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# Gaps and bands of one dimensional periodic Schrödinger operators

JOHN GARNETT AND EUGENE TRUBOWITZ

## 1. Introduction

Let $q(x)$ be the periodic extension to the whole line of a function in $L^2_R[0, 1]$, the Hilbert space of all real valued square integrable functions on the unit interval. The spectrum of the Schrödinger operator $-(d^2/dx^2)+q(x)$ acting on $L^2(R^1)$ is the union of purely absolutely continuous bands $B_n(q)$, $n \geq 1$. The $n$th band $B_n$ is the set

$$\{\nu_n(k, q): -\tfrac{1}{2} \leq k \leq \tfrac{1}{2}\}.$$

Here $\nu_n(k, q)$, $n \geq 1$, the $n$th eigenvalue (counted with multiplicities when $k = 0, \pm\frac{1}{2}$) of the boundary value problem

$$\begin{aligned} &-y''+q(x)y = \lambda y \\ &y(x+1) = e^{2\pi i k}y(x), \qquad -\infty < x < \infty. \end{aligned} \tag{1.1}$$

The eigenvalue $\nu_n(k)$ is a continuous function of $k$ so that $B_n$ is a closed subinterval of $R^1$. The purpose of this paper is to study the following question: When is a collection of closed subintervals of $R^1$ the set of bands corresponding to a function $q$ in $L^2_R[0, 1]$?

It is well known that the bands may touch but never overlap. This property makes it possible to reformulate the question posed above in a more suggestive way. A tile is a closed interval. Tiles can be arranged in any way on the line so long as they never overlap. They are, however, permitted to touch. Suppose we are given a sequence of tiles. Can we place them in order on the line so that they coincide with the sequence of bands for a $q$ in $L^2_R[0, 1]$?

Let $\alpha_n(q)$, $n \geq 1$, be the length of $B_n(q)$. It is a routine fact that

$$\alpha_n(q) = (2n-1)\pi^2 + l^2(n).$$

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The notation $a_n = b_n + l^2(n)$ means that $\sum_{n>1}(a_n - b_n)^2 < \infty$. What is more interesting is that for each $q$ in $L^2_{\mathbf{R}}[0, 1]$ the inequality

$$\alpha_n(q) \leq (2n-1)\pi^2$$

holds for all $n \geq 1$. The result is even stronger. If a single one of the inequalities is an equality, then they are all equalities and $q$ is constant. These universal bounds on the lengths of the bands will be established in Section 3 where they are shown to be equivalent to facts about conformal mappings of slit domains. J. Moser [3] also found them while studying the spectrum of certain limit periodic potentials.

Judging by the last paragraph, it would seem that a sequence of tiles must satisfy rather subtle conditions in order to be a candidate for a set of bands. Also, we do not know that the individual inequalities and the asymptotic restriction on the lengths exhaust all necessary conditions. For these reasons we take a different point of view towards characterizing the spectra of one dimensional periodic Schrödinger operators. We hope to return, at another time, to the problem of finding a complete set of necessary conditions on the bands.

From now on we assume that the bottom of the first band is at 0. All other sets of bands are obtained from these by translation. The complement of the spectral bands is a sequence of open subintervals of $(0, \infty)$ called the forbidden bands or the gaps. It is well known that for most potentials $q$ (a set of the second category in $L^2_{\mathbf{R}}[0, 1]$) no bands touch, so that there is a nontrivial gap between every two bands. To each set of bands $B_n(q)$, $n \geq 1$, we associate the sequence of nonnegative numbers

$$\gamma_1(q), \gamma_2(q), \ldots$$

where $\gamma_n(q)$ is the distance between the top of the $n$th band and the bottom of the next.

An open title of length $\gamma$ is an open interval of length $\gamma$ when $\gamma$ is positive and a point when $\gamma$ is zero. Open tiles may be arranged in any manner on $(0, \infty)$ as long as none of them overlap. Now let $\gamma_n \geq 0$, $n \geq 1$, be a sequence of nonnegative numbers. We ask whether it is possible to place the sequence of open tiles of length $\gamma_n$, $n \geq 1$, in order on the positive axis $(0, \infty)$ such that the complement (we regard points simply as marking places where two bands touch—they are not removed) is the band spectrum of a $q$ in $L^2_{\mathbf{R}}[0, 1]$? Our goal in this paper is to describe the set of all possible configurations of bands by understanding the distribution of gaps.

There is a simple necessary condition on the length of the gaps corresponding to a $q$ in $L^2_{\mathbf{R}}[0, 1]$. The sequence $\gamma_n(q)$, $n \geq 1$, is in $l^2$ i.e., $\sum_{n\geq 1} \gamma_n^2 < \infty$. It is also sufficient.

THEOREM 1. *Let $\gamma_n$, $n \geq 1$, be any sequence of nonnegative numbers satisfying*

$$\sum_{n\geq 1} \gamma_n^2 < \infty.$$

*Then, there is a way of placing the sequence of open tiles of lengths $\gamma_n$, $n \geq 1$, in order on the positive axis $(0, \infty)$ so that the compliment is the set of bands for a function $q$ in $L^2_{\mathbf{R}}[0, 1]$. In other words, the map*

$$q \to \gamma(q) = (\gamma_n(q), n \geq 1),$$

*from $L^2_{\mathbf{R}}[0, 1]$ to $(l^2)^+$, is onto.*

Here, $(l^2)^+$ is the space of the nonnegative, square summable sequences $\gamma_n$, $n \geq 1$.

Theorem 1 tells us that there is no obstruction to a sequence of nonnegative numbers being an actual gap sequence other than an explicit asymptotic condition. This is in marked contrast to the set of band lengths.

It is natural to ask how many different ways a sequence of open tiles, whose lengths are $\gamma_n$, $n \geq 1$, can be placed so that the complement is a set of bands. For example, suppose that the tiles are properly arranged. If the first tile is moved, even a very small amount, the complement may no longer be an actual band spectrum. However, we can slide the (infinitely many) other tiles to try to compensate for this. There could be a great deal of freedom.

THEOREM 2. *There is just one way to place a sequence of open tiles, satisfying the hypothesis of Theorem 1, on the positive real axis so that they are genuine gaps.*

Thus, we have shown that $(l^2)^+$ is a moduli space for all band configurations. Equivalently, a band spectrum is uniquely determined by its gap lengths and all gap sequences in $(l^2)^+$ occur as gap lengths.

Theorems 1 and 2 are proved in Section 5. We are going to use a characterization of bands due to Marčenko and Ostrovskiĭ [2]. They identify band configurations with slit quarter planes. In Section 4, we give a new approach to their beautiful theory with the improvements that are necessary for our purposes.

Let $\mu_n(q)$, $n \geq 1$, and $\nu_n(q)$, $n \geq 0$, be the Dirichlet and Neumann spectrum of $q$ in $L^2_{\mathbf{R}}[0, 1]$, that is, the spectra of (1.1) for the boundary conditions

$$y(0) = 0, \qquad y(1) = 0$$

and

$$y'(0)=0, \qquad y'(1)=0$$

respectively. If $q$ is an even function $(q(1-x)=q(x))$ then $\gamma_n(q)=|\mu_n(q)-\nu_n(q)|$, $n\geq 1$. We define the signed gap lengths of $q$ in $E_0$, the subspace of even functions in $L^2_{\mathbb{R}}[0,1]$ with mean 0, to be the sequence $(\mu_n(q)-\nu_n(q), n>1)$. In Section 5 we prove

THEOREM 3. *The map from $q$ to its signed gap lengths is a real analytic isomorphism between $E_0$ and $l^2$.*

To indicate why this theorem is true we calculate the derivative at $q=0$. The gradient of a Dirichlet or Neumann eigenvalue is the square of its corresponding normalized eigenfunction. Consequently, the directional derivative of the $n$th signed gap length at $q=0$ in the direction of the function $v\in R_0$ is given by 2 $\langle \sin^2 n\pi x-\cos^2 n\pi x, v\rangle=-2\ \langle \cos 2\pi nx, v\rangle$. We see that the derivative of the map from $q$ to signed gap lengths is a boundedly invertible linear map between $E_0$ and $l^2$. The inverse function theorem shows that our map is a real analytic isomorphism in a neighborhood of $q=0$. We are using the fact that the Dirichlet and Neumann eigenvalues are real analytic functions of $q$. This is proved in [4].

To prove the global Theorem 3 we have to show that the conformal map of a quarter plane with infinitely many slits to the upper half plane is a real analytic function of the infinitely many slits. In fact we obtain three real analytic isomorphisms between the three spaces $E_0$, $l^2$ and $l^2_1$, the space of real sequences $\{h_n\}$ satisfying $\sum n^2h_n^2<\infty$. In Section 2 we introduce the conformal mapping $\delta(\lambda, q)$ from the upper half plane to the quarter plane with excised slits $T_n=\{n\pi+iy: 0\leq y\leq |h_n|\}$, and $\gamma_n(q)$ is the length of $\delta^{-1}(T_n)$. When $q\in E_0$, we define $h_n(q)=\operatorname{sgn}(\mu_n(q)-\nu_n(q))\,|h_n(q)|$, where $|h_n(q)|$ is the length of the $n$-th slit $T_n$ determined by $\delta(\lambda, q)$. Then all three maps in the diagram

$$\begin{tikzcd}
E_0\ni q \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \{\gamma_n=\mu_n(q)-\nu_n(q)\}\in l^2 \\
\{h_n\}\in l^2_1 \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

are real analytic, one-to-one, onto, and have real analytic inverses.

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## 2. Preliminaries

In this section we introduce some notation and derive some simple facts which will be used later.

Let $y_1(x, \lambda, q)$ and $y_2(x, \lambda, q)$ be the solutions of

$$-y''+q(x)y=\lambda y \tag{2.1}$$

satisfying

$$y_1(0, \lambda)=y_2'(0, \lambda)=1$$
$$y_1'(0, \lambda)=y_2(0, \lambda)=0,$$

and set

$$\Delta(\lambda)=\Delta(\lambda, q)=y_1(1, \lambda)+y_2'(1, \lambda).$$

The sequence of roots

$$\lambda_0<\lambda_1\le\lambda_2<\lambda_3\le\lambda_4<\cdots$$

of $\Delta^2(\lambda)-4=0$ is the *spectrum* of equation (2.1) with periodic boundary conditions of period 2, i.e. $y(x+2)=y(x)$, $-\infty<x<\infty$. Here equality means that $\lambda_{2n-1}=\lambda_{2n}$ is a double root or eigenvalue. The lowest eigenvalue $\lambda_0$ is simple, $\Delta(\lambda_0)=2$, and the corresponding eigenfunction has period 1. The eigenfunctions corresponding to $\lambda_{2n-1}, \lambda_{2n}$ have period 1 when $n$ is even and they are antiperiodic $(y(x+1)=-y(x))$ when $n$ is odd. Also, $\Delta(\lambda_{2n-1})=\Delta(\lambda_{2n})=2(-1)^n$, $n\ge1$. We have the estimate[1]

$$\lambda_{2n-1}, \lambda_{2n}=n^2\pi^2+\int_0^1 q(x)\,dx+l^2(n)$$

Finally, $\lambda_0$ and $\lambda_1$ are the bottom and top of $B_1$, while $\lambda_2$ and $\lambda_3$ are the bottom and top of $B_2$, and so on.

We see from the discussion above that the problem of describing band configurations is equivalent to the characterization of all periodic spectra, or in another guise, all functions $\Delta(\lambda)=\Delta(\lambda, q)$.

From now on, unless otherwise stated, we adopt the normalization $\lambda_0(q)=0$.

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[1] $a_n=b_n+l^2(n)$ means $\sum_{n\ge1}(a_n-b_n)^2<\infty$.

