The proof of this lemma can be carried out as in [5, 3.4]. Fix Q as above. We extend φ to a continuous map $\psi : \bar{Q} \to \bar{R}^n$ by reflection in A and ∂B. Then ψ maps $P \cup TP \cup A$ quasiconformally onto $B \cup (\bar{R}^n \setminus \bar{B}) \cup \varphi A$. Let $\{b_1, \ldots, b_q\}$ be the set of vertices of P in ∂B and let $a_j = \varphi(b_j)$. We extend ψ to a quasimeromorphic mapping h of B by setting

$$h \mid g(\bar{Q} \cap B) = \psi \circ g^{-1} \mid g(\bar{Q} \cap B), \qquad g \in G.$$

Then h is a branched covering onto $Y = \bar{R}^n \setminus \{a_1, \ldots, a_q\}$, it is automorphic with respect to the group G, and it is injective in each fundamental set.

The map h induces from the hyperbolic metric ρ in B a metric τ on Y defined by

$$\tau(y, z) = \min\{\rho(u, v) \mid u \in h^{-1}(y), v \in h^{-1}(z)\}.$$

3.2. PROPOSITION. *There exist a constant $a(n, K)$, depending only on n and $K = K(h)$, and a number $\beta > 0$ such that*

$$\left|\tau(y_1, y_2) - \left|\log \frac{\log(1/\sigma(a_j, y_1))}{\log(1/\sigma(a_j, y_2))}\right|\right| \leq a(n, K) \tag{3.3}$$

whenever $y_1, y_2 \in D(a_j, \beta) \setminus \{a_j\}$, $j = 1, \ldots, q$.

Proof. Let $y_1, y_2 \in Y$ and let $x_i \in h^{-1}(y_i)$ be such that $\tau(y_1, y_2) = \rho(x_1, x_2)$. Suppose that for some j x_i belongs to the horosphere $S((1-r_i)b_j, r_i)$, $i = 1, 2$. We may assume $r_1 \geq r_2$. Since b_j is a parabolic fixed point for G, [4, 6.16] implies that for some $s_j > 0$

$$C_1 e^{-\gamma/r_i} \leq \sigma(a_j, y_i) \leq C_2 e^{-\delta/r_i} \quad \text{if} \quad \sigma(a_j, y_i) \leq s_j, \tag{3.4}$$

where C_1, C_2, γ, and δ are positive constants with $1 \leq \gamma/\delta \leq b(n, K)$. A similar statement is included also in [4, 6.17(ii)] where, however, the r in the exponent should be replaced by $1/r$.

The inequalities (3.4) give for $\sigma(a_j, y_i) \leq s_j$, $i = 1, 2$,

$$\frac{-\log C_2 + \delta/r_2}{-\log C_1 + \gamma/r_1} \leq \frac{\log(1/\sigma(a_j, y_2))}{\log(1/\sigma(a_j, y_1))} \leq \frac{-\log C_1 + \gamma/r_2}{-\log C_2 + \delta/r_1}.$$

By choosing s_j smaller if necessary we get

$$\log\frac{r_1}{r_2} - \log\frac{2\gamma}{\delta} \leq \log\frac{\log(1/\sigma(a_j, y_2))}{\log(1/\sigma(a_j, y_1))} \leq \log\frac{r_1}{r_2} + \log\frac{2\gamma}{\delta}$$

and $\log(r_1/r_2) - br_1 \leq \rho(x_1, x_2) \leq \log(r_1/r_2) + br_1$ where b is some positive constant. The proposition follows with $\beta = \min(s_1, \ldots, s_q)$.

Sources for examples of groups G of the type above are mentioned for instance in [13]. The possible configurations of the set $\{a_1, \ldots, a_q\}$ depend on G and the dilatation of h. We shall in the following give an example in dimension three where the set $\{a_1, \ldots, a_q\}$ is arbitrarily large and h has an absolute bound for its dilatation.

3.5. *Example.* We shall give the definition of a hyperbolic polyhedron in $H^3 = \{x \in R^3 \mid x_3 > 0\}$. Let Σ be the set of spheres $S(x, 1)$ in R^3 where x runs through the points of the lattice $\{x \in \partial H^3 \mid x = j\sqrt{3}\, e_1 + k(\sqrt{3}\, e_1/2 + 3e_2/2),\ j, k \in Z\}$. Here e_i is the ith standard coordinate vector. We let m be a positive integer and define planes

$$A_1 = \{x \in R^3 \mid x_2 = 0\},$$
$$A_2 = \{x \in R^3 \mid x_2 - \sqrt{3}\, x_1 = 0\},$$
$$A_3 = \{x \in R^3 \mid x_2 + \sqrt{3}\, x_1 = 3m\}.$$

Let Δ be the bounded open triangle in ∂H^3 bounded by the planes A_i. Let Σ_m be the subset of Σ consisting of spheres which meet Δ, and let P' be the open hyperbolic convex polyhedron in H^3 bounded by the spheres in Σ_m and the planes A_i. Let T be a Möbius transformation which maps H^3 onto B^3 and $T(\sqrt{3}\, m/2, m/2, m) = 0$. Set $P = TP'$. The dihedral angle between any two adjacent faces of P is $\pi/3$ or $\pi/2$. Hence P defines a group G as described before. We shall next give a more detailed definition of the map $\varphi : \bar{P} \to \bar{B}^3$.

Let b be a vertex of P in ∂B, let $8r = 8r_b$ be the Euclidean distance from b to the set of other vertices of P. Let $U = U_b$ be the component of $P \cap (B \setminus \bar{B}(1-2r))$ such that $b \in \bar{U}$. In the following K_1 and K_2 are some absolute constants >1. By the technique in [5, p. 128] we first construct a K_1-quasiconformal mapping $g = g_b$ of $V = V_b = P \cap B(b, 4r)$ onto $V \setminus \bar{U}$ such that

(1) g is the identity on $\partial V \cap B(1-2r)$,

(2) $U \cap B(b, r)$ is mapped onto $W_b = (V \setminus \bar{U}) \cap B(b', r/32)$ and $b' = g(b)$ is a point in $S(1-2r) \cap \bar{U}$ such that $d(b', \partial P) \geq r/8$,

(3) $|g(x) - b'| = c \exp(-1/|x-b|)$ if $x \in U \cap B(b, r)$ for some constant c.

Let φ_1 be the map of P such that $\varphi_1 \mid V_b = g_b$ if b is a vertex of P in ∂B and identity elsewhere. Furthermore, there exists a K_2-quasimeromorphic mapping φ_2 of $E = \varphi_1 P$ onto B such that $\varphi_2 \mid W_b$ is the radial stretching $x \mapsto (1-2r_b)^{-1}x$ for each vertex b in ∂B. The required map $\varphi \mid P$ is defined as $\varphi_2 \circ \varphi_1$.

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Department of Mathematics
University of Helsinki
Helsinki, Finland

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