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Torsion in K_2 of fields and 0-cycles on rational surfaces

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Introduction

Let L/K be a cyclic extension with generating automorphism σ. In [10] it was shown that $\operatorname{Ker}\operatorname{Tr}_{L/K}=K_2(L)^{1-\sigma}$, when $[L:K]$ is prime to the characteristic of K (analogue of Hilbert 90). We remove this assumption on [L:K] in Theorem 2.1; this we do by "lifting" to characteristic 0.

Let K be a field and $\omega\in K$ a primitive nth root of unity. Merkurjev and Suslin [10] have proved the remarkable result that the map $K^*\xrightarrow{(\omega,.)}K_2(K)_n$ given by $f\to(\omega,f)_K$ is surjective (the subscript n denotes the n-torsion in a group). Here we show (Theorem 3.1) that if k is a subfield of K, with k algebraically closed in K, then the map $(\omega,.)_K$ induces an isomorphism $K^*/k^*K^{*n}\xrightarrow{\approx}K_2(K)_n/\operatorname{Im}K_2(k)_n$. In particular, if $K_2(k)$ happens to be torsion-free (e.g. k finite or algebraically closed), we have $K^*/k^*K^{*n}\approx K_2(K)_n$. When $\operatorname{tr.deg}_k K=1$, this isomorphism is due to Tate [13] when k is finite, and due to Bloch [3] when k is an algebraically closed field of characteristic 0.

We apply Theorem 3.1 to study the well-known question about the injectivity of $K_2(k)\xrightarrow{i}K_2(K)$ when k is algebraically closed in K. We show that $\operatorname{Ker} i=0$ if char $k=0$, and that $\operatorname{Ker} i$ is p-primary if char $k=p>0$ (Theorem 4.6). We also prove a relative version of this result (Theorem 4.8).

In §§5–6 we apply the results of the previous sections to obtain some informations about Galois cohomology of K_2 and the group $A_0(X)$ of 0-cycles of degree 0 of a rational surface X. For example, we show that for a local field of characteristic 0, $A_0(X)$ is finite.

We are grateful to C. S. Seshadri for informing us about Weil's theorem (see Lemma 3.4 below). Thanks are also due to R. C. Cowsik, N. Mohan Kumar and M. V. Nori for stimulating discussion.

Finally, we record our deep appreciation for A. A. Suslin for his lectures at T.I.F.R. on [10] and making a preliminary version of [10] available to us, without which this work would not have been possible. In fact, in proving Theorems 4.6 and 4.8 we use the method of "killing the Brauer group" by Severi–Brauer extensions developed in [10]. Further, quite often our proofs rely on results of [10].

While this manuscript was in its final stages we received a preprint of A. A. Suslin entitled "Torsion in K_2 of fields" which contains all our main results. In fact, he proves the injectivity Theorems 4.6 and 4.8 even in positive characteristic by proving the remarkable result that K_2 of a field of characteristic $p>0$ has no p-torsion. However, since our methods are different and may be of independent interest, we thought it might be worthwhile to publish this paper.

§1. Results of Merkurjev, Suslin and Tate

In this section we collect some results of Merkurjev, Suslin and Tate which are crucial for us.

For a ring A, the standard map $A^* \times A^* \to K_2(A)$ will be denoted by $(,)_A$ or simply by $(,)$; here A^* denotes the group of units of A. If B is an A-algebra which is finitely generated free as an A-module, the transfer map $K_2(B) \to K_2(A)$ will be denoted by $\mathrm{Tr}_{B/A}$ or simply by Tr.

THEOREM 1.1. (Merkurjev–Suslin, [10], Theorem 14.1). *Let F'/F be a finite cyclic extension of fields of degree prime to the characteristic of F and let σ be a generator of* Gal (F'/F), *Then*

$$\mathrm{Ker}\,(K_2(F') \xrightarrow{\mathrm{Tr}} K_2(F)) = K_2(F')^{1-\sigma}.$$

THEOREM 1.2 (Merkurjev–Suslin, [10], Theorem 14.2). *Let F be a field containing a primitive nth root of unity ω, where n is an integer prime to the characteristic of F. Then*

$$K_2(F)_n = (\omega, F^*).$$

THEOREM 1.3 (Tate, [13], Theorem 6.3). *Let F be a function field in one variable over a finite field k with k algebraically closed in F. Assume that k contains a primitive* nth *root of unity ω with n prime to the characteristic of F. Then the map $F^* \to K_2(F)$ given by $f \to (\omega, f)$ induces an isomorphism*

$$F^*/k^*F^{*n} \xrightarrow{\approx} (\omega, F^*).$$

§2. Hilbert's theorem 90

The following theorem removes the restriction of $[F':F]$ being prime to char F in Theorem 1.1:

THEOREM 2.1 (Merkurjev–Suslin). *Let F'/F be a finite cyclic extension and let σ be a generator of* Gal (F'/F). *Then*

$$\mathrm{Ker}\,(K_2(F') \xrightarrow{\mathrm{Tr}} K_2(F)) = K_2(F')^{1-\sigma}.$$

For the proof of this theorem we need a few lemmas.

LEMMA 2.2. *Let F'/F be a separably generated field extension and let A be a complete discrete valuation ring of characteristic* 0 *with F as its residue class field. Then there exists a complete discrete valuation ring A' containing A such that A' is unramified over A and A' has F' as its residue class field.*

Further, if F'/F is a finite Galois extension and L (resp. L') denotes the field of fractions of A (resp. A'), then L'/L is a finite Galois extension with the natural map Gal $(L'/L) \to$ Gal (F'/F) *an isomorphism.*

We omit the proof of this lemma.

If A is a discrete valuation ring with field of fractions L, then we shall identify $K_2(A)$ with its natural image in $K_2(L)$ ([6],Theorem 2.2).

LEMMA 2.3. *Let A be a discrete valuation ring with π a uniformizing parameter and F the residue class field. Then* Ker $(K_2(A) \to K_2(F))$ *is generated by symbols of the type $(1+a\pi, u)$ with $a \in A$ and $u \in A^*$.*

Proof. Let C be the subgroup of $K_2(A)$ generated by symbols of the type $(1+a\pi, u)$. We have to show that the natural map $K_2(A)/C \xrightarrow{\varphi} K_2(F)$ is an isomorphism.

Let $\bar{u}$, $\bar{v}$ be arbitrary elements of F^* and let $u, v \in A^*$ be lifts of $\bar{u}$, $\bar{v}$. Then the class of the symbol (u, v) in $K_2(A)/C$ is independent of the lifts. Define a map $F^* \times F^* \to K_2(A)/C$ by sending $(\bar{u}, \bar{v})$ to the class of the symbol (u, v) in $K_2(A)/C$. It is easily seen that this gives rise to a homomorphism $K_2(F) \to K_2(A)/C$ which is the inverse of φ.

Proof of Theorem 2.1. In view of Theorem 1.1, we may assume char $F > 0$. We may further assume F to be finitely generated over the prime field.