LEMMA 2.1. *Let*[2]

$$\delta(\lambda)=\delta(\lambda, q)=\int_0^\lambda \frac{-\dot{\Delta}(\mu)}{\sqrt{4-\Delta^2(\mu)}}\,d\mu$$

*Then* $\delta(\lambda)$ *is a conformal mapping of the upper half plane* $\{\operatorname{Im}\lambda>0\}$ *to the slit quarter plane*

$$\Omega(h)=\{\operatorname{Re} z>0, \operatorname{Im} z>0\}\Big\backslash \bigcup_{n=1}^{\infty} T_n,$$

*where*

$$T_n=\{n\pi+y:0<y\le h_n\}$$

*and*

$$\sum_{n\ge 1} n^2h_n^2<\infty.$$

*Moreover,*

$$\Delta(\lambda)=2\cos\delta(\lambda).$$

*Proof.* Let $\dot{\lambda}_n$, $n\ge 1$, be the zeros of $\dot{\Delta}$. It follows from Laguerre's theorem [5 p. 266], that

$$\lambda_{2n-1}\le\dot{\lambda}_n\le\lambda_{2n}, \qquad n\ge 1,$$

because $\Delta(\lambda)$ is entire of order 1/2 and the roots of $\Delta(\lambda)=\pm 2$ coincide with the real sequence $\lambda_n$, $n\ge 0$. Since

$$\frac{d}{d\lambda}\cos^{-1}\left(\frac{\Delta(\lambda)}{2}\right)=\frac{-\dot{\Delta}(\lambda)}{\sqrt{4-\Delta^2(\lambda)}},$$

we have

$$\Delta(\lambda)=2\cos\delta(\lambda).$$

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[2] $\dot{\Delta}$ is an abbreviation for $d\Delta/d\lambda$.

We now see that $\delta(\lambda)$ is a conformal mapping from the upper half plane to some quarter plane $\Omega(h)$ and that $\delta$ maps the gap $(\lambda_{2n-1}, \lambda_{2n})$ onto the slit $T_n$, the band $(\lambda_{2n}, \lambda_{2n+1})$ onto the interval $(n\pi, (n+1)\pi)$ and the segment $(-\infty, 0)$ onto the imaginary axis. It remains to check the estimate on the slit heights.

From the product representations [See 4]

$$4-\Delta^2(\lambda)=4(\lambda-\lambda_0)\prod_{n\geq 1}\frac{(\lambda_{2n-1}-\lambda)(\lambda_{2n}-\lambda)}{n^4\pi^4}$$

and

$$\dot{\Delta}(\lambda)=\prod_{u\geq 1}\frac{\dot{\lambda}_n-\lambda}{n^2\pi^2}$$

we obtain the estimates

$$4-\Delta^2(\lambda)=(\lambda_{2n}-\lambda)(\lambda-\lambda_{2n-1})O(1/n^2),\ \lambda_{2n-1}<\lambda<\lambda_{2n}$$

and

$$n\sup_{\lambda_{2n-1}\leq\lambda\leq\lambda_{2n}}|\dot{\Delta}(\lambda)|=l^2(n)$$

Hence

$$\begin{aligned} h_n &= \int_{\lambda_{2n-1}}^{\lambda_n}\frac{-\dot{\Delta}(\mu)}{(4-\dot{\Delta}^2(\mu))^{\frac{1}{2}}}\,\mathrm{d}\mu \\ &= O(n)\int_{\lambda_{2n-1}}^{\lambda_n}\frac{|\dot{\Delta}(\mu)|\,d\mu}{\sqrt{(\lambda_{2n}-\mu)(\mu-\lambda_{2n-1})}} \end{aligned}$$

so that

$$\begin{aligned} |h_n| &\leq O(n)\sup_{\lambda_{2n-1}\leq\lambda\leq\lambda_{2n}}|\dot{\Delta}(\lambda)|\int_{\lambda_{2n-1}}^{\lambda_{2n}}\frac{d\mu}{\sqrt{(\lambda_{2n}-\mu)(\mu-\lambda_{2n-1})}} \\ &= O(n)\sup_{\lambda_{2n-1}\leq\lambda\leq\lambda_{2n}}|\dot{\Delta}(\lambda)| \end{aligned}$$

and $\sum n^2h_n^2<\infty$. □

The idea of Marčenko and Ostrowskĭi is to use the slit heights as a set of moduli.

## 3. Lengths and harmonic measures

Write $(l^2)^+$ for the space of sequences $a_n$, $n \geq 1$, such that $\sum a_n^2 < \infty$ and $a_n \geq 0$, and denote by $(l_1^2)^+$ the space of sequences $h_n$, $n \geq 1$, such that $\sum_{n\geq 1} n^2 h_n^2 < \infty$ and $h_n \geq 0$. Say $h \in (l_1^2)^+$ is *finite* if $h_n = 0$ for $n$ sufficiently large.

For $h \in (l_1^2)^+$ let $\Omega(h)$ be the slit quarter plane

$$\Omega(h) = \{\operatorname{Re} z > 0, \operatorname{Im} z > 0\} \Big\backslash \bigcup_{n=1}^{\infty} T_n$$

where

$$T_n = \{n\pi + iy : 0 < y \leq h_n\}$$

is the $n$-th slit in $\partial\Omega(h)$, and let $z = \varphi_h(\lambda)$ be a conformal mapping from the upper half plane $\mathcal{U} = \{\operatorname{Im} \lambda > 0\}$ onto $\Omega(h)$. By Carathéodory's theorem [6], $\varphi_h$ extends to a continuous mapping from the closure $\overline{\mathcal{U}} \cup \{\infty\}$ and the extended $\varphi_h$ is two-to-one over each non-trivial $T_n$ and one-to-one over the remainder of $\partial\Omega(h) \cup \{\infty\}$. We normalize $\varphi_h$ by

$$\begin{cases} \varphi_h(0) = 0 \\ \varphi_h(\infty) = \infty, \end{cases} \tag{3.1}$$

which determines $\varphi_h$ uniquely to within a positive multiple. When $h$ is finite, $\varphi_h^{-1}$ is by reflection meromorphic at $\infty$ and

$$\varphi_h^{-1}(z) = az^2 + b + O\left(\frac{1}{|z|^2}\right), \qquad |z| \text{ large},$$

with $a > 0$. Replacing $\varphi_h(\lambda)$ by $\varphi_h(\lambda/a)$, we may further normalize $\varphi_h$ so that

$$\varphi_h^{-1}(z) = z^2 + b + O(1/|z|^2), \qquad |z| \text{ large}, \tag{3.2}$$

which makes $\varphi_h$ unique when $h$ is finite. If $h$ is not finite, the truncations

$$h_n^{(k)} = \begin{cases} h_n & n \leq k \\ 0, & n > k \end{cases} \tag{3.3}$$

have domains $\Omega_k = \Omega(h^{(k)})$ decreasing to $\Omega(h)$ and by Courant's theorem (and its proof [6 p. 383]), their mappings $\varphi_{h^{(k)}}(\lambda)$, when normalized by (3.1) and (3.2), converge on $\mathcal{U} \cup \{\infty\}$, uniformly with respect to the spherical metric, to conformal map $\varphi_h : \mathcal{U} \to \Omega(h)$. In this way we have a uniquely determined map $\varphi_h$ for all $h \in (l_1^2)^+$. From now on $\varphi_h$ denotes this unique conformal map. Set $\lambda_0 = 0$; and for $n \geq 1$ define

$$\lambda_{2n-1} = \lambda_{2n-1}(h) = \varphi_h^{-1}(n\pi -) = \lim_{\varepsilon \downarrow 0} \varphi_h^{-1}(n\pi - \varepsilon)$$

$$\lambda_{2n} = \lambda_{2n}(h) = \varphi_h^{-1}(n\pi +) = \lim_{\varepsilon \downarrow 0} \varphi_h^{-1}(n\pi + \varepsilon)$$

$$\alpha_n = \alpha_n(h) = \lambda_{2n-1} - \lambda_{2n-2}$$

and

$$\gamma_n = \gamma_n(h) = \lambda_{2n} - \lambda_{2n-1}$$

Thus $\alpha_n$ is the length of $\varphi^{-1}([(n-1)\pi+, n\pi-])$ and $\gamma_n$ is the length of $\varphi_h^{-1}(T_n)$. When $\lambda_n$, $n \geq 0$, is the periodic spectrum of (2.1), translated so that $\lambda_0 = 0$, $\varphi_h(\lambda)$ is the same as the map $\delta(\lambda)$ defined in Lemma 2.1, and then $\alpha_n$ is the length of the $n$-th band $B_n$ and $\gamma_n$ is the length of the $n$-th gap.

Most of our estimates of lengths depend on the following simple lemma.

LEMMA 3.1. *Assume $h$ is finite. Let $u(z)$ be a bounded harmonic function on $\Omega(h)$ such that*

$$u(z) = 0, \qquad z \in \partial\Omega(h), \quad |z| \text{ large}$$

*and let $U(\lambda) = u(\varphi_h(\lambda))$. Then for Lebesgue almost all $t \in \mathbb{R}$, the limit*

$$U(t) = \lim_{\eta \downarrow 0} U(t + i\eta)$$

*exists and is integrable, and*

$$\int_{-\infty}^{\infty} U(t)\, dt = \lim_{x \to \infty} 2\pi x^2 u(x + ix). \tag{3.4}$$

*In particular, the limit in (3.4) is finite and it is strictly positive if $u(z)$ is nonnegative but not identically zero.*

Notice that if $u(z)$ is the harmonic measure of a bounded Borel set $E \subset \partial\Omega(h)$, then $U(t)$ agrees almost everywhere with the characteristic function of $\varphi_h^{-1}(E)$ and the limit in (3.4) evaluates the length of $\varphi_h^{-1}(E)$.

*Proof.* The boundary value exists by Fatou's theorem because $U(\lambda)$ is a bounded harmonic function on $\mathcal{U}$; it is integrable, in fact bounded and compactly supported, because $U(t)=U(\varphi_h(t))=0$ if $t\in\mathbb{R}$ and $|t|$ is large. Moreover, for $\lambda=\xi+i\eta$,

$$U(\lambda)=\frac{1}{\pi}\int\frac{\eta}{(\xi-t)^2+\eta^2}\,U(t)\,dt,$$

so that by dominated convergence

$$\int_{-\infty}^{\infty}U(t)\,dt=\lim_{\eta\to\infty}\pi\eta U(\xi+i\eta),$$

uniformly in $|\xi|\le C$. By (3.2)

$$\eta=\operatorname{Im}\varphi_h^{-1}(x+ix)=2x^2+0(1),\qquad x\to\infty$$

$$\xi=\operatorname{Re}\varphi_n^{-1}(x+ix)=0(1),\qquad x\to\infty,$$

and since $u(x+ix)\to 0$ $(x\to\infty)$, we therefore have

$$\lim_{\eta\to\infty}\pi\eta U(\xi+i\eta)=\lim_{\eta\to\infty}2\pi x^2u(x+ix)$$

and (3.4). The limit is finite becuase the integral converges. If $u(z)$ is nonnegative but not identically zero, then $U(t)\ge 0$ and $U(\lambda)>0$ for all $\lambda\in\mathcal{U}$, and the integral representation of $U(\lambda)$ shows that $\int U(t)\,dt>0$. $\square$

We shall later need this refinement of the lemma:

$$\int_{-\infty}^{\infty}U(t)\,dt=\lim_{\eta\to\infty}2\pi x^2u(x+i(x+c))\tag{3.5}$$

for any constant $c$. The proof is the same.

THEOREM 3.2. *For all $h\in(l_1^2)^+$ and all $n\ge 1$,*

$$\alpha_n(h)\le(2n-1)\pi^2.$$

*Equality holds for a single $n$ if and only if $h=0$.*

*Proof.* If $h=0$ then $\varphi_h^{-1}(z)=z^2$, so that $\lambda_{2n-1}=n^2\pi^2$, $\lambda_{2n-2}=(n-1)^2\pi^2$ and $\alpha_n=(2n-1)\pi^2$. Fix $n$, let $h^{(k)}$ be the truncation (3.3) of $h$ and let $u_k(z)$ be the harmonic measure of $((n-1)\pi, n\pi)\subset\partial\Omega_k=\partial\Omega(h^{(k)})$. By the maximum principle $u_k(z)-u_{k+1}(z)$ is harmonic, nonnegative and bounded on $\Omega_{k+1}$ and it is strictly positive if $h_{k+1}>0$. The lemma then applies to $u_k-u_{k+1}$ to give

$$\alpha_n(h^{(k)})-\alpha_n(h^{(k+1)})\geq 0$$

and

$$\alpha_n(h^{(k)})-\alpha_n(h^{(k+1)})\gneqq 0$$

if $h_{k+1}>0$. Thus $\alpha_n(h^{(k)})$ is nonincreasing in $k$ and it jumps down at each $k$ with $h_k>0$. Hence $\alpha_n(h^{(k)})\leq\alpha_n(0)=(2n-1)\pi^2$, with equality if and only if $h^{(k)}=0$. The theorem now follows because by Courant's theorem $\alpha_n(h)=\lim_{k\to\infty}\alpha_n(h^{(k)})$. □

THEOREM 3.3. *If $h\in(l_1^2)^+$ then*

$$\gamma_n(h)\leq 4\operatorname{Max}(2\pi n h_n, h_n^2), \tag{3.6}$$

*and*

$$\sum_{N\geq 1}(\gamma_N(h))^2\leq 64\pi^2\sum_{n\geq 1}n^2h_n^2+16\Big(\sum_{n\geq 1}n^2h_n^2\Big)^2$$

Note that if

$$\gamma_n^*(h)=\sup_k\gamma_n(h^{(k)})$$

where $h^{(k)}$ is the truncation of $h$ defined by (3.3), then by Theorem 3.3, we have

$$\gamma_n^*\leq 4\operatorname{Max}(2\pi n h_n, h_n^2),$$

so that $\sum(\gamma_n^*)^2<\infty$.

*Proof.* The $l^2$ estimate follows from the pointwise estimate because

$$\sup_n h_n^2/n^2\leq\sum_{n\geq 1}n^2h_n^2 \quad\text{and}\quad \sum_{n\geq 1}h_n^4\leq\Big(\sup_n h_n^2/n^2\Big)\sum_{n\geq 1}n^2h_n^2$$

In proving (3.6) we may, by Courant's theorem, assume $h$ is finite. So let $h$ be finite and let $\omega_n(z)$ be the harmonic measure of $T_n$ in $\Omega(h)$. By the lemma

$$\gamma_n(h)=\lim_{x\to\infty} 2\pi x^2\omega_n(x+ix)$$

and by the maximum principle and the lemma, $\gamma_n(h)\le\gamma_n(h')$ where $h'_m=\delta_{n,m}h_m$, because replacing $h$ by $h'$ does not decrease $\omega_n(z)$. So replace $h$ by $h'$. Then $z\to z^2$ maps $\Omega(h)$ into $\mathcal{U}\backslash\Gamma_n$ where $\Gamma_n$ is the parabolic arc

$$\Gamma_n=\{(n^2\pi^2-s^2)+2\pi ins:0\le s\le h_n\}$$

and $\omega_n(z)=W_n(\sqrt{z})$, where $W_n$ is the harmonic measure of $\Gamma_n$ in $\mathcal{U}\backslash\Gamma_n$. Enclose $\Gamma_n$ in a closed disc $D_n$ with center $n^2\pi^2-h_n^2$ and smallest radius

$$r_n=\text{Max}\,(2\pi h_n,\ h_n^2)$$

On $\mathcal{U}\backslash D_n$ the harmonic measure of the orthogonal semicircle $\mathcal{U}\cap\partial D_n$ is

$$W'_n(\zeta)=\frac{2}{\pi}\int_{\mathbb{R}\cap D_n}\frac{\eta}{(\xi-t)^2+\eta^2}\,dt,\qquad \zeta=\xi+i\eta$$

which is $2/\pi$ times the angle of visibility of $\mathbb{R}\cap D_n$ at the point $\zeta$. By the maximum principle $W_n(\zeta)\le W'_n(\zeta)$, $\zeta\in\mathcal{U}\backslash D_n$, and by the lemma

$$\begin{aligned}\gamma_n(h)&=\lim_{\eta\to\infty}\pi\eta W_n(i\eta)\\ &\le\lim_{\eta\to\infty}\pi\eta W'_n(i\eta)\\ &=2\,\text{meas}\,(\mathbb{R}\cap D_n)=4r_n\end{aligned}$$

which is (3.6). □

For the Marčenko-Ostrovskiĭ characterization of spectra we need two further estimates.

THEOREM 3.4. *Let* $h\in(l_1^2)^+$. *Then*
(a) *There is a constant* $c=c(h)$ *such that*

$$\lambda_{2n-1}(h)=n^2\pi^2+c+l^2(n)$$
$$\lambda_{2n}(h)=n^2\pi^2+c+l^2(n)$$

*and*

(b) $\lim_{k\to\infty} \sum_n (\lambda_{2n}(h^{(k)}) - \lambda_{2n}(h))^2 = 0$

*where $h^{(k)}$ is the truncation of $h$.*

*Proof.* Part (a). Since $\gamma_n = \lambda_{2n} - \lambda_{2n-1} \in l^2$, it is enough to consider $\lambda_{2n}$. We first reduce the proof to showing

$$\lambda_{2n}^{(n)} = \lambda_{2n}(h^{(n)}) = n^2\pi^2 + c(h) + l^2(n), \tag{3.7}$$

where $h^{(n)}$ is the truncation (3.3). Write $u_n^{(N)}(z)$ for the harmonic measure in $\Omega_N = \Omega(h^{(N)})$ of $\partial\Omega_N \cap \{0 < \operatorname{Re} z \le n\pi\}$, so that

$$\lambda_{2n}^{(N)} = \lambda_{2n}(h^{(N)}) = \lim_{x\to\infty} 2\pi x^2 u_n^{(N)}(x+ix),$$

and let $d\omega^{(N)}(z, \zeta)$ be the element of harmonic measure for $z \in \Omega_N$, $\zeta \in \partial\Omega_N$. Comparing boundary values, we see that for $N > n$,

$$u_n^{(n)}(z) = u_n^{(N)}(z) + \sum_{k=n+1}^{N} \int_{T_k} u_n^{(n)}(\zeta)\, d\omega^{(N)}(z, \zeta),$$

$z \in \Omega_n$, from which Lemma 3.1 and Courant's theorem give

$$\begin{aligned} 0 \le \lambda_{2n}^{(n)} - \lambda_{2n} &= \lim_{N\to\infty} (\lambda_{2n}^{(n)} - \lambda_{2n}^{(N)}) \\ &\le \sum_{k=n+1}^{\infty} \left(\sup_{\zeta\in T_k} u_n^{(n)}(\zeta)\right)\left(\sup_{N\ge n} \gamma_k(h^{(N)})\right). \end{aligned}$$

For $\zeta \in T_k$, $k > n$,

$$u_n^{(n)}(\zeta) \le \frac{2}{\pi} \arctan\left(\frac{\operatorname{Im}\zeta}{(k-n)\pi}\right) \le \text{Const.}\, \frac{h_k}{k-n},$$

because the middle term is the harmonic measure at $\zeta$ of $\{n\pi + iy : 0 < y < \infty\}$ in the quarter plane $\{y > 0, x > n\pi\}$ and this harmonic measure dominates $u_n^{(n)}(\zeta)$ on $\partial\Omega_n$. Also, by Theorem 3.3,

$$\sup_{N\ge n} \gamma_k(h^{(N)}) \le \gamma_k^* \in l^2,$$

and hence

$$|\lambda_{2n}-\lambda_{2n}^{(n)}|\le \text{Const.}\sum_{k=n+1}^{\infty}\frac{h_k\gamma_k^*}{k-n}=\text{Const. }\delta_n. \tag{3.8}$$

But $\delta_n\in l^2$ since

$$\delta_n\le\frac{1}{n}\sum_{k=n+1}kh_k\gamma_k^*\le\frac{1}{n}\left(\sum k^2h_k^2\right)^{1/2}\left(\sum(\gamma_k^*)^2\right)^{1/2}.$$

Therefore proving (3.7) will prove part (a).

Now consider

$$v_n(z)=\frac{1}{\pi}\int_0^{n^2\pi^2}\frac{\operatorname{Im}(z^2)\,dt}{(t-\operatorname{Re}(z^2))^2+(\operatorname{Im}(z^2))^2},$$

which is the harmonic measure of $\{0<x<n\pi\}$ in the quarter plane $\Omega(0)=\{x>0,\ y>0\}$. By the lemma

$$\lambda_{2n}^{(n)}-n^2\pi^2=\lim_{x\to\infty}2\pi x^2(u_n^{(n)}(x+ix)-v_n(x+ix)),$$

while by integrating boundary values, we have

$$u_n^{(n)}(z)-v_n(z)=\sum_{k=1}^{n}\int_{T_k}(1-v_n(\zeta))\,d\omega^{(n)}(z,\zeta).$$

Write $1-v_n(\zeta)=V_1(\zeta)+V_2^{(n)}(\zeta)$, where

$$V_1(\zeta)=\frac{2}{\pi}\arg\zeta$$

is the harmonic measure of $\{iy: y>0\}$ in $\Omega(0)$, and

$$V_2^{(n)}(\zeta)=\frac{1}{\pi}\int_{n^2\pi^2}^{\infty}\frac{\operatorname{Im}(\zeta^2)\,dt}{(t-\operatorname{Re}(\zeta^2))^2+(\operatorname{Im}(\zeta^2))^2},$$

and let

$$A_n=\lim_{x\to\infty}2\pi x^2\sum_{k=1}^{n}\int_{T_k}V_1(\zeta)\,d\omega^{(n)}(x+ix,\zeta), \tag{3.9}$$

and

$$B_n = \lim_{x\to\infty} 2\pi x^2 \sum_{k=1}^{n} \int_{T_k} V_2^{(n)}(\zeta)\, d\omega^{(n)}(x+ix, \zeta). \tag{3.10}$$

Then $\lambda_{2n}^{(n)} - n^2\pi^2 = A_n + B_n$, and (3.7) will be proved by establishing that $B_n \in l^2$ and that for some constant $c = c(h)$,

$$\{A_n - c(h)\} \in l^2.$$

First consider $B_n$. At $\zeta \in T_k$ we have

$$V_2^{(n)}(\zeta) \le \text{Const.} \frac{h_k}{n-k}, \qquad k < n$$

and

$$V_2^{(n)}(\zeta) \le \tfrac{1}{2}, \qquad k = n.$$

Consequently

$$B_n \le \frac{\gamma_n^*}{2} + \text{Const.} \sum_{k=1}^{n-1} \frac{h_k \gamma_k^*}{n-k}.$$

By Theorem 3.3, $\gamma_n^* \in l^2$, and since $k(n-k) \ge n-1$, $1 \le k \le n-1$,

$$\begin{aligned}\sum_{k=1}^{n-1} \frac{h_k\gamma_k^*}{n-k} &\le \frac{1}{n-1} \sum k h_k \gamma_k^* \\ &\le \frac{1}{n-1} \left(\sum k^2 h_k^2\right)^{1/2} \left(\sum (\gamma_k^*)^2\right)^{1/2},\end{aligned}$$

and hence $B_n \in l^2$.

To study $A_n$, observe first that because $V_1(\zeta) \le 2h_k/\pi^2 k$, $\zeta \in T_k$, we have

$$\lim_{x\to\infty} 2\pi x^2 \int_{T_k} V_1(\zeta)\, d\omega^{(n)}(x+ix, \zeta) \le \frac{2h_k}{\pi^2 k} \gamma_k^*.$$

By the maximum principle the limit is nonnegative and it is nonincreasing in $n$.

Therefore the limit

$$a_k = \lim_{N\to\infty} \lim_{x\to\infty} 2\pi x^2 \int_{T_k} V_1(\zeta)\, d\omega^{(N)}(x+ix, \zeta)$$

exists. The constant in (3.7) will be $c(h) = \sum_{k=1}^{\infty} a_k$. Now

$$A_n - c(h) = \sum_{k=1}^{n} \left(-a_k + \lim_{x\to\infty} 2\pi x^2 \int_{T_k} V_1(\zeta)\, d\omega^{(n)}(x+ix, \zeta)\right)$$

$$- \sum_{k=n+1}^{\infty} a_k = C_n + D_n. \tag{3.11}$$

Since $a_k \le (2h_k/\pi^2 k)\gamma_k^*$, Theorem 3.3 shows that both series $D_n$ and $c(n) = -D_0$ are convergent and also that

$$\sum_{n\ge 1} D_n^2 < \infty,$$

because

$$D_n \le \frac{\text{Const.}}{(n+1)^2} \sum_{k=n+1}^{\infty} (kh_k)\gamma_k^* \le \frac{\text{Const.}}{(n+1)^2}$$

Finally, we have

$$C_n = \sum_{k=1}^{n} \lim_{N\to\infty} \lim_{x\to\infty} 2\pi x^2 \int_{T_k} V_1(\zeta)(d\omega^{(n)}(x+ix, \zeta) - d\omega^{(N)}(x+ix, \zeta)),$$

and by the maximum principle $d\omega^{(n)}(z, \zeta) - d\omega^{(N)}(z, \zeta) \ge 0$ on $T_k$, so that

$$C_n \le \text{Const.} \sum_{k=1}^{n} \frac{h_k}{k} \lim_{N\to\infty} (\gamma_k^{(n)} - \gamma_k^{(N)}).$$

LEMMA 3.5. *For $n \ge k$,*

$$\sup_{N>n} (\gamma_k^{(n)} - \gamma_k^{(N)}) \le \text{Const.} \sum_{j=n+1}^{\infty} \frac{h_j \gamma_j^*}{j-k} = \text{Const.}\ \delta_n$$

*where $\delta_n$ is defined in* (3.8).