By Lemma 2.2 choose an unramified extension of complete discrete valuation rings $A \to A'$ of characteristic 0 with residue class fields F and F'. Let π denote a uniformizing parameter in A and let L, L' denote the fields of fractions of A, A' respectively. Let C (resp. C') denote the kernel of the natural map $K_2(A) \to K_2(F)$ (resp. $K_2(A') \to K_2(F')$). We note that the transfer map $K_2(A') \to K_2(A)$ maps C' *onto* C. This follows from Lemma 2.3 and the fact that $1+\pi A'$ is cohomologically trivial ([12], Lemma 2, p. 193).

Let $\bar{x} \in \operatorname{Ker}(K_2(F') \longrightarrow K_2(F))$. From the surjectivity of $C' \xrightarrow{\text{Tr}} C$ it follows that $\bar{x}$ lifts to an $x \in \operatorname{Ker}(K_2(A') \longrightarrow K_2(A))$. By Theorem 1.1, $x = y^{1-\sigma}$ for some $y \in K_2(L')$. We can write $y = (u, \pi)z$, where $z = \prod (u_i, v_i)$, with u, u_i, $v_i \in A'^*$. Hence $(u, \pi)^{1-\sigma}$ belongs to $K_2(A')$. Therefore it follows from the tame symbol exact sequence $1 \to K_2(A') \to K_2(L') \to F'^* \to 1$ that $u^{1-\sigma} = 1+\pi a'$ for some $a' \in A'$. Since $1+\pi A'$ is cohomologically trivial, we have $u^{1-\sigma} = (1+\pi b')^{1-\sigma}$ for some $b' \in A'$. So $x \in K_2(A')^{1-\sigma}$ and hence $\bar{x} \in K_2(F')^{1-\sigma}$.

COROLLARY 2.4. *If $A \to A'$ is an unramified extension of complete discrete valuation rings which is cyclic with Galois group generated by σ, then*

$$\operatorname{Ker}(K_2(A') \xrightarrow{\text{Tr}} K_2(A)) = K_2(A')^{1-\sigma}.$$

§3. Torsion in K_2 of fields

The main result of this section is the following

THEOREM 3.1. *Let k be a field with a primitive* nth *root of unity ω, where n is prime to* char k, *and let K be an extension field such that k is separably algebraically closed in K. Then the map $K^* \to K_2(K)_n$ given by $f \to (\omega, f)$ induces an isomorphism*

$$\varphi : K^*/k^*K^{*n} \xrightarrow{\approx} K_2(K)_n/\operatorname{Im} K_2(k)_n.$$

Proof. By Theorem 1.2 the map φ is surjective. To prove its injectivity it suffices to show that

$$f \in K^*, \qquad (\omega, f) = 1 \Rightarrow f \in k^*K^{*n}. \tag{3.2}$$

We divide the proof of (3.2) into several steps.

Step 1. We may assume that K/k is a regular extension with both of these fields finitely generated over the prime field.

Proof. Let $f \in K^*$ be such that $(\omega, f)_K = 1$. Let L be a subfield of K, finitely generated over the prime field such that $\omega, f \in L$ and $(\omega, f)_L = 1$. Let F denote the algebraic closure in L of the prime field. Since k is separably algebraically closed in K, we have $F \subset k$, proving Step 1.

Step 2. Let X be a projective normal model over k, with K as its function field. To prove (3.2) it suffices to prove:

If D is a Weil divisor on X such that D is not linearly equivalent to 0 and such that $nD = (f)$ is a principal divisor, then $(\omega, f)_K \neq 1$. (3.3)

Proof. We have a commutative diagram with exact row, where $\mathrm{Cl}(X)$ denotes the class group of X. The map τ is induced by the tame symbols and ψ is defined

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Cl}(X)_n & \xrightarrow{\psi} & K^*/k^*K^{*n} & \longrightarrow & \coprod_{\substack{x \in X \\ \mathrm{codim}\, x = 1}} k(x)^* \\ & & & & \searrow^{\varphi} \quad \nearrow_{\tau} & & \\ & & & & (\omega, K^*)_K/(\omega, k^*)_K & & \end{array}$$

as follows: let D be a Weil divisor such that $nD = (f)$. Then $\psi(\mathrm{Cl}(D)) =$ the class of f in K^*/k^*K^{*n}. The exactness of the row follows from the fact that if $f \in K^*$, then $\tau\varphi(\bar{f}) = 1$ if and only if $n \mid v_x(f)$ (where v_x denotes the discrete valuation at x) for all $x \in X$ of codimension 1.

Hence $\mathrm{Ker}\, \varphi$ is a subgroup of $\mathrm{Cl}(X)_n$ and so it suffices to prove (3.3).

Step 3. To prove (3.3) it suffices to prove it when $\dim X = 1$.

Let $X = \mathrm{Proj}\, k[x_0, x_1, \ldots, x_m]$ be a normal, geometrically integral scheme over k. Let Γ denote the "incidence" variety, i.e. Γ is the subscheme of $\mathbb{P}_k^m \times_k X$ defined by the equation $\sum_{i=0}^m t_i x_i = 0$, where $(t_0, t_1, \ldots, t_m)$ denotes the coordinates in $\mathbb{P}_k^m$. Let $\pi : \Gamma \to X$ and $p : \Gamma \to \mathbb{P}_k^m$ be the natural projections.

LEMMA 3.4. (Weil, [15], Théorème 2). *With the notation as above, let* $D \not\sim 0$ *be a Weil divisor on* X *with* $nD \sim 0$ *for some* $n > 0$ *and let* η *denote the generic point of* $\mathbb{P}_k^m$. *If* $\dim X \geq 2$, *then* $\pi^*(D)$ *restricted to* $p^{-1}(\eta)$ *is not linearly equivalent to* 0.

Proof of Step 3. Since $k(\Gamma)$ is a pure transcendental extension of $k(X)$, the map $K_2(k(X)) \to K_2(k(\Gamma))$ is injective. Hence it suffices to show that $(\omega, f)_{k(\Gamma)} \neq 1$. But $\pi^*(D)|_{p^{-1}(\eta)} \not\sim 0$ and $\pi^*(nD)|_{p^{-1}(\eta)} = (f)|_{p^{-1}(\eta)}$. By Bertini's theorem $p^{-1}(\eta)$ is a normal, projective, geometrically integral scheme of dimension $\dim X-1$ over $k(\mathbb{P}_k^m)$ and $k\Gamma)=k(p^{-1}(\eta))$. Replacing k (resp. X) by $k(\mathbb{P}_k^m)$ (resp. $p^{-1}(\eta)$) and repeating this process we reduce to the 1-dimensional case.

Step 4. Proof for the case $\dim X=1$.

Now X is a smooth projective curve over k. Choose a finitely generated regular $\mathbb{Z}$-algebra A with k as its field of fractions, an A-scheme $\tilde{X}$, and a Cartier divisor $\tilde{D}$ on $\tilde{X}$ such that we have a diagram

$$\begin{array}{ccc} \operatorname{Spec} k \times_{\operatorname{Spec} A} \tilde{X} & \xrightarrow{\theta} & \tilde{X} \\ \downarrow & & \downarrow \pi \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} A \end{array} \tag{3.5}$$

and such that the following hold:

(i) $\operatorname{Spec} k \times_{\operatorname{Spec} A} \tilde{X}=X$ as k-schemes

(ii) π is smooth and for every $\wp\in\operatorname{Spec} A$, $\pi^{-1}(\wp)$ is smooth and geometrically integral over $k(\wp)$.