Accepting this lemma for a moment, we use it to note that

$$C_n \leq \text{Const.} \left( \sum_{k=1}^{\infty} \frac{h_k}{k} \right) \cdot \delta_n$$

and hence that $C_n \in l^2$ because, as we showed above, $\delta_n \in l^2$.

To summarize, we now conclude that $A_n - c(h) = C_n + D_n \in l^2$ and consequently that $\lambda_{2n}^{(n)} - n^2\pi^2 - c(h) \in l^2$. That proves (3.7) and part (a).

*Proof of Lemma* 3.5. Write $\omega_k^{(n)}(\zeta)$ for the harmonic measure at $\zeta \in \Omega_n = \Omega(h^{(n)})$ of the set $T_k \subset \partial\Omega_n$, $k \leq n$. Then

$$\gamma_k^{(n)} - \gamma_k^{(N)} = \lim_{x \to \infty} 2\pi x^2 \sum_{j=n+1}^{N} \int_{T_j} \omega_k^{(n)}(\zeta)\, d\omega^{(N)}(x + ix, \zeta).$$

For $\zeta \in T_j$, $j > n$, we see, comparing $\omega_k^{(n)}(\zeta)$ to the harmonic measure of $\{k\pi + iy, 0 < y < \infty\}$ in the quarter plane $\{x > k\pi, y > 0\}$, that

$$\omega_k^{(n)}(\zeta) \leq \text{Const.} \frac{h_j}{j-k}, \qquad \zeta \in T_j.$$

Consequently

$$\gamma_k^{(n)} - \gamma_k^{(N)} \leq \text{Const.} \sum_{j=n+1}^{\infty} \frac{h_j \gamma_j^*}{j-k} = \text{Const.}\ \delta_n.$$

Part (b). As in the reduction of part (a) to (3.7), we have for $k \geq n$,

$$\begin{aligned} |\lambda_{2n}^{(k)} - \lambda_{2n}| &\leq \sum_{j=k+1}^{\infty} \left( \sup_{\zeta \in T_j} u_n^{(k)}(\zeta) \right) \gamma_j^* \\ &\leq \sum_{j=k+1}^{\infty} \text{Const.}\ h_j \gamma_j^*. \end{aligned}$$

And for $k < n$ we have

$$|\lambda_{2n}^{(k)} - \lambda_{2n}| \leq \sum_{j=n+1}^{\infty} \left( \sup_{\zeta \in T_j} u_n^{(k)}(\zeta) \right) \gamma_j^* + \sum_{j=k+1}^{n} \sup_{\zeta \in T_j} (1 - u_n^{(k)}(\zeta)) \gamma_j^*$$

If $k$ is so large that $h_j \leq 1$, $j > k$, then $\sup_{\zeta \in T_j} (1 - u_n^{(k)}(\zeta)) \leq \text{Const.}\, h_j$, because $1 - u_n^{(k)}(\zeta)$ reflects to be harmonic on $\{\zeta : |\zeta - j\pi| < \pi\}$ and $u_n^{(k)}(\pi) = 1$. Hence we have

$$|\lambda_{2n}^{(k)} - \lambda_{2n}| \leq \text{Const.} \sum_{j=k+1}^{\infty} h_j \gamma_j^*$$

for all $n$ if $k$ is sufficiently large, and since

$$\left( \sum_{j=k+1}^{\infty} h_j \gamma_j^* \right)^2 \leq \frac{1}{(k+1)^2} \left(\sum j^2 h_j^2\right)\left(\sum (\gamma_j^*)^2\right),$$

(b) is proved. □

## 4. The Marčenko-Ostrovskiĭ Theorem

We say $q \in L^2_{\mathbb{R}}[0, 1]$ is *even* if $q(x) = q(1-x)$ and we let $E$ denote the subspace of even functions in $L^2_{\mathbb{R}}[0, 1]$.

THEOREM 4.1. *Let $h \in (l_1^2)^+$ and let $\varphi_h$ be the conformal mapping from the upper half plane to the slit quarter plane $\Omega(h)$ (normalized as in §3 above). Then there exists $q \in E$ such that*

$$\varphi_h(\lambda) = \delta(\lambda, q).$$

Except for the fact that the potential $q$ is even, this theorem was proved by Marčenko and Ostrovskiĭ (see Theorem 5.1 of [2]) by a different method. In this section we give an alternative proof, using the estimates of §3,and some ideas from [4], and we prove that $q$ can be chosen from $E$.

We first consider the roots

$$\mu_1(q) < \mu_2(q) < \cdots$$

of $y_2(1, \lambda, q) = 0$. The sequences $\mu_n$, $n \geq 1$, is called the *Dirichlet* spectrum of $q$, it is the set of eigenvalues of (2.1) with Dirichlet boundary conditions $y(0) = y(1) = 0$. It is well known [7] that $\lambda_{2n-1} \leq \mu_n \leq \lambda_{2n}$, and so the Dirichlet spectrum satisfies the estimate

$$\mu_n = n^2\pi^2 + \int_0^1 q(x)\, dx + l^2(n).$$

It is also well [7] known that $q(x)$ is even if and only if $\mu_n(q)=\lambda_{2n-1}(q)$ or $\lambda_{2n}(q)$ for all $n\geq 1$. We need the following characterization of Dirichlet spectra. Let $S$ be the Hilbert manifold of all increasing sequences $\sigma_n=n^2\pi^2+l^2(n)$, $n\geq 1$, and let $E_0\subset L^2_{\mathbb{R}}([0,1])$ be the subspace of even functions with mean 0. The fact we need is that all Dirichlet spectra are obtained by translating sequences in $S$:

THEOREM 4.2. *The map from $E_0$ to $S$ defined by*

$$E_0\ni q\to(\mu_1(q),\mu_2(q),\ldots)\in S$$

*is one-to-one, onto and bianalytic.*

*Proof.* See the Appendix for a proof of Theorem 4.2.

We next make a list of all possible functions $\Delta(\lambda, q)$.

LEMMA 4.3. *Let $\sigma\in S$, i.e. $\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\ldots)$ is any strictly increasing sequence of real numbers satisfying*

$$\sigma_n=n^2\pi^2+l^2(n).$$

*Then the series*

$$\Delta_\sigma(\lambda)=2\cos\sqrt{\lambda}+\sum_{n\geq 1}2[(-1)^n-\cos\sqrt{\sigma_n}]\prod_{m\neq n}\frac{\sigma_m-\lambda}{\sigma_m-\sigma_n}$$

*converges, uniformly on bounded subsets of $\mathbb{C}$, to an entire function. Moreover there is an even function $q(x)\in L^2_{\mathbb{R}}([0,1])$ with $\int_0^1 q(x)\,dx=0$ such that $\Delta(\lambda,q)=\Delta_\sigma(\lambda)$ and $\sigma_n=\mu_n(q)$, $n\geq 1$. Conversely, if $q\in L^2$* is even and *$\int_0^1 q\,dx=0$, then $\Delta(\lambda,q)=\Delta_\mu(\lambda)$ where $\boldsymbol{\mu}=(\mu_1,\mu_2,\ldots)$ is the Dirichlet spectrum of $q$.*

*Proof*: First suppose $q(x)$ is even and $\int_0^1 q(x)\,dx=0$. Then

$$\Delta(\lambda,q)-2\cos\sqrt{\lambda}=0\left(\frac{e^{|\mathrm{Im}\sqrt{\lambda}|}}{\lambda}\right)$$

and $\mu_m=\mu_m(q)=m^2\pi^2+l^2(m)$, so that the contour integral

$$I_N(\lambda)=\frac{1}{2\pi i}\int_{|z|=(N+1/2)^2\pi^2}\frac{\Delta(z,q)-2\cos\sqrt{z}}{z-\lambda}\left(\prod_{m\geq 1}\frac{m^2\pi^2}{\mu_m(q)-z}\right)dz$$

tends to 0 as $N \to \infty$. Direct application of the residue theorem yields

$$0 = (\Delta(\lambda) - 2\cos\sqrt{\lambda}) \prod_{m \geq 1} \frac{m^2\pi^2}{\mu_m - \lambda}$$

$$- \sum_{n \geq 1} (\Delta(\mu_n) - 2\cos\sqrt{\mu_n}) \frac{n^2\pi^2}{\mu_n - \lambda} \prod_{m \neq n} \frac{m^2\pi^2}{\mu_m - \mu_n}.$$

Multiplying both sides by $\prod_{m \geq 1} ((\mu_m - \lambda)/m^2\pi^2)$, we obtain

$$\Delta(\lambda, q) = 2\cos\sqrt{\lambda} + \sum_{n \geq 1} (\Delta(\mu_n) - 2\cos\sqrt{\mu_n}) \prod_{m \neq n} \frac{\mu_m - \lambda}{\mu_m - \mu_m},$$

$$= 2\cos\sqrt{\lambda} + \sum_{n \geq 1} 2((-1)^n - \cos\sqrt{\mu_n}) \prod_{m \neq n} \frac{\mu_m - \lambda}{\mu_m - \mu_m},$$

because $\mu_n(q) = \lambda_{2n-1}(q)$ or $\lambda_{2n}(q)$, $n \geq 1$. Therefore, $\Delta(\lambda, q) = \Delta_\mu(\lambda)$.

Conversely, if $\sigma_n$, $n \geq 1$, is a sequence satisfying the hypothesis of the lemma, then, by Theorem 4.2 there is a $q \in E_0$ such that $\sigma_n = \mu_n(q)$, $n \geq 1$. It follows from what we have already shown that

$$\Delta_\sigma(\lambda) = \Delta_{\mu(q)}(\lambda) = \Delta(\lambda, q).$$

The proof is finished. □

Unfortunately, the manifold $S$ of all sequences $\sigma$ which satisfy the hypotheses of Lemma 4.3 is not a moduli space for functions $\Delta(\lambda)$, nor a fortiori spectra, because many sequences in $S$ yield the same function. In fact, $\Delta_{\lambda^*}(\lambda) = \Delta(\lambda, q)$, $q \in E_0$, for any sequence $\boldsymbol{\lambda}^* = (\lambda_n^* = \lambda_{2n}(q)$ for $\lambda_{2n-1}(q), n \geq 1)$. It is for this reason that we must consider the conformal mappings $\delta(\lambda)$ and $\varphi_h(\lambda)$.

*Proof of Theorem* 4.1. We first treat the case of finite $h$. Let $\lambda_n^k = \lambda_n(h^{(k)})$, where $h^{(k)}$ is the truncation (3.3) of $h$. By Theorem 3.4 there is a constant $c_k = c(h^{(k)})$ such that $\lambda_{2n}^{(k)} = n^2\pi^2 + c_k + l^2(n)$. By Theorem 4.2 there is an even function $q_k \in L^2_{\mathbb{R}}[0, 1]$ with $\int_0^1 q_k\,dx = c_k$ such that

$$\mu_n = \mu_n(q_k) = \lambda_{2n}^k, \qquad n \geq 1$$

and

$$\Delta(\lambda, q_k) = \Delta(\lambda, \mu_n).$$

We now show

$$\lambda_n^k = \lambda_n(q_k).$$

(So far all we know is that $\lambda_{2n}^k$ is either $\lambda_{2n-1}(q_k)$ or $\lambda_{2n}(q_k)$.)