(iii) $\theta^*(\tilde{D})=D$ (so that $\tilde{D}\not\sim 0$)

(iv) $n\tilde{D}=(f)$ on $\tilde{X}$.

Further, by shrinking Spec A, we can arrange

(v) $\tilde{D}|_{\pi^{-1}(\wp)}\not\sim 0$ for all $\wp\in\operatorname{Spec} A$.

(Since $D\not\sim 0$, this follows from the semi-continuity theorem).

If $\dim A=0$, then $A=k$ is a finite field and (3.3) follows from Tate's theorem (1.3).

So we assume $\dim A>0$ and proceed by induction on $\dim A$.

Choose a prime ideal $\wp$ of height 1 in A such that $n\notin\wp$ and such that, W denoting $V(\wp)$, $\pi^{-1}(W)$ is not an irreducible component of the divisor $(f)=n\tilde{D}$ on $\tilde{X}$ (note that $\pi^{-1}(W)$ is integral by our assumptions). Hence f is a unit in the discrete valuation ring $\mathcal{O}=\mathcal{O}_{\tilde{X},\pi^{-1}(W)}$. Since $K_2(\mathcal{O})\to K_2(k(X))$ is injective, it suffices to show that (ω,f) is not 0 as an element of $K_2(\mathcal{O})$. This will follow if we can show that $(\omega,\bar{f})_L\neq 0$, where $L=k(\pi^{-1}(\wp))$ is the residue class field of $\mathcal{O}$ and $\bar{f}$ denotes the image of f in L. We have a diagram

$$\begin{array}{ccc} \pi^{-1}(\wp) & \longrightarrow & \pi^{-1}(W) \\ \downarrow & & \downarrow \\ \operatorname{Spec} k(\wp) & \longrightarrow & W=\operatorname{Spec} A/\wp \end{array} \tag{3.6}$$

similar to diagram (3.5). Let $\bar{D}$ denote the restriction of $\tilde{D}$ to $\pi^{-1}(\wp)$. Then $\bar{D} \not\sim 0$ (by (v)) and $(n\bar{D}) = (\bar{f})$. Since $\dim A/\wp < \dim A$, by induction $(\omega, \bar{f})_L \neq 0$. This completes the proof of Theorem 3.1.

COROLLARY 3.7. *In the situation of Theorem* 3.1, *let L be a field extension of K such that K is separably algebraically closed in L. Then the natural map*

$$K_2(K)_n/\mathrm{Im}\, K_2(k)_n \to K_2(L)_n/\mathrm{Im}\, K_2(k)_n$$

is injective.

COROLLARY 3.8. *With notation as in Theorem* 3.1, *assume k to be algebraically closed and K any field containing k. Then there is an isomorphism*

$$K^*/K^{*n} \xrightarrow{\approx} K_2(K)_n$$

induced by $f \mapsto (\omega, f)$.

Proof. Observe that $K_2(k)$ is torsion-free.

COROLLARY 3.9. *With notation as in Theorem* 3.1, *assume k to be algebraic over a finite field. Then there is an isomorphism*

$$K^*/k^*K^{*n} \xrightarrow{\approx} K_2(K)_n$$

induced by $f \to (\omega, f)$.

Proof. $K_2(k) = 0$.

Remark 3.10. Bloch ([3], Corollary 1.24) proved the assertion of Corollary 3.8 when k is algebraically closed of characteristic 0 and $\mathrm{tr.deg}_k\, K = 1$. Corollary 3.9 in the case $\mathrm{tr.deg} k\, K = 1$ is due to Tate (Theorem 1.3). We note that Theorem 3.1 was proved by reducing to this result of Tate.

§4. Injectivity theorems about K_2

Let $k \subset K$ be fields with k algebraically closed in K. We shall prove here (Theorem 4.6) that the map $K_2(k) \xrightarrow{i} K_2(K)$ is injective if char $k = 0$, and that Ker i is p-primary if char $k = p > 0$ (see [5], Problem 10). We shall also prove a relative version (Theorem 4.8) of Theorem 4.6 which will be used in §5.

We start with a few preliminaries.

THEOREM 4.1. (Merkurjev–Suslin [10], Theorem 11.5). *Let k be a field containing a primitive* mth *root of unity, where m is prime to* char k. Then there is an isomoprhism

$$K_2(k)/mK_2(k) \xrightarrow{R_{k,m}} \mathrm{Br}\,(k)_m$$

"*given by cyclic algebras*".

For a scheme X let $\mathcal{K}_2(X)$ denote the sheaf associated to the presheaf $U \to K_2(\Gamma(U, \mathcal{O}_X))$.

THEOREM 4.2. (Merkurjev–Suslin [10], Proposition 8.7.6). *Let k be a field and l a prime different from* char k. *Let X be the Severi–Brauer variety corresponding to a central simple k-algebra of dimension* l^2 *(over k) and let* $k(X)$ *denote the function field of X. Then the map* $K_2(k) \to H^0(X, \mathcal{K}_2(X))$ *is an isomorphism. In particular,* $K_2(k) \to K_2(k(X))$ *is injective.*

LEMMA 4.3. *Let* k'/k *be a finite separable extension and let* K/k *be a field extension, with k algebraically closed in K. Let* $K' = k'K$ *and n any positive integer. Then the natural map*

$$K^*/k^*K^{*n} \to K'^*/k'^*K'^{*n}$$

is injective.

Proof. We may assume k to be finitely generated over k. Further, passing to the normal closure of k'/k, we may assume k'/k to be Galois, say with Galois group G.

Let X be a projective, normal model for K/k and let $X' = k' \otimes_k X$. Let P (resp. P') denote the group of principal divisors on X (resp. X'). Since $P = K^*/k^*$, $P' = K'^*/k'^*$, we have to show that $P/nP \to P'/nP'$ is injective for all n.

From the exact sequence

$$0 \to k'^* \to K'^* \to P' \to 0$$

and Hilbert's theorem 90 we get an exact sequence

$$0 \to k^* \to K^* \to P'^G \to 0,$$

i.e. $P \approx P'^G \subset P'$.

Let $Z^1(\ )$ denote the group of Weil divisors on a variety .

Clearly $Z^1(X')^G = Z^1(X)$. So, from the exact sequence

$$0 \to P' \to Z^1(X') \to \mathrm{Cl}\,(X') \to 0,$$

we get by applying $H^0(G, .)$, an exact sequence

$$0 \to P \to Z^1(X) \to \mathrm{Cl}\,(X')^G.$$

Hence $\mathrm{Cl}\,(X) \to \mathrm{Cl}\,(X')$ is injective. So $P'/P \to Z^1(X')/Z^1(X)$ is injective. But $Z^1(X')/Z^1(X)$ is free and hence P'/P is free.

Recall that a subgroup N of an abelian group M is called *l-pure* if $lM \cap N = lN$.

LEMMA 4.4. *Let $k \subset K$ be fields with k algebraically closed in K. Let $l \neq$ char k be a prime and let k contain a primitive* lth *root of unity. Then* $\mathrm{Ker}\,(K_2(k) \to K_2(K))$ *is l-pure in $K_2(k)$.*

Proof. Denote the map $K_2(k) \to K_2(K)$ by i. Let ω be a primitive lth root of unity in k.

Let $x \in \mathrm{Ker}\, i$ such that $x = ly$ for some $y \in K_2(k)$. We have to find a $z \in \mathrm{Ker}\, i$ such that $x = lz$. Since $\mathrm{Ker}\, i$ is a torsion group, we may assume x (and hence y) to be l-primary, say $l^r y = 0$. Let $k' = k(\zeta)$, where ζ is a primitive (l^r)th root of unity with $\zeta^{l^{r-1}} = \omega$. Also let $K' = k'K$. Note that k' is algebraically closed in K'.

Let $y_{k'} = (\zeta, a')_{k'}$ with $a' \in k'$ and $y_K = (\omega, f)_K$ with $f \in K$ (see Theorem 1.2).

In $K_2(K')$, we have $(\zeta, a')_{K'} = (\omega, f)_{K'} = (\zeta, f^{l^{r-1}})_{K'}$. Hence, by Theorem 3.1, there exists $b' \in k'^*$ and $g' \in K'^*$ such that $a'/f^{l^{r-1}} = b'(g')^{l^r}$, i.e. $(fg'^l)^{l^{r-1}} = a'/b'$. Since k' is algebraically closed in K', $fg'^l \in k'^*$. Now by Lemma 4.3 we can find $a \in k^*$ and $g \in K^*$ such that $f = ag^l$. So $(\omega, f)_K = (\omega, a)_K$. Setting $z = y - (\omega, a)_k$, we have $x = lz$ with $z \in \mathrm{Ker}\, i$.

LEMMA 4.5. *Let k be a field containing a primitive* lth *root of unity ω, where l is a prime $\neq$* char k. *Given $x \in K_2(k)$, there exists a finite tower of fields $k = L_0 \subset L_1 \subset \cdots \subset L_n = L$ such that $x_L \in lK_2(L)$ and L_{i+1}/L_i is the function field of a Severi–Brauer variety (over L_i) of a central simple L_i-algebra of dimension l^2.*

Proof. Let $x = \prod_{j=1}^m (a_j, b_j)$, a_j, $b_j \in k^*$. We shall proceed by induction on m. Let L_1 be the function field of the Severi–Brauer variety corresponding to the cyclic k-algebra associated with the symbol (a_1, b_1) and ω. By Theorem 4.1, $(a_1, b_1)_{L_1} \in lK_2(L_1)$. By the induction hypothesis, we can find a tower of fields $L_1 \subset L_2 \subset \cdots \subset L_n$ such that L_{i+1}/L_i is the function field of a Severi–Brauer variety (over L_i) of a central simple L_i-algebra of dimension l^2 and such that $\prod_{j=2}^m (a_j, b_j) \in lK_2(L_n)$. Hence $x_{L_n} \in lK_2(L_n)$.

THEOREM 4.6. *Let $k \subset K$ be fields with k algebraically closed in K. Then*
(i) $K_2(k) \to K_2(K)$ *is injective if* char $k = 0$.
(ii) Ker $(K_2(k) \to K_2(K))$ *is p-primary if* char $k = p > 0$.

Proof. Let i denote the map $K_2(k) \to K_2(K)$. To prove the theorem it suffices to show that Ker i has no l-torsion for all primes l different from char k.

We may assume K to be finitely generated over k. Also by a transfer argument we may assume that k contains a primitive lth root of unity.

Let $(t_1, \ldots, t_s)$ be a transcendence basis of K over k and let l^r be the largest power of l dividing $[K : k(t_1, \ldots, t_s)]$. Then, for all regular extensions k' of k, the l-primary part of Ker $(K_2(k') \to K_2(k'K))$ is annihilated by l^r (here $k'K$ stands for the field of fractions of $k' \otimes_k K$).

Let x be an element of Ker i such that $lx = 0$. To show that $x = 0$, it suffices to find a regular extension k' of k such that $x = l^r z$ for some $z \in$ Ker $(K_2(k') \to K_2(k'K))$ and such that $K_2(k) \to K_2(k')$ is injective.

By Lemma 4.5 there exists a field $k_1 \supset k$ such that $K_2(k) \to K_2(k_1)$ is injective and $x_{k_1} = ly_1$ for some $y_1 \in K_2(k_1)$. Further, by Lemma 4.4, we may assume y_1 to be in Ker $(K_2(k_1) \to K_2(k_1K))$. Continuing the argument we get fields $k_1 \subset k_2 \subset \cdots$ and elements $y_i \in$ Ker $(K_2(k_i) \to K_2(k_iK))$ such that $(y_i)_{k_{i+1}} = ly_{i+1}$. Since by construction, $K_2(k) \to K_2(k_1) \to \cdots \to K_2(k_i) \to \cdots$ are injective maps, we are done by taking $k' = k_r$ and $z = y_r$.

COROLLARY 4.7. *Let k be a field of transcendence degree* 1 *over a finite field and let $K \supset k$ be a field with k algebraically closed in K. Then $K_2(k) \to K_2(K)$ is injective.*

Proof. Immediate from Theorem 4.6 and a result of Bass and Tate ([1], Theorem 2.1) that $K_2(k)$ has no p-torsion ($p =$ char k).

Remark. Let K be the completion of a global field k_0 at a non-archimedian prime and let k be the algebraic closure of k_0 in K. Tate ([14], Theorem 5.5) has shown that $K_2(k) \to K_2(K)_{\text{tors}}$ is surjective. Theorem 4.6 and Corollary 4.7 show that the above map is an isomorphism.

In what follows, for fields $k \subset K$, we shall denote the group $K_2(K)/\text{Im}\, K_2(k)$ by $K_2(K/k)$.

THEOREM 4.8. *Let $k \subset K$ be fields with k algebraically closed in K. Let k' be another extension of k which is algebraically disjoint from K over k. Let l be a prime $\neq$ char k. Then the natural map*

$$K_2(K/k)_l \to K_2(Kk'/k')_l$$

is injective.

Proof. We may assume that k contains a primitive lth root of unity.

By a direct limit argument we may assume K and k' to be finitely generated over k.

If k'/k is purely inseparable then we are through by a transfer argument since $l \neq \operatorname{char} k$. This means that we may assume k'/k to be separably generated.

Next we claim that we may assume K to be a regular extension of k. For, let L/k be a separably generated subextension of K with K/L purely inseparable. Then we have a commutative diagram

$$\begin{array}{ccc} K_2(L/k)_l & \longrightarrow & K_2(K/k)_l \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ K_2(Lk'/k')_l & \longrightarrow & K_2(Kk'/k')_l \end{array}$$

where the horizontal maps are injective (use transfer, noting that $[K:L]$ is a power of char k). Let $x \in \operatorname{Ker} \psi$. Then $x^{p^r} \in \operatorname{Ker} \varphi$ for r large ($p = \operatorname{char} k$). So $x^{p^r} = 0$ since L/k is regular. But x being l-torsion we have $x = 0$.