Let $\varphi_k(\lambda) = \varphi_{h^{(k)}}(\lambda)$ be the (normalized) conformal map from the upper half plane to $\Omega(h^{(k)})$. Then $\cos \varphi_k(\lambda)$ is entire and by (3.2)

$$\varphi_k(\lambda) = (\lambda - c_k)^{1/2} + O\left(\frac{1}{|\lambda|^{3/2}}\right), \qquad |\lambda| \text{ large},$$

so that

$$2 \cos \varphi_k(\lambda) = 2 \cos \sqrt{(\lambda - c_k)} + O\left(\frac{e^{|\operatorname{Im} \lambda|}}{|\lambda|^{3/2}}\right),$$

for large $|\lambda|$. Consequently, just as in the proof of Lemma 4.3,

$$\frac{1}{2\pi i} \int_{|z|=(N+1/2)^2\pi^2} \frac{2 \cos \varphi_k(z) - 2 \cos \sqrt{(z - c_k)}}{z - \lambda} \left(\prod_{j \geq 1} \frac{j^2\pi^2}{\lambda_{2j}^k - z}\right) dz$$

tends to 0 as $N \to \infty$, and

$$2 \cos \varphi_k(z) = 2 \cos \sqrt{z - c_k} + \sum_{j \geq 1} 2[(-1)^j - \cos \sqrt{\lambda_{2j}^k - c_k}] \prod_{m \neq j} \frac{\lambda_{2m}^k - z}{\lambda_{2m}^k - \lambda_{2j}^k}$$

since $\cos \varphi_k(\lambda_{2j}^k) = (-1)^j$. But applying Lemma 4.3 to $q_k - c_k$, which has zero mean and Dirichlet spectrum $\mu_j(q_k - c_k) = \mu_j(q_k) - c_k = \lambda_{2j}^k - c_k$, we obtain

$$\begin{aligned} \Delta(z, q_k) &= \Delta(z - c_k, q_k - c_k) \\ &= 2 \cos \sqrt{z - c_k} + \sum_{j \geq 1} 2[(-1)^j - \cos \sqrt{(\lambda_{2j}^k - c_k)} \prod_{m \neq j} \frac{\lambda_{2m}^k - z}{\lambda_{2m}^k - \lambda_{2j}^k}. \end{aligned}$$

Therefore $2 \cos \varphi_k(z) = \Delta(z, q_k)$ and $\lambda_j^k = \lambda_j(q_k)$ for all $j$, and from this it follows that

$$\varphi_k(\lambda) = \delta(\lambda, q_k).$$

The general case now follows by approximation. By part (b) of Theorem 3.4, the sequences

$$\boldsymbol{\mu}^k = (\mu_j^k = \mu_j(q_k) = \lambda_{2j}^k, j \geq 1)$$

converge in the space $S$ to

$$\boldsymbol{\mu} = (\lambda_{2j}(h), j \geq 1).$$

Hence by Theorem 4.2, $q_k$ converges to an even function $q \in L^2_{\mathbb{R}}[0, 1]$ and $\mu_j(q) = \lambda_{2j}(h)$, $j \geq 1$. But then since $\delta(\lambda, q_k)$ converges to $\delta(\lambda, q)$ uniformly on compact subsets of the upper half plane we have

$$\varphi_h(\lambda) = \lim_k \delta(\lambda, q_k) = \delta(\lambda, q)$$

and the proof is finished. □

The theorem can also be proved without using the approximations $q_k$. The proof of Theorem 3.4 shows that

$$\varphi_h^{-1}(z) = z^2 + c(h) + 0\left(\frac{1}{|z|}\right)$$

when $h \in (l_1^2)^+$ and $z \in \partial\Omega(h)$, Re $z > 0$. A reflection across the positive imaginary axis and a Phragmén-Lindelöf argument then gives

$$\varphi_h(\lambda) = (\lambda - c(h))^{1/2} + O\left(\frac{1}{|\lambda|}\right), \qquad \lambda \to \infty,$$

and hence we have

$$2 \cos \varphi_h(\lambda) = 2 \cos \surd(\lambda - c(h)) + O\left(\frac{e^{|\mathrm{Im} \surd\lambda|}}{|\lambda|}\right)$$

even when $h$ is not finite. The proof now follows as in the finite case.

## 5. Proofs of Theorems 1, 2 and 3

Write $\mathbb{R}_+^N = \{x \in \mathbb{R}^N : x_n > 0, 1 \leq n \leq N\}$ and regard $\mathbb{R}_+^N$ both as the subspace $\{h_n = 0,\ n > N;\ h_n > 0,\ n \leq N\}$ of $(l_1^2)^+$ and as the subspace $\{\gamma_n = 0,\ n > N;$

$\gamma_n > 0, n \leq N\}$ of $(l^2)^+$. Then we have defined a mapping $h \to \gamma_n(h)$ from $\mathbb{R}_+^N$ into $\mathbb{R}_+^N$ because, as we have seen, $\gamma_n = 0$ if and only if $h_n = 0$. Theorems 1 and 2 are consequences of

LEMMA 5.1. *From $\mathbb{R}_+^N$ to $\mathbb{R}_+^N$ the map $h \to \gamma_n(h)$ is real analytic. It satisfies*

$$\frac{\partial \gamma_n}{\partial h_n} > 0, \qquad \frac{\partial \gamma_k}{\partial h_n} < 0, \qquad k \neq n \tag{5.1}$$

$$\frac{\partial \gamma_n}{\partial h_n} + \sum_{k;\, k \neq n} \frac{\partial \gamma_k}{\partial h_n} > \text{Const. } n e^{-(M+h_n)/2}, \tag{5.2}$$

*where* $M = \max\{h_n : n = 1, 2, \ldots\}$.

The main use of the lemma is the observation that the Jacobian of $\gamma_n(h)$ is never zero, since by (5.1) and (5.2) the diagonal entry of each column dominates the absolute sum of the rest of that column.

*Proof.* Real analyticity will be proved in the next section. For $h \in \mathbb{R}_+^N$, $z \in \Omega(h)$ and $1 \leq n \leq N$, write

$$\omega_n(z) = \omega_n(h, z) = \omega(\Omega(h), T_n, z),$$

the harmonic measure of $T_n$ at $z$, relative to the domain $\Omega(h)$, so that by Lemma 3.1,

$$\gamma_n(h) = \lim_{x \to \infty} 2\pi x^2 \omega_n(x + ix).$$

By the maximum principle, an increase in $h_n$ will increase $\omega_n(z)$ and thus $\gamma_n$, but it will decrease $\omega_k(z)$ and $\gamma_k$, $k \neq n$. Hence we have the weak form of (5.1),

$$\frac{\partial \gamma_n}{\partial h_n} \geq 0, \qquad \frac{\partial \gamma_k}{\partial h_n} \leq 0, \qquad k \neq n.$$

Fix $n$, let $e_n$ be the unit vector $(e_n)_j = \delta_{n,j}$, let $t > 0$ and consider the positive harmonic function

$$V_t(h, z) = \frac{1}{t} \sum_{k=1}^{N} (\omega_k(h + te_n, z) - \omega_k(h, z))$$

$z \in \Omega(h+te_n)$. We bound $V_t(h, z)$ from below. Let $I_t$ be the segment

$$I_t = \{n\pi + i(h_n + s) : 0 < s \leq t\}$$

and let $\Omega_n^*(b)$, $b > 0$, be the slit strip

$$\Omega_n^*(b) = \{|x - n\pi| < \pi,\ y > 0\} \backslash \{n\pi + iy : 0 < y \leq b\}$$

with base

$$B_n = \{(n-1)\pi < x < (n+1)\pi\}.$$

On $\partial\Omega_n^*(h_n + t)$ we have

$$\begin{aligned} V_h(t, \zeta) &\geq \frac{1}{t}\left(1 - \sum_{k=1}^{N} \omega_k(h, \zeta)\right)\chi_{I_t}(\zeta) \\ &\geq \frac{1}{t}\, \omega(\Omega_n^*(h_n), B_n, \zeta)\chi_{I_t}(\zeta). \end{aligned} \tag{5.3}$$

where $\chi_E$ denotes the characteristic function of $E$. The estimate of $\omega(\Omega_n^*(h_n), B_n, \zeta)$, $\zeta \in I_t$, is in two cases.

*Case* 1. $h_n + t \leq \pi/2$. In terms of the coordinate $w = (z/\pi) - n$, $\Omega_n^*(h_n)$ contains the slit half disc

$$D = \{|w| < 1, \operatorname{Im} w > 0\} \backslash \{iy : 0 < y < h_n/\pi\},$$

which has diameter $B_n$ and which contains $I_t$. The mapping $\tau(w) = \{(\pi^2 w^2 + h_n^2)/(\pi^2 + h_n^2 w^2)\}^{1/2}$ sends the slit half disc into the full half disc $\{|\tau| < 1, \operatorname{Im} \tau > 0\}$ so that $B_n$ corresponds to the two segments

$$C_n = \tau(B_n) = [-1, -h_n/\pi] \cup [h_n/\pi, 1]$$

and so that $Z = \zeta(s) = n\pi + i(h_n + s) \in I_t$ falls on

$$w = i\sigma(s) = i\pi\sqrt{s}\left\{\frac{2h_n + s}{\pi^4 - h_n^2(h_n + s)^2}\right\}^{1/2}.$$

Therefore

$$\omega(\Omega_n^*(h_n), B_n, \zeta(s)) \geq \omega(D, C_n, i\sigma(s)) \geq \frac{2}{\pi}\arctan\frac{\pi\sigma(s)}{h_n} - \frac{4}{\pi}\arctan\sigma(s).$$

$$\geq \text{Const.}\left(\frac{s}{h_n}\right)^{1/2} + O\left(\left(\frac{s}{h_n}\right)^{3/2}\right),$$

with a positive constant.

*Case* 2. $h_n + t \geq \pi/2$. From a comparison with the half strips $\{j\pi < x < (j+1)\pi, 0 < y < \infty\}$, $j = n-1, n$, we have $\omega(\Omega_n^*(h_n), B_n, z) \geq e^{-y}|\sin x|$, and hence $\omega(\Omega_n^*(h_n), B_n, z) \geq \text{const.}\, e^{-h_n}$ on the two horizontal segments $\{\pi/4 < |x - n\pi| < 3\pi/4,\ y = h_n - 1\}$. Repeating the argument of Case 1 with the slit half disc

$$\{|z - (n\pi + i(h_n - 1))| < \pi, y, y > h_n - 1\}\backslash\{n\pi + iy : y \leq h_n\}$$

then yields

$$\omega(\Omega_n^*(h_n), B_n, \zeta(s)) \geq \text{Const.}\, e^{-h_n}\sqrt{s}.$$

Together the two cases give us

$$\omega(\Omega_n^*(h_n), B_n, \zeta(s)) \geq A(h_n)\sqrt{s}$$

with

$$A(h_n) = \begin{cases} \text{Const.}/\sqrt{h_n}, & h_n \text{ small} \\ \text{Const.}\, e^{-h_n}, & h_n \text{ large.} \end{cases}$$

Now take $z_n = n\pi + i(h_n + 1)$, $0 < t < h_n/2$ and let

$$g(t) = \omega(\Omega_n^*(h_n + t), I_t\backslash I_{t/2}, z_n).$$

If $h_n \leq \pi/2$, a comparision with the slit half disc $\{|z - n\pi| < \pi\}\backslash\{n\pi + iy : 0 < y < h_n + t\}$, gives

$$g(t) \geq \text{Const.}\sqrt{h_n t}.$$

If $h_n > \pi/2$, a comparison to the slit disc

$$\{|z - (n\pi + ih_n)| < \pi/2\}\backslash\{n\pi + iy : h_n - \pi/2 < y < h_n + t\},$$

yields $g(t) \geq \text{Const.}\sqrt{t}$. Therefore (5.3) gives us

$$V_t(h, z_n) \geq \frac{1}{t} \cdot \inf_{I_t \setminus I_{t/2}} \omega(\Omega_n^*(h_n), B_n, \zeta) g(t)$$
$$\geq \text{Const.}\, e^{-h_n},$$

in both cases. Harnack's inequality gives the same lower bound, with a somewhat smaller constant, on $\{|z - z_n| < \frac{1}{2}\}$ and a final comparison with the strip

$$\{|x - n\pi| < \pi, y > h_n + 1\}$$

then yields

$$V_t(h, z) \geq \text{Const.}\, e^{-(y+h_n)/2}, \qquad |x - n\pi| < \pi/2, \qquad y > h_n + 1.$$

Finally, let $W$ be the quarter plane $\{x > 0,\ y > 1 + M = 1 + \max h_k\}$. Applying Lemma 3.1 to $W$, we see that

$$\lim_{x \to \infty} 2\pi x^2 V_t(h, x + i(x + M)) \geq \text{Const.}\, ne^{-(M+h_n)/2},$$

and hence by the remark following the statement of Lemma 3.1

$$\lim_{x \to \infty} 2\pi x^2 V_t(h, x + ix) \geq \text{Const.}\, ne^{-(M+h_n)/2},$$

which proves (5.2).