So we are reduced to the case where K is regular over k.

Before proceeding further we record a lemma.

LEMMA 4.9. *The assertion of Theorem* 4.8 *is valid when* $k' = k(X)$, *where* X *is* $\mathbb{P}^r_k$ *(for some* r*) or* X *is the Severi–Brauer variety of a central simple algebra of dimension* l^2 *over* k.

Proof. We have $H^0(X, \mathcal{K}_2(X)) = K_2(k)$ and $H^0(K \otimes_k X, \mathcal{K}_2(K \otimes_k X)) = K_2(K)$; this is well-known when $X = \mathbb{P}^r_k$ and is Theorem 4.2 when X is a Severi–Brauer variety. So we have a commutative diagram with exact rows (denoting $K \otimes_k X$ by Y)

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_2(k) & \longrightarrow & K_2(k(X)) & \xrightarrow{\partial} & \coprod_{\substack{x \in X \\ \operatorname{codim} x = 1}} k(x)^* \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ 0 & \longrightarrow & K_2(K) & \longrightarrow & K_2(K(Y)) & \xrightarrow{\partial'} & \coprod_{\substack{y \in Y \\ \operatorname{codim} y = 1}} K(y)^* \end{array}$$

(∂ and ∂' are the tame maps and the square on the right is commutative because K/k is regular). Since γ is injective, it follows that $\operatorname{Coker} \alpha \to \operatorname{Coker} \beta$ is injective.

Proof of Theorem 4.8 (continued). Recall that we were reduced to the case k'/k separably generated. Choosing a separating transcendence basis for k'/k and using Lemma 4.9 (with $X = \mathbb{P}_k^r$) we may further reduce to the case where k'/k is finite separable.

Let $\bar{z} \in \operatorname{Ker}(K_2(K/k)_l \to K_2(Kk'/k')_l)$ and let $z \in K_2(K)$ be a representative of $\bar{z}$. Let $x \in K_2(k)$ such that $x_K = lz$. By Lemma 4.5 choose an extension L of k such that $x_L = ly$ for some $y \in K_2(L)$. So $l(z_{KL} - y_{KL}) = 0$. We have a commutative diagram

$$\begin{array}{ccccc}
K_2(K/k)_l & \xrightarrow{\alpha} & K_2(KL/L)_l & \xleftarrow{\beta} & K_2(KL)_l/K_2(L)_l \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \gamma} \\
K_2(Kk'/k')_l & \longrightarrow & K_2(Kk'L/k'L)_l & \xleftarrow{\delta} & K_2(Kk'L)_l/K_2(k'L)_l
\end{array}$$

The maps β and δ are obviously injective. The map α is injective by Lemma 4.9. The map γ is injective by Theorem 3.1 and Lemma 4.3. Since $\alpha(\bar{z}) \in \operatorname{Im} \beta$, it follows that $\bar{z} = 0$. This completes the proof of Theorem 4.8.

COROLLARY 4.10. *Let k be a field of characteristic p (which may be 0). Let X be a smooth, geometrically integral k-scheme of finite type. Let k' be any field extension of k and let $X' = k' \otimes_k X$. If the natural map $K_2(k') \to H^0(X', \mathcal{K}_2(X'))$ is an isomorphism up to p-torsion* (i.e. *an isomorphism when $p = 0$*), *then so is the map $K_2(k) \to H^0(X, \mathcal{K}_2(X))$.*

Proof. The corollary follows immediately from Theorem 4.6, Theorem 4.8 and the commutative diagram

$$\begin{array}{ccc}
\dfrac{H^0(X, \mathcal{K}_2(X))}{\operatorname{Im} K_2(k)} & \longrightarrow & \dfrac{H^0(X', \mathcal{K}_2(X'))}{\operatorname{Im} K_2(k')} \\
\big\downarrow & & \big\downarrow \\
K_2(k(X)/k) & & K^2(k'(X')/k')
\end{array}$$

COROLLARY 4.11. *Let X be the Severi–Brauer variety corresponding to a central simple algebra of dimension n^2 over k with n prime to* char k. *Then the natural map $K_2(k) \to H^0(X, \mathcal{K}_2(X))$ is an isomorphism. In particular, $K_2(k) \to K_2(k(X))$ is injective.*

Proof. It is easily seen by a transfer argument that the kernel of $K_2(k) \to K_2(k(X))$ is n-torsion. So, it follows from Theorem 4.6 that $K_2(k) \to K_2(k(X))$ is injective. Corollary 4.11 now follows from Corollary 4.10 by taking k' to be the algebraic closure of k.

Remark 4.12. The corollary above extends a result of Merkurjev–Suslin who proved it when n is a prime. In fact Merkurjev–Suslin's result was crucial in proving Theorems 4.6 and 4.8.

We end this section by making some remarks about torsion in K_2 and Galois cohomology.

Let k be a field and k_s its separable closure. For any $G(k_s/k)$-module M, the Galois cohomology groups $H^i(G(k_s/k), M)$ will be denoted by $H^i(k, M)$. For any integer n, prime to char k, we shall denote by μ_n, the group of nth roots of unity in k_s. Further, $\mu_n^{\otimes 2}$ will denote the $G(k_s/k)$-module $\mu_n \otimes \mu_n$ with diagonal action.

Let $K \supset k$ be a field with k algebraically closed in K and let l be a prime $\neq$ char k. Assume that k contains a primitive lth root of unity. There is a natural surjective map $\mu_l \otimes (k^*/k^{*l}) \to K_2(k)_l$ given by $\omega \otimes f \to (\omega, f)$. Since $\mu_l \subset k$, we have natural isomorphisms

$$\begin{aligned} H^1(k, \mu_l^{\otimes 2}) &= \mu_l \otimes H^1(k, \mu_l) \\ &= \mu_l \otimes \frac{k^*}{k^{*l}} \end{aligned}$$

and consequently a natural surjective map

$$\alpha_l(k): H^1(k, \mu_l^{\otimes 2}) \to K_2(k)_l.$$

Theorem 3.1 can now be interpreted as the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(k, \mu_l^{\otimes 2}) & \longrightarrow & H^1(K, \mu_l^{\otimes 2}) & \longrightarrow & \dfrac{K_2(K)_l}{K_2(k)_l} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mu_l \otimes \dfrac{k^*}{k^{*l}} & \longrightarrow & \mu_l \otimes 2 \dfrac{K^*}{K^{*l}} & \longrightarrow & \dfrac{K_2(K)_l}{K_2(k)_l} & \longrightarrow & 0 \end{array}$$

Now let k be arbitrary, i.e. k need not contain μ_l. We are going to show that $K_2(K)_l/K_2(k)_l$ can still be described by means of an exact sequence as above.

PROPOSITION 4.13. *Let $k \subset K$ be fields with k algebraically closed in K and let l be a prime $\neq$* char k. *Then there is a natural exact sequence*

$$0 \to H^1(k, \mu_l^{\otimes 2}) \to H^1(K, \mu_l^{\otimes 2}) \to \frac{K_2(K)_l}{K_2(k)_l} \to 0$$

Proof. Let $k' = k(\mu_l)$, $K' = K(\mu_l)$ and $\Delta = G(k'/k) = G(K'/K)$.
First we note that

$$\text{(i)}\quad H^1(k', \mu_l^{\otimes 2})^\Delta = H^1(k, \mu_l^{\otimes 2})$$
$$H^1(K', \mu_l^{\otimes 2})^\Delta = H^1(K, \mu_l^{\otimes 2})$$

and

$$\text{(ii)}\quad \left(\frac{K_2(K')_l}{K_2(k')_l}\right)^\Delta = \frac{K_2(K)_l}{K_2(k)_l}.$$

A proof of (i) (for instance for k') follows easily from Hochschild–Serre for the Galois tower $k \subset k' \subset k_s$ and noting that the order of Δ is prime to l.

To prove (ii) we first observe that $\hat{H}^0\left(\Delta, \frac{K_2(K')_l}{K_2(k')_l}\right) = 0$. Now (ii) follows from the commutative diagram

$$\begin{array}{ccc} \dfrac{K_2(K')_l}{K_2(k')_l} & \xrightarrow{\text{Norm}} & \dfrac{K_2(K')_l}{K_2(k')_l} \\ & {\scriptstyle \text{Tr}}\searrow \quad \nearrow{\scriptstyle i} & \\ & \dfrac{K_2(K)_l}{K_2(k)_l} & \end{array}$$

and the facts that Tr is surjective and i injective (for the last two statements use transfer).

Since $k' \supset \mu_l$, by the discussion preceding Proposition 4.13 we have an exact sequence

$$0 \to H^1(k, \mu_l^{\otimes 2}) \to H^1(K', \mu_l^{\otimes 2}) \to \frac{K_2(K')_l}{K_2(k')_l} \to 0$$

and the maps are Δ-linear. Now Proposition 4.13 follows by taking Δ-invariants.

Let k, k', Δ etc. be as above. We have seen earlier that there is a natural surjective map $\alpha_l(k'): H^1(k', \mu_l^{\otimes 2}) \to K_2(k')_l$, which is in fact Δ-linear. Taking Δ-invariants we get a natural surjective map

$$\alpha_l(k): H^1(k, \mu_l^{\otimes 2}) \to K_2(k)_l.$$

By Proposition 4.13 and Theorem 4.6 we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(k, \mu_l^{\otimes 2}) & \longrightarrow & H^1(K, \mu_l^{\otimes 2}) & \longrightarrow & \dfrac{K_2(K)_l}{K_2(k)_l} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_l(k)} & & \downarrow{\scriptstyle \alpha_l(K)} & & \| & & \\ 0 & \longrightarrow & K_2(k)_l & \longrightarrow & K_2(K)_l & \longrightarrow & \dfrac{K_2(K)_l}{K_2(k)_l} & \longrightarrow & 0 \end{array}$$

Hence we have

PROPOSITION 4.14. *In the situation of Proposition* 4.13

$$\operatorname{Ker} \alpha_l(K) = \operatorname{Ker} \alpha_l(k).$$

COROLLARY 4.15. *If k is an algebraic number field, then* $\operatorname{Ker} \alpha_l(K)$ *is finite. In particular, if k contains μ_l, then* $\operatorname{Ker} (K^*/K^{*l} \xrightarrow{(\omega,\,)} K_2(K)_l)$ *is finite.*

Proof. By ([13], Theorem 6.3) $\operatorname{Ker} \alpha_l(k)$ is finite. In fact Tate's theorem describes the order of $\operatorname{Ker} \alpha_l(k)$ explicitly.

Let K be a field of characteristic $p>0$ and let k be the algebraic closure of $\mathbb{F}_p$ in K. Let $k' = k(\mu_l)$, $d = [k' : k]$ and $\Delta = G(k'/k)$. In this case $\operatorname{Ker} \alpha_l(K)$ is either 0 or $\mathbb{Z}/(l)$. More precisely, we have

COROLLARY 4.16. (I) *If $d \geq 3$, then* $\operatorname{Ker} \alpha_l(K) = 0$.

(ii) *If $d \leq 2$, then* $\operatorname{Ker} \alpha_l(K) \approx k'^*/k'^{*l}$ *(the latter group is $\mathbb{Z}/(l)$ or 0 according as k has l-extensions or not).*

Proof. Since $K_2(k) = 0$,

$$\begin{aligned} \operatorname{Ker} \alpha_l(K) &= \operatorname{Ker} \alpha_l(k) \\ &= H^1(k, \mu_l^{\otimes 2}) \\ &= H^1(k', \mu_l^{\otimes 2})^{\Delta} \\ &= (\mu_l \otimes H^1(k', \mu_l))^{\Delta} \\ &= \left(\mu_l \otimes \frac{k'^*}{k'^{*l}}\right)^{\Delta}. \end{aligned}$$

Now computing these Δ-invariants we get the corollary.

When K has transcendence degree 1 over $\mathbb{F}_p$, our corollary is due to Tate ([13], Theorem 6.3).

§5. Galois cohomology of K_2

In this section we make some remarks about Galois cohomology of K_2 which we shall use in the next section to get some information about 0-cycles on rational surfaces over certain fields.

Let k'/k be a cyclic extension with generating automorphism σ. We shall denote by p the characteristic of k; p may be 0. Let K be a field containing k, with k algebraically closed in K and let k' denote Kk'.

Recall that $K_2(K/k)$ denotes the group $K_2(K)/\mathrm{Im}\, K_2(k)$. The transfer map $K_2(K'/k') \to K_2(K/k)$ will be denoted by $\mathrm{Tr}_{k,k',K}$. The natural map $K_2(K/k) \to K_2(K'/k')$ will be denoted by i. Finally, N will denote the norm map.

The following lemma will be used to compute $H^1 = H^1(G(k'/k), K_2(K'/k'))$.

LEMMA 5.1. *There are exact sequences*

$$\text{(i)}\quad 0 \to \frac{\mathrm{Ker}\, \mathrm{Tr}_{k,k',K}}{K_2(K'/k')^{1-\sigma}} \to H^1 \xrightarrow{\alpha} \mathrm{Im}\, \mathrm{Tr}_{k,k',K} \cap \mathrm{Ker}\, i \to 0$$

and

$$\text{(ii)}\quad 0 \to \frac{\mathrm{Ker}\, \mathrm{Tr}_{k,k',K}}{K_2(K'/k')^{1-\sigma}} \xrightarrow{\beta} \frac{K_2(k)}{\mathrm{Tr}\, K_2(k')} \to \frac{K_2(K)}{\mathrm{Tr}\, K_2(K')}$$

Proof. The maps α and β are induced by transfer and the other two maps are obvious. The commutative diagram

$$\begin{array}{ccc} K_2(K'/k') & \longrightarrow & K_2(K'/k') \\ & {}_{\mathrm{Tr}_{k,k',K}}\searrow \quad \swarrow_{i} & \\ & K_2(K/k) & \end{array}$$

gives an exact sequence

$$0 \to \mathrm{Ker}\, \mathrm{Tr}_{k,k',K} \to \mathrm{Ker}\, N \to \mathrm{Im}\, \mathrm{Tr}_{k,k',K} \cap \mathrm{Ker}\, i \to 0$$

with the second map induced by transfer.