The proof that the inequalities (5.1) are strict is a very similar argument, with $V_t(h, z)$ replaced by $(1/t)(\omega_k(h + te_n, z) - \omega_k(h, z))$, and we omit the details.

Notice that the proof of (5.2) remains valid if we permit $h_j = 0$, for some $j \neq n$, and just delete the term $\partial\gamma_j/\partial h_n$, which is zero anyway.

The proof of Lemma 5.1 can also be used to show that $\gamma_n(h)$ is Lipschitz. Since we will need that fact, as well as the upper bound for $\partial\gamma_n/\partial h_n$, in the next section, we pause to prove it now. By (5.1) and (5.2)

$$\sum_{k \neq n} |\gamma_k(h + te_n) - \gamma_k(h)| \leq \gamma_n(h + te_n) - \gamma_n(h),$$

$t > 0$, so we only consider $\gamma_n(h + te_n) - \gamma_n(h)$. If $h_n = 0$, then by (3.6)

$$\gamma_n(h + te_n) - \gamma_n(h) \leq 8\pi nt, \qquad t > 0.$$

Assume $h_n > 0$. Then, for $t > 0$,

$$\gamma_n(h + te_n) - \gamma_n(h) = \lim_{x \to \infty} 2\pi x^2 \{\omega_n(h + te_n, x + ix) - \omega_n(h, x + ix)\}.$$

The difference is the harmonic function on $\Omega(h + te_n)$ with boundary value

$$(1 - \omega_n(h, \zeta))\chi_{I_t}(\zeta) \le \omega(\Omega_n^*(h_n), \partial\Omega_n^*(h_n) \setminus T_n, \zeta)\chi_{I_t}(\zeta).$$

Comparing $\Omega_n^*(h_n)$ to the slit disc

$$\{|z - (n\pi + ih_n)| \le \operatorname{Min}(1, h_n)\} \setminus \{n\pi + y : 0 < y \le h_n\}$$

gives

$$\sup_{\zeta \in I_t} \omega(\Omega_n^*(h_n), \partial\Omega_n^*(h_n) \setminus T_n, \zeta)$$
$$\le \text{Const.} \operatorname{Max}(1, h_n^{-1/2}) t^{1/2}.$$

Let $\delta = \operatorname{Min}(1, h_n/2)$ and let $z \in \Omega(h + te_n)$, $|z - (n\pi + ih_n)| = \delta$. Then

$$\omega(\Omega(h + te_n), I_t, z) \le \text{Const.} (t/\delta)^{1/2}, \qquad t < \delta/2,$$

by a comparison with a slit half plane. For the same choice of $z$ we also have $\omega_n(h, z) \ge \text{const.}$, and hence by the maximum principle

$$\omega_n(h + te_n, x + ix) - \omega_n(h, x + ix) \le \text{Const.} \operatorname{Max}(1, 1/h_n) \omega_n(h, x + ix) \cdot t.$$

Therefore by (3.6), we have

$$\frac{\gamma_n(h + te_n) - \gamma_n(h)}{t} \le \text{Const.} \operatorname{Max}(n, nh_n, h_n^2), \tag{5.4}$$

and $\gamma_n$ is Lipschitz.

*Proof of Theorem* 1. Let $\gamma_n$ be any sequence in $(l^2)^+$ and set

$$\gamma_n^N = \begin{cases} \operatorname{Max}\left(\gamma_n, \dfrac{1}{Nn^2}\right), & n \le N \\ 0 & n > N. \end{cases}$$

By the lemma the Jacobian of the map

$$h \to \gamma(h) = (\gamma_1(h), \ldots, \gamma_N(h))$$

from $\mathbb{R}_+^N$ to $\mathbb{R}_+^N$ is never zero. Hence, by the Inverse Function Theorem $\gamma$ is a diffeomorphism in some neighborhood of every point of $\mathbb{R}_+^N$ so that $\gamma$ is an open mapping from $\mathbb{R}_+^N$ to $\mathbb{R}_+^N$. Also, the map is proper because

$$(\text{const.})nh_n \le \gamma_n \le 4\pi \max(nh_n, h_n^2). \tag{5.5}$$

It follows that $\gamma$ maps onto $\mathbb{R}_+^N$, because a proper open map is onto. In particular, there is $h^{(N)} \in \mathbb{R}_+^N$ such that

$$\gamma_n(h^{(N)}) = \gamma_n^N$$

for all $n \ge 1$. By (5.5) the sequence $\{h^{(N)}, N \ge 1\}$ is bounded in the Hilbert space $l_1^2$. If $h \in l_1^2$ is a weak limit point of the sequence, then for some subsequence,

$$h_n^{(N_j)} \to h_n, \qquad (j \to \infty),$$

for all $n$, so that $h \in (l_1^2)^+$ and by Courant's theorem

$$\gamma_n(h) = \lim_j \gamma_n^{N_j} = \gamma_n$$

for all $n$.

*Proof of Theorem* 2. Fix distinct $h$ and $\tilde{h}$ in $(l_1^2)^+$. We show $\gamma(h) \ne \gamma(\tilde{h})$. Now because $\gamma_n = 0$ if and only if $h_n = 0$ and because (5.2) remains valid when we delete these indices $j$ for which $h_j = \gamma_j = 0$, we may assume $h_n \ne 0$ for all $n$. Choose $N$ so large that $h^{(N)} \ne \tilde{h}^{(N)}$, and let $\alpha \in \mathbb{R}^N$ be the unit vector

$$\alpha = \frac{\tilde{h}^{(N)} - h^{(N)}}{\|\tilde{h}^{(N)} - h^{(N)}\|} = (\alpha_1, \alpha_2, \ldots, \alpha_N)$$

where $\| \ \|$ is the euclidean norm $(\sum h_n^2)^{1/2}$ in $\mathbb{R}^N$. Set

$$\gamma^{(\alpha)}(h) = \sum_{n=1}^{N} \frac{\alpha_n}{|\alpha_n|} \gamma_n(h),$$

$h \in \mathbb{R}_+^N$.

Then

$$\gamma^{(\alpha)}(\tilde{h}^{(N)})-\gamma^{(\alpha)}(h^{(N)})=\int_0^1 \frac{d}{dt}(\gamma^{(\alpha)}(h^{(N)}+t(\tilde{h}^{(N)}-h^{(N)}))\,dt$$

$$=\|\tilde{h}^{(N)}-h^{(N)}\|\int_0^1\left(\sum_{k=1}^{N}\frac{\alpha_k}{|\alpha_k|}\sum_{n=1}^{N}\alpha_n\frac{\partial\gamma_k}{\partial h_n}(h^{(N)}+t(\tilde{h}^{(N)}-h^{(N)}))\right)dt$$

$$\geq\|\tilde{h}^{(N)}-h^{(N)}\|\int_0^1\sum_{n=1}^{N}|\alpha_n|\left(\frac{\partial\gamma_n}{\partial h_n}+\sum_{k,\,k\neq n}\frac{\partial\gamma_k}{\partial h_n}\right)(h^{(N)}+t(\tilde{h}^{(N)}-h^{(N)})\,dt$$

$$\geq \text{Const. } e^{-m_N}\|h^{(N)}-\tilde{h}^{(N)}\|\sum_{n=1}^{N}n\,|\alpha_n|$$

$$\geq \text{Const. } e^{-m_N}\|h^{(N)}-\tilde{h}^{(N)}\|,$$

by the lemma, where

$$m_N=\text{Max}\,\{\max\,(h_n,\tilde{h}_n),\ 1\leq n\leq N\}.$$

Thus the maps is one-to-one over $\mathbb{R}_+^N$, and the Cauchy-Schwarz inequality gives the estimate

$$\|\gamma(h^{(N)})-\gamma(\tilde{h}^{(N)})\|\geq \text{Const. }\frac{e^{-m_N}}{\sqrt{N}}\|h^{(N)}-\tilde{h}^{(N)}\|. \tag{5.6}$$

LEMMA 5.2. *If $h\in(l_1^2)^+$ and if $N$ is large, then*

$$\left\{\sum_{n=1}^{N}(\gamma_n(h)-\gamma_n(h^{(N)}))^2\right\}^{1/2}\leq\frac{\text{Const.}}{\sqrt{N}}\left\{\sum_{k=N}^{\infty}k^2h_k^2\right\}^{1/2}.$$

Accepting Lemma 5.2 temporarily, we see that for constants $C_1$ and $C_2$,

$$\left\{\sum_{n=1}^{N}(\gamma_n(h)-\gamma_n(\tilde{h}))^2\right\}^{1/2}\geq C_1\frac{e^{-m_\infty}}{\sqrt{N}}\left\{\sum_{n=1}^{N}(h_n-\tilde{h}_n)^2\right\}^{1/2}$$
$$-\frac{C_2}{\sqrt{N}}\left\{\sum_{n=N}^{\infty}(n^2h_n^2+n^2\tilde{h}_n^2)\right\}^{1/2} \tag{5.7}$$

If $N$ is large the second term is smaller than the first term and that proves Theorem 2.

*Proof of Lemma* 5.2. This resembles part of the proof of Theorem 3.4. Let $M > N$. Then $\gamma_n(h^{(M)}) - \gamma_n(h^{(N)})$ corresponds via Lemma 3.1 to the harmonic function on $\Omega(h^{(M)})$ having boundary values

$$v_n(\zeta) = \sum_{k=N+1}^{M} \omega(\Omega(h^{(N)}), T_n, \zeta)\chi_{T_k}(\zeta).$$

In Section 3 we saw that for $k > n$,

$$\sup_{\zeta\in T_k} \omega(\Omega(h^{(N)}), T_n, \zeta) \le \text{Const.} \frac{h_k}{k-n},$$

and that if $N$ is so large that $h_k \le k$, $k > N$,

$$\gamma_k(h^{(M)}) \le 8\pi k h_k.$$

Therefore

$$\gamma_n(h^{(M)}) - \gamma_n(h^{(N)}) \le \text{Const.} \sum_{k=N+1}^{\infty} \frac{kh_k^2}{k-n},$$

and by Courant's theorem,

$$\gamma_n(h) - \gamma_n(h^{(N)}) \le \text{Const.} \sum_{k=N+1}^{\infty} \frac{kh_k^2}{k-n}; \qquad 1 \le n \le N.$$

Let $\sum_1^N t_n^2 = 1$. Then

$$\begin{aligned}\sum_{n=1}^{N} t_n(\gamma_n(h) - \gamma_n(h^{(N)}) &\le \text{Const.} \sum_{n=1}^{N} t_n \sum_{k=N+1}^{\infty} \frac{kh_k^2}{k-n} \\ &= \text{Const.} \sum_{k=N+1}^{\infty} kh_k^2 \sum_{n=1}^{N} \frac{t_n}{k-n} \\ &\le \text{Const.} \sum_{k=N+1}^{\infty} kh_k^2 \sum_{n=1}^{N} \left\{\frac{1}{(k-n)^2}\right\}^{1/2} \\ &\le \text{Const.} \sum_{k=N+1}^{\infty} \frac{k^2h_k^2}{N},\end{aligned}$$

and the lemma follows.

*Proof of Theorem* 3. It is shown in [4] that the maps

$$q \to \mu(q) = (\mu_n(q), n \geq 1)$$

and

$$q \to \nu(q) = (\nu_n(q), n \geq 1)$$

from $E_0$ to $S$ are real analytic. So, $\gamma$ is real analytic.