Dividing the first two terms of this sequence by $K_2(K'/k')^{1-\sigma}$, we get the exact sequence (i).

The exactness of (ii) follows easily from Theorem 2.1.

COROLLARY 5.2. *If* $\mathrm{Tr}: K_2(k') \to K_2(k)$ *is surjective, then*

$$H^1 \approx \mathrm{Im}\,\mathrm{Tr}_{k,k',K} \cap \mathrm{Ker}\, i$$

and is p-primary (because Ker i *is p-primary by Theorem* 4.8).

Notation. We shall say that a field is C_1' if it has cohomological dimension ≤ 1. For instance, C_1 fields and quasi-finite fields are C_1'.

Remark 5.3. The hypothesis of Corollary 5.2 is satisfied when k is a local field, a C_1' field or a C_2 field. The case of local fields is due to Bass and Tate (see Milnor [11], Corollary A.15), the C_1' case is classical, and the C_2 case is due to Kato ([7], Proposition 1 of §4).

Convention. In what follows we shall use phrases like a certain group is "p-torsion" or it is finite "up to p-torsion" etc. These statements, when $p = 0$, will have to be interpreted as the group is "trivial" or "finite" etc.

THEOREM 5.4. (i) *With the notation as above, if k is a* C_1', C_2 *or a local field, then* $H^1 = H^1(G(k'/k), K_2(K'/k'))$ *is p-torsion. If k, is a global field, then* H^1 *is finite up to p-torsion.*

(ii) *Further, if* K' *is a pure transcendental extension of* k', *then* $H^1 = 0$ *for k a* C_1' *or a local field, and* H^1 *is finite for k a global field.*

Remark. Note that, in view of (i), the statement (ii) is relevant only when $p > 0$.

Proof. The statement (i) in the non-global case is Remark 5.3. The global case follows from Lemma 5.1, the following lemma, and the fact that Ker i is p-primary.

LEMMA 5.5. *If k is a global field, then* $K_2(k)/\mathrm{Tr}\, K_2(k')$ *is finite.*

Proof. Let A and A' be the ring of integers in k and k' respectively. We have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
K_2(A') & \longrightarrow & K_2(k') & \longrightarrow & \coprod_{ht\wp'=1} k(\wp')^* & \longrightarrow & 1 \\
\downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle N} & & \\
K_2(A) & \longrightarrow & K_2(k) & \longrightarrow & \coprod_{ht\wp=1} k(\wp)^* & \longrightarrow & 1
\end{array}$$

Since Coker N is finite and $K_2(A)$ is finite by theorems of Garland and Tate, we are through.

Proof of Theorem 5.4(ii). We may assume $p>0$. In view of Remark 5.3 and Lemma 5.5 it suffices to show that the map $i: K_2(K/k) \to K_2(K'/k')$ is injective when k is C_1', local or global. Observe that in all these three case, $K_2(k)$ is p-divisible: for the C_1' case see Bass–Tate [1]; the local case follows from Moore's theorem because the group of roots of unity of k is divisible by p; finally, the global case follows from the fact $K_2(k)$ is torsion and a theorem of Tate asserting that $K_2(k)$ has no p-torsion.

Let "bar" denote "reduction mod p". So $\overline{K_2(K/k)} = \overline{K_2(K)}$ and $\overline{K_2(K'/k')} = \overline{K_2(K')}$. By a recent result of Bloch–Gabber–Karo (see addendum in [8]) we have a commutative diagram

$$\begin{array}{ccc} \overline{K_2(K)} & \longrightarrow & \overline{K_2(K')} \\ \big\downarrow & & \big\downarrow \\ \Omega^2_{K/\mathbb{Z}} & \hookrightarrow & \Omega^2_{K'/\mathbb{Z}} \end{array}$$

with the vertical maps injective. The lower horizontal map is injective since K'/K is separable. So $\overline{K_2(K)} \to \overline{K_2(K'/k')}$ is injective. It follows that $\overline{K_2(K/k)} \to \overline{K_2(K'/k')}$ is injective.

Since K' is purely transcendental over k', say $K' = k'(\mathbb{P}^r_{k'})$ the exact sequence

$$0 \to K_2(k') \to K_2(K') \xrightarrow{\text{tame}} \coprod_{\substack{x \in \mathbb{P}^r_{k'} \\ \operatorname{codim} x = 1}} k'(x)^*$$

shows that $K_2(K'/k')$ has no p-torsion. From the injectivity of $\bar{i}: \overline{K_2(K/k)} \to \overline{K_2(K'/k')}$ we derive that $\operatorname{Ker} i$ is p-divisible. But $\operatorname{Ker} i$ has bounded p-torsion. Hence $\operatorname{Ker} i = 0$.

Remark 5.6. In general H^1 need not be 0. For instance, take $k = \mathbb{R}$, $k' = \mathbb{C}$, $K = \mathbb{R}(x, y)$ with $x^2 + y^2 = -1$. Let G denote $\operatorname{Gal}(\mathbb{C}/R)$. Since $K_2(\mathbb{C})$ is uniquely divisible, $H^1(G, K_2(K'/\mathbb{C})) = H^1(G, K_2(K'))$. But $H^1(G, K_2(K')) = \operatorname{Im} \operatorname{Tr}_{K'/K} \cap \operatorname{Ker}(K_2(K) \to K_2(K'))$ (by Theorem 2.1) and the second member of this equation contains the non-zero element $(-1, -1)_K$.

§6. Zero-cycles on rational surfaces

Let k be a field. Let X be a rational surface over k, i.e. $k_s \otimes_k X$ is birational to $\mathbb{P}^2_{k_s}$. Let $A_0(X)$ denote the group of 0-cycles of degree 0 on X modulo rational equivalence. In [2] Bloch and conjectured that $A_0(X)$ should be finite for k a local or a global field. Here we settle this conjecture in the affirmative when k is a local field of characteristic 0.

THEOREM 6.1. *Let k be a C_1' field or a local field of characteristic p (p may be 0), and let X be a rational surface on k. Then $A_0(X)$ is finite up to p-torsion.*

Let k'/k be a finite Galois extension such that $X' = k' \otimes_k X$ is birational to $\mathbb{P}^2_{k'}$. Let $G = \mathrm{Gal}(k'/k)$ and let $K' = Kk'$. By Bloch ([2], p. 7.11), $A_0(X)$ is finite if $H^1(G, K_2(K'/k'))$ and the image of $A_0(X)$ in $H^1(G, H^1(X', \mathcal{K}_2(X')))$ are finite (here $H^1(X', \mathcal{K}_2(X')) \approx \mathrm{Pic}\, X' \otimes k'^*$). As in [2], for local, global or C_1' fields, the image of $A_0(X)$ in the group $H^1(G, H^1(X', \mathcal{K}_2(X')))$ is finite. Hence it suffices to show that $H^1(G, K_2(K'/k'))$ is finite.