Suppose $\gamma(q) = \gamma(\tilde{q})$ for some $q, \tilde{q} \in E_0$. Then, by Theorem 2, $q$ and $\tilde{q}$ have the same periodic spectrum since $|\gamma_n(q)| = |\gamma_n(\tilde{q})|$ $n \geq 1$. Using the additional information $\operatorname{sgn} \gamma_n(q) = \operatorname{sgn} \gamma_n(\tilde{q})$, $n \geq 1$, we may conclude that $\mu_n(q) = \mu_n(\tilde{q})$, $n \geq 1$, because they must both lie at the same end of the $n$th gap. However, as noted in Theorem 4.2, two even functions with the same Dirichlet spectrum are equal. Therefore the map is one to one.

Let $\gamma \in l^2$. By Theorems 1 and 2 there is a unique periodic spectrum $\lambda_0 = 0$, $\lambda_n$, $n \geq 1$ with $\lambda_{2n} - \lambda_{2n-1} = |\gamma_n|$, $n \geq 1$. For each $n \geq 1$ choose $\mu_n = \lambda_{2n}$ or $\lambda_{2n-1}$ and $\nu_n = \lambda_{2n-1}$ or $\lambda_{2n}$ so that $\gamma_n = \mu_n - \nu_n$. It is shown in [4] that there exists a unique even function whose Dirichlet and Neumann spectrum are $\mu_n$, $n \geq 1$ and $0, \nu_n$, $n \geq 1$ respectively. Thus, the map $\gamma$ is onto $l^2$.

It remains to show that $\gamma^{-1}$ is real analytic. Let $\gamma \in l^2$ and let $\lambda_0 = 0$, $\lambda_n$ be the endpoints of the gaps for the conformal map corresponding to $|\gamma| = (|\gamma_n|, n \geq 1)$. Set

$$\mu_n(\gamma) = \begin{cases} \lambda_{2n} & \gamma_n \geq 0 \\ \lambda_{2n-1} & \gamma_n \leq 0 \end{cases}. \tag{5.8}$$

We will show in Section 6 that $\mu(\gamma)$ is a real analytic map from $l^2$ to $S$. Let $e(\mu)(x)$ be the unique even function with Dirichlet spectrum $\mu$. It is shown in [4] that $e$ is a real analytic function $\mu$. Therefore, $e_0(\gamma) = e(\mu(\gamma))(x) - [e(\mu(\gamma))]$, where $[f] = \int_0^1 f\,dx$, is a real analytic map from $l^2$ to $E_0$. By construction, $e_0$ is the inverse of $\gamma$. The proof is finished. $\square$

## 6. Analyticity

To complete the proof of Theorem 3 we must show the map $\mu_n(\gamma)$ from $l^2$ to $S$, defined by (5.8), is real analytic. This will be done first by mapping $\gamma$ to the slit lengths $h_n$. For $h \in l_1^2$ defined $|h| \in (l_1^2)^+$ by $|h|_n = |h_n|$ and define

$$\gamma_n(h) = \operatorname{sgn}(h_n)\gamma_n(|h|).$$

Because $\gamma_n(|h|)=0$ if and only if $h_n=0$, the proofs of Theorem 1 and Theorem 2 show that $h \leftrightarrow \gamma(h)$ is a homeomorphism from $l_1^2$ onto $l^2$ (bicontinuity follows from (5.6) and (5.7)). Also define

$$\mu_n(h)=\begin{cases}\lambda_{2n}(|h|), & h_n \geq 0\\ \lambda_{2n-1}(|h|), & h_n<0\end{cases} \tag{6.1}$$

Then because $\lambda_{2n}=\lambda_{2n-1}$ if and only if $h_n=0$, $\mu_n$ is continuous on $l_1^2$. In this section we prove: (See Note added in proof p. 312).

THEOREM 6.1. (a) *The map $h \to \gamma(h)$ and its inverse are real analytic.*

(b) *The map $h \to \{\mu_n(h)-c(h)\}$ where $c(h)$ is defined by Theorem* 3.4, *is a real anlytic map from $l_1^2$ into S.*

Together (a) and (b) complete the proof of Theorem 3; and they show that all three of the maps we have defined between $E_0$, $l^2$ and $l_1^2$ are bianalytic.

Recall that a map $F$ from an open subset $V$ of a complex Hilbert space $K_1$ to a complex Hilbert space $K_2$ is analytic if at each $x_0 \in V$ there is a ball $\{x:\|x-x_0\|<\varepsilon\} \subset V$ on which $F$ is bounded and if, whenever $y \in K_2$ and $x \in K_1$, $\|x\|<\varepsilon$, the $K_2$-inner product

$$z \to \langle F(x_0+zx), y\rangle \tag{6.2}$$

is analytic on $\{z \in \mathbb{C}:|z|<1\}$. A map from one real Hilbert space $H_1$ to another $H_2$ is real analytic if it can be extended to an analytic mapping from a neighborhood $V$ of $H_1$ in $\mathbb{C}\otimes H_1$ into $\mathbb{C}\otimes H_2$. The map is bianalytic if it is a bijection and if both it and its inverse have such extensions. By the Inverse Function Theorem, a bijective real analytic map from $H_1$ to $H_2$ is bianalytic if its Jacobian is invertible at each point of $H_1$.

We begin the proof by showing that the harmonic function which gives rise to $\gamma_n(h)$ is analytic in any finite number of variables, using a Schwarz iteration. Fix $n$ and fix $N>n$, and write

$$\omega_n(h, w)=\omega(\Omega(h), T_n, w),$$

$w \in \Omega(h)$, $h \in \mathbb{R}_+^N$. Also fix numbers $0<\delta_1'<\delta_2'<\delta_1<\delta_2<1$, to be determined later, and set $\varepsilon_k=\delta_1'/2k$.

LEMMA 6.2. *The function*

$$\mathbb{R}_+^N \ni h \to \omega_n(h, w)$$

*extends to a function* $\omega_n(t, w)$ *analytic in*

$$\{t \in \mathbb{C}^N : |\text{Im } t_k| < \varepsilon_k\}$$

*and harmonic in*

$$w \in W_N = \Omega(|\text{Re } t|) \Big\backslash \bigcup_{k=1}^{N} \Delta_k(t)$$

*where* $|\text{Re } t| = (|\text{Re } t_1|, |\text{Re } t_2|, \ldots, |\text{Re } t_N|) \in \mathbb{R}_+^N$, *and* $\Delta_k(t)$ *is the disc in the* $w$ *plane*

$$\Delta_k(t) = \begin{cases} |w - k\pi| \leq \delta_2/k, & |\text{Re } t_k| < \delta_1'/k \\ |w - (k\pi + i\,|\text{Re } t_k|)| < \delta_2'/k & |\text{Re } t_k| \geq \delta_1'/k \end{cases}$$

*The extension is odd as a function of* $t_n$ *and even as a function of* $t_k$, $k \neq n$. *It is bounded in* $w \in W_N(t)$ *and it vanishes on*

$$\partial W_N(t) \Big\backslash \left( T_n \cup \bigcup_{k=1}^{N} \partial\Delta_k(t) \right)$$

*Proof.* Fix $h \in \mathbb{R}_+^N$. We extend $\omega_n(\cdot, w)$ one variable at a time, beginning with $h_n$. Let $r_n = \delta_2/n$ and $b_n = \delta_1/n$

*Case* 1. $h_n < b_n$. Let $\Delta_t$ be the slit half disc

$$\Delta_t = \{z : |z| < r_n, \text{Im } z > 0\} \backslash \{iy : 0 < y \leq t\},$$

$0 \leq t < b_n$, and let $P_t(\zeta, z)\,|d\zeta|$ be the element of harmonic measure for $z \in \Delta_t$ on the semicricle

$$\Gamma_1 = \{|z| = r_n, \text{Im } z > 0\} \subset \partial\Delta_t.$$

with respect to $\Delta_t$. Then

$$w_t(z) = \left\{ \frac{(z/r_n)^2 + (t/r_n)^2}{1 + (z/r_n)^2 (t/r_n)^2} \right\}^{1/2}$$

is a conformal map from $\Delta_t$ to the half disc $D = \{|\lambda| < 1, \text{ Im } \lambda > 0\}$ and consequently

$$P_t(\zeta, z) = K(w_t(\zeta), w_t(z)) \left|\frac{dw_t(\zeta)}{d\zeta}\right|$$

where $K(\sigma, \lambda)$ is the Poisson kernel for $\lambda \in D$. Using the map $((1+\lambda)/(1-\lambda))^2$ from $D$ to the upper half plane, we see that

$$K(\sigma, \lambda) = \frac{1}{\pi} \operatorname{Im} \left( \frac{1}{\left(\frac{1+\lambda}{1-\lambda}\right)^2 - \left(\frac{1+\sigma}{1-\sigma}\right)^2} \right) \frac{2}{1-\sigma^2},$$

$\lambda \in D$, $\sigma \in \partial D$, so that by inspection $P_t(\zeta, z)$, $\zeta \in \Gamma_1$, is the sum of a power series convergent in $\{t \in \mathbb{C} : |t| < |z|\}$. By continuity

$$\sup_{|t|<b_n/2} \sup_{|z|=b_n} \int_{\Gamma_1} |P_t(\zeta, z)|\,|d\zeta| \leq 1$$

if $\delta_1/\delta_2 = b_n/r_n$ is small. Now set

$$W = (-n\pi + \Omega(h)) \cap \{|z| > b_n\}$$

and

$$\Gamma_0 = \{|z| = b_n, \operatorname{Im} z > 0\} \subset \partial W,$$

and let $Q(z, w)\,|dz|$ be the element of harmonic measure for $w \in W$ on $\Gamma_0$ relative to the domain $W$. Comparing $\Omega(h)$ to a half plane gives

$$\sup_{w \in \Gamma_1} \int_{\Gamma_0} Q(z, w)\,|dz| \leq C\delta_1/\delta_2, \tag{6.3}$$

with constant $C$ independent of $n$ and $h$. Therefore the operator

$$A_t f(w) = \int_{\Gamma_0} \int_{\Gamma_1} f(\zeta) P_t(\zeta, z)\,|d\zeta|\, Q(z, w)\,|dz|$$

from $L^\infty(\Gamma_1, |dz|)$ to the space of bounded harmonic functions on $W$ has a power series expansion in $\{t \in \mathbb{C} : |t| < b_n/2\}$ and

$$\sup_{w \in \Gamma_1} |A_t f(w)| \leq \text{Const.}\,(\delta_1/\delta_2) \sup_{\Gamma_1} |f(\zeta)| \tag{6.4}$$

The extended function $A_t f(w)$ is jointly continuous in $t$ and $w$, and since $A_t$ is an integral, $s \to A_t(f_s)(w)$ remains analytic in any complex parameter for which $s \to f_s$ is analytic. Hence by Hartog's Theorem

$$A_t^k f(w) = A_t \cdot \cdots \cdot A_t f(w)$$

is analytic in $\{t \in \mathbb{C} : |t| < b_n/2\}$.

Now let $v(t, z) = \omega(\Delta_t, [0, it], z)$ and $v_0(t, w) = \int_{\Gamma_0} v(t, z) Q(z, w)\, |dz|$. Then for $0 \leq t \leq \delta_1/2n$ and $w \in W$,

$$\begin{aligned} u(t, w) &= \omega(\Omega(h^{(N)} - h_n e_n + t e_n), T_n, w + n\pi) \\ &= \int_{\Gamma_0} u(t, z) Q(z, w)\, |dz| \\ &= v_0(t, w) + \int_{\Gamma_0} \int_{\Gamma_1} u(t, \zeta) P_t(z, \zeta)\, |d\zeta|\, Q(z, w)\, |dz|, \end{aligned}$$

because $u(t, z) - v(t, z) = \int_{\Gamma_1} u(t, \zeta) P_t(z, \zeta)\, |d\zeta|$, $z \in \Gamma_0$. Therefore

$$u(t, w) = \sum_{k=0}^{\infty} A_t^k v_0(t, \cdot)(w),$$

where the series converges uniformly $\{|t| < \delta_1/2n\}$ by (6.4). Because

$$v(t, z) = \int_{-t/r_n}^{t/r_n} K(x, w_t(z))\, dx, \tag{6.5}$$

$v(t, z)$ is analytic in $\{t \in \mathbb{C} : |t| < |z|\}$. Hence $v_0(t, z)$, $z \in \Gamma_0$ and $u(t, w)$, $w \in W$, have power series representations convergent in $\{t \in \mathbb{C} : |t| < b_n/2\}$. By (6.5), $\sup_{\Gamma_0} |v(t, z)| \leq C\, |t|/r_n$, so that by (6.3)

$$\sup_{|w| \geq r_n} |v_0(t, w)| \leq C\, |t|\, b_n/r_n^2$$

and thus

$$\sup_{w \in \Gamma_1} |u(t, w)| \leq |t|/r_n, \tag{6.6}$$

by (6.4) if $\delta_1/\delta_2 = b_n/r_n$ is small. Notice that on $\partial W$, $w \to u(t, w)$ vanishes except an $\Gamma_0$ and that by (6.5) and by the form of $w_t(z)$, $v(t, z)$, and hence $u(t, w)$, are odd functions of $t$.