Clearly one may assume that G is an l-group for some prime $l \neq p$ and then by a Hochschild–Serre argument and doing the cyclic case (as in Theorem 5.4), it suffices to show that

$$M = \mathrm{Coker}\,(K_2(K/k) \to K_2(K'/k')^G)$$

is finite when G is cyclic of order l but $k' \otimes_k X$ not necessarily birational to $\mathbb{P}^2_{k'}$.

We have a proof that M is indeed finite with k as in Theorem 6.1. But we have just learnt that Suslin has proved that M is actually 0 for arbitrary smooth varieties X and arbitrary G. This in particular shows that $A_0(X)$ is finite when k is local or global, and $A_0(X) = 0$ when k is C_1'. In view of this we shall only sketch our proof of finiteness of M (with k local or C_1').

Sketch of proof of finiteness of M

We may assume $k \supset \mu_l$

$$\text{Let } Z = \mathrm{Ker}\Big(\coprod_{\substack{x \in X \\ \mathrm{codim} = x = 1}} k(x)^* \to \coprod_{\substack{x \in X \\ \mathrm{codim}\, x = 2}} \mathbb{Z}\Big).$$

From the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_2(K'/k')^C & \longrightarrow & Z & \longrightarrow & H^1(X', \mathcal{K}_2(X'))^G & & \\ & & \uparrow & & \| & & \uparrow & & \\ 0 & \longrightarrow & K_2(K/k) & \longrightarrow & Z & \longrightarrow & H^1(X, \mathcal{K}_2(X)) & \longrightarrow & 0 \end{array}$$

the finiteness of M will follow if we can show that

$$\mathrm{Ker}\,(H^1(X, \mathcal{K}_2(X))_l \to H^1(X', \mathcal{K}_2(X'))_l)$$

is finite. This, in turn, will follow from the two statements below:

(a) $\operatorname{Ker}(H^1(X, \mathcal{K}_2(X)_l) \to H^1(X', \mathcal{K}_2(X')_l))$ is finite.

(b) the natural map $H^1(X, \mathcal{K}_2(X)_l) \xrightarrow{\varphi} H^1(X, \mathcal{K}_2(X))_l$ has finite kernel and cokernel.

Proof of (a). It is clear from the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & \left(\dfrac{K_2(K')_l}{K_2(k')_l}\right)^G & \longrightarrow & \left(\coprod\limits_{\substack{x' \in X' \\ \operatorname{codim} x'=1}} \mu_l\right)^G & \longrightarrow & H^1(X', \mathcal{K}_2(X')_l)^G & \\
& \uparrow{\scriptstyle \alpha} & & \| & & \uparrow & \\
0 \longrightarrow & \dfrac{K_2(K)_l}{K_2(k)_l} & \longrightarrow & \coprod\limits_{\substack{x \in X \\ \operatorname{codim} x=1}} \mu_l & \longrightarrow & H^1(X, \mathcal{K}_2(X)_l) & \longrightarrow 0
\end{array}$$

that (a) will follow if we show that $\operatorname{Coker} \alpha$ is finite. But by Theorem 3.1, $K_2(K)_l/K_2(k)_l = P/lP$, where P denotes the group of principal divisors on X, and similarly for K'. Now $\operatorname{Coker}(P/lP \to (P'/lP')^G) = H^1(G, P') \to H^2(G, k'^*)$ and the last group is finite for the fields in question.

Proof of (b). The finiteness of $\operatorname{Ker} \varphi$ follows from the commutative diagram with exact rows and column

$$\begin{array}{ccccccc}
0 \longrightarrow & \dfrac{K_2(K)_l}{K_2(k)_l} & \longrightarrow & \coprod\limits_{\substack{x \in X \\ \operatorname{codim} x=1}} \mu_l & \longrightarrow & H^1(X, \mathcal{K}_2(X)_l) & \longrightarrow 0 \\
& \downarrow & & \| & & \downarrow & \\
0 \longrightarrow & K_2(K/k)_l & \longrightarrow & \left(\coprod\limits_{\substack{x \in X \\ \operatorname{codim} x=1}} k(x)^*\right)_l & \longrightarrow & H^1(X, \mathcal{K}_2(X))_l & \\
& \downarrow & & & & & \\
& \dfrac{K_2(k)}{lK_2(k)} = \operatorname{Br}(k)_l & & & & &
\end{array}$$

and the fact that $\operatorname{Br}(k)_l$ is finite for the fields in question.

Write the multiplication by l in $H^1(X, \mathcal{K}_2(X))$ as the composite

$$H^1(X, \mathcal{K}_2(X)) \xrightarrow{\psi} H^1(X, l\mathcal{K}_2(X)) \xrightarrow{\eta} H^1(X, \mathcal{K}_2(X)).$$

In view of the exact sequence

$$0 \to \operatorname{Ker} \psi \to \operatorname{Ker}(\eta \circ \psi) \xrightarrow{\psi} \operatorname{Ker} \eta$$

the finiteness of Coker φ will follow once we show that Ker η is finite. But this follows from the existence of surjective maps $H^2_{et}(X, \mu_l) \to H^0(X, \mathcal{K}_2(X)/l\mathcal{K}_2(X)) \twoheadrightarrow \text{Ker}\, \eta$ (the first map is due to Bloch ([4], Proposition 3.5) and the fact that the first group is finite since X is rational.

Remark 6.2. It is easily seen that when $k = \mathbb{R}$ and $k' = \mathbb{C}$, then $A_0(X)$ is finite. First we note that Ker $(K_2(K) \to K_2(K'))$ is of order at most 2 (use Corollary 4.10 and the tame sequence). Since $H^1(G, K_2(K')) \to H^1(G, K_2(K'/k'))$ is an isomorphism, it follows that $H^1(G, K_2(K'/k'))$ is of order at most 2. Hence $A_0(X)$ is finite.

Added in proof

After this work was prepared for publication we noticed a paper by J. L. Colliot-Thelene, J.-J. Sansuc and C. Soule in C.R. Acad. Sci. Paris vol. *294* (28 June 1982), Ser I, pp. 749–752, where th. 6.1 is proved in the case when k is a global field of positive characteristic by different methods. Thereafter, we received a letter from J. L. Colliot-Thelene along with his works which showed us that he had obtained a th. 6.1 when k is a local or global field and X has a rational point. We also learned that the group M = Coker $(K_2(K/k) \to K_2(K'/k')^G)$ discussed in Section 6 was proved to be 0 by Colliot-Thelene for arbitrary smooth X (in char. 0) with a rational point. The result of Suslin mentioned in Section 6, namely $M = 0$, is a generalization of this result of Colliot-Thelene. We also learned that Th. 3.1 was proved by Colliot-Thelene when char. $k = 0$ (these results of Colliot-Thelene are in Inv. Math. vol. *71* (1983), 1–20).

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