*Case* 2. $h_n \ge b_n = \delta_1/n$. Let $a_n = \delta_1'/n$, $s_n = \delta_2'/n$ and let $\Delta_t$ be the slit disc $\{|z - ih_n| < s_n\} \backslash \{iy : h_n - s_n < y < t\}$, $h_n - a_n < t < h_n + a_n$, and let $P_t(\zeta, z)\,|d\zeta|$ be harmonic measure for $z \in \Delta_t$ on the curve $\Gamma_1 = \{|\zeta - ih_n| = s_n\} \subset \partial\Delta_t$ relative to $\Delta_t$. Using

$$w_t(z) = \left\{ \frac{\left(\frac{h_n + iz}{s_n}\right) + \left(\frac{t - h_n}{s_n}\right)}{1 + \left(\frac{t - h_n}{s_n}\right)\left(\frac{h_n + iz}{s_n}\right)} \right\}^{1/2}$$

for the conformal map from $\Delta_t$ to the half disc $D$, we see that $P_t(\zeta, z)$ is the sum of a power series convergent in $\{t \in \mathbb{C} : |t - h_n| < |z - ih_n|\}$ and that

$$\sup_{|t - h_n| < a_n/2} \; \sup_{|z - ih_n| = a_n} \int_{\Gamma_1} |P_t(\zeta, z)|\,|d\zeta| \le 1$$

if $\delta_1'/\delta_2'$ is small. Set

$$W = (-n\pi + \Omega(h)) \cap \{|z - ih_n| > a_n\}$$

and

$$\Gamma_0 = \{z : |z - ih_n| = a_n\} \subset \partial W,$$

and let $Q(z, w)\,|dz|$ be harmonic measure for $w \in W$ on $\Gamma_0$. Then

$$\sup_{w \in \Gamma_1} \int_{\Gamma_0} Q(z, w)\,|dz| \le \text{Const.}\,(a_n/s_n)^{1/2}$$

and

$$A_t f(w) = \int_{\Gamma_0} \int_{\Gamma_1} f(\zeta) P_t(\zeta, z)\,|d\zeta|\,Q(z, w)\,|dz|,$$

$f \in L^\infty(\Gamma_1, |d\zeta|)$, is analytic in $\{t \in \mathbb{C} : |t - h_n| < a_n/2\}$, and harmonic in $w \in W$, and

$$\sup_{w \in \Gamma_1} |A_t f(w)| \le \text{Const.}\,(a_n/s_n)^{1/2} \sup_{\Gamma_1} |f(\zeta)|.$$

Let

$$u(t, w) = \omega(\Omega(h + (t - h_n)e_n), T_n, w + n\pi)$$

and

$$u_0(w) = \omega(W, [0, i(h_n - a_n)], w).$$

Then for $|t - h_n| < a_n/2$,

$$\begin{aligned} u(t, w) &= u_0(w) + \int_{\Gamma_0} \int_{\Gamma_1} u(t, \zeta) P_t(\zeta, z)\, |d\zeta|\, Q(z, w)\, |dz| \\ &= \sum_{k=0}^{\infty} A_t^k u_0(w) \end{aligned}$$

with convergence uniform in $t$. Thus $u(t, w)$ extends to be analytic on $\{t \in \mathbb{C} : |t - h_n| < a_n/2\}$ and harmonic on $W$. If $a_n/s_n$ is small, then

$$\sup_{|w - ih_n| \geq s_n} |u(t, w)| \leq 3/2$$

and $u(t, w) = 0$, $w \in \partial W \backslash \Gamma_0$, so that if $\delta_2'$ is small

$$\sup_{|w - ih_n| \geq s_n} |u(t, w)| \leq 2u(h_n, w). \tag{6.7}$$

uniformly in $t$.

That extends $\omega(\Omega(h), T_n, z)$ to $\{t_n \in \mathbb{C} : |\mathrm{Im}\, t_n| < \varepsilon_n,\ \mathrm{Re}\, t_n > -\varepsilon_n\}$ $\varepsilon_n = \delta_1'/2n$, because if two of the power series constructed have intersecting domains, they coincide on the positive reals and hence everywhere. Since $u(-t, w) = -u(t, w)$, $|t|$ small, a reflection defines the function on $\{t_n \in \mathbb{C} : |\mathrm{Im}\, t_n| < \varepsilon_n\}$.

Next let $k \neq n$ and let $u_1(h, w)$, $h_j \geq 0$, $j \neq n$, $|\mathrm{Im}\, h_n| < \varepsilon_n$, be the analytic continuation of $\omega(h^N, T_n, w)$, made already. We repeat the above reasoning to obtain analyticity in $t = h_k$.

*Case* 3. $h_k < b_k = \delta_1/k$. As in Case 1 we have a slit half disc $\Delta_t = \{|z| < r_k = \delta_2/k,\ \mathrm{Im}\, z > 0\} \backslash \{(y : 0 < y \leq t\}$, $0 < t < b_k$, semicircles $\Gamma_1 = \{|z| = r_k,\ \mathrm{Im}\, z > 0\}$, and $\Gamma_0 = \{|z| = b_k,\ \mathrm{Im}\, z > 0\}$, a domain

$$W = ((n - k)\pi + W_0) \cap \{|z| > b_k\},$$

where

$$W_0=\begin{cases}(-n\pi)+\Omega(h))\cap\{|z|>r_n\}, & h_n<b_n\\ (-n\pi+\Omega(h))\cap\{|z-ih_n|>s_n\}, & h_n\geq b_n\end{cases}$$

is contained in the domain of the first extension $u_1(h, w)$, and kernels $P_t(\zeta, z)$ for $\Delta_t$ and $Q(z, w)$ for $W$. The operator

$$A_t f(w)=\int_{\Gamma_0}\int_{\Gamma_1} f(\zeta)P_t(\zeta, z)\,|d\zeta|\,Q(z, w)\,|dz|$$

again satisfies (6.4) and $A_t f(w)$ analytic in $\{t\in\mathbb{C}:|t|<b_k/2\}$.

Now let $u_0(w)$ be the solution to the Dirichlet problem in $W$ with boundary value $u_1(z)=u_1(h-h_k e_k, z+k\pi)$ on $\partial W\backslash\Gamma_0$ and $u_0=0$ on $\Gamma_0$. When $0\leq t\leq b_k$, we then have

$$\begin{aligned}u(t, w)&\equiv u_1(h-h_k e_k+te_k, w+k\pi)\\ &=u_0(w)+\int_{\Gamma_0} u(t, w)Q(z, w)\,|dz|\end{aligned}$$

because both sides have the same values on $\partial W$. Then since $u(t, z)$ is harmonic on $\Delta_t$ and $u(t, \zeta)=0$ on $\partial\Delta_t\backslash\Gamma_1$,

$$\begin{aligned}u(t, w)&=u_0(w)+\int_{\Gamma_0}\int_{\Gamma_1} u(t, \zeta)P_t(\zeta, z)\,|dz|\,Q(z, w)\,|dz|\\ &=\sum_{j=0}^{\infty} A_t^j u_0(w),\end{aligned}$$

is analytic in $\{t\in\mathbb{C}:|t|<b_k/2\}$. And since $u_1(z)=0$ on $\partial\Omega(h)\cap\{\operatorname{Re} w-k\pi\,|<\pi\}$, we have by (6.3)

$$\sup_{\Gamma_1}|u_0(z)-u_1(z)|\leq C(\delta_1/\delta_2)\sup_{\Gamma_1}|u_2(z)|.$$

Therefore (6.4) gives the estimate

$$\sup_{\Gamma_1}|u(t, w)|\leq(1+C\delta_1/\delta_2)\sup_{\Gamma_1}|u_1(z)|, \tag{6.8}$$

uniformly in $\{t\in\mathbb{C}:|t|<b_k/2\}$, if $\delta_1/\delta_2=b_k/r_k$ is small. Note also that $u(t, w)=$

$u_1(w)$ on $\{|w|>r_k\}\cap\partial W$. In this case $u(t, w)$ is even as a function of $t$, because $u_0(w)$ is independent of $t$ and because, by the formula for $w_t(z)$, $P_t(\zeta, z)$ is even in $t$.

*Case* 4. $h_k \geq b_k$. We take $a_k = \delta_1'/k$, $s_k = \delta_2'/k$ and proceed as in Case 2, but with domain

$$W=((n-k)\pi + W_0)\cap\{|z-ih_k|>a_k\},$$

where $W_0$ is as defined in Case 3, and with $u_0(w)$ the solution to the Dirichlet problem on $W$ with boundary value 0 on $\Gamma_0=\{|z-ih_k|=a_k\}$ and $u_1(h, w+k\pi)$ on $\partial W\backslash\Gamma_0$. Then

$$\begin{aligned} u(t, w) &= u_0(w)+\int_{\Gamma_0}\int_{\Gamma_1} u(t, \zeta)P_t(\zeta, z)\,|d\zeta|\,Q(z, w)\,|dz| \\ &= \sum_{j=0}^{\infty}(A_t^j u_0)(w) \end{aligned}$$

is analytic in $\{t\in\mathbb{C}:|t-ih_k|<a_k/2\}$ if $\delta_1'/\delta_2'=a_k/s_k$ is small. We now have the estimate

$$\sup_{\Gamma_1}|u(t, w)|\leq(1+c(\delta_1'/\delta_2')^{1/2})\sup_{\Gamma_1}|u_1(z)|, \tag{6.9}$$

uniformly in $t$. In this case $u(t, w)=u_1(w)$ on $\{|w-ih_k|>s_k\cap\partial W$.

By reflection the even function $u(t, w)$ has now been defined and is analytic in $\{t_k\in\mathbb{C}:|\mathrm{Im}\, t_k|<\varepsilon_k=\delta_1'/2k\}$. By Hartog's theorem $\omega_n(h, w)=\omega(\Omega(h), T_n, w)$ has been extended to be analytic in $\{(t_k, t_n)\in\mathbb{C}^2:|\mathrm{Im}\, t_k|<\varepsilon_k, |\mathrm{Im}\, t_n|<\varepsilon_n\}$ and harmonic in $w\in W$. Now repeat the arguments of Case 3 and Case 4 for the remaining variables $h_j$. The continuation is well-defined because it agrees with $\omega_n(h, w)$ when $h\in\mathbb{R}_+^N$ and because an analytic function in $\{t\in\mathbb{C}^N:|\mathrm{Im}\,|\,t_k|<\varepsilon_k, 1\leq k\leq N\}$ is determined by its values on $\mathbb{R}_+^N$. The construction shows that $\omega_n(t, w)$ is bounded and harmonic in $w\in W_N(t)$ and that its boundary values vanish except on $T_n$ and the circles or half circles $\partial\Delta_k(t)\cap\partial W_N(t)$. $\square$

By Lemma 6.2, and a normal families argument,

$$\gamma_n(h)=\lim_{x\to\infty} 2\pi x^2\omega_n(h, x+ix)$$

has analytic extension from $\mathbb{R}_+^N$ to $\{t\in\mathbb{C}^N:|\mathrm{Im}\, t_k|<\varepsilon_k\}$. We now make some estimates which will permit us to send $N$ to $\infty$ and simultaneously control $\sum|\gamma_n(t)|^2$.

LEMMA 6.3. *Let $M>0$, and let $1\le n\le N$. If $h\in\mathbb{R}^N$ and if $\sum k^2h_k^2\le M^2$, then there are $\delta_1(M)<\delta_2(M)$, independent of $n$ and $N$, and $\delta_1'(h)<\delta_2'(h)<\delta_1(M)$, depending on $h$ but not on $n$ and $N$, such that on $\{t\in\mathbb{C}^N:|t_k-h_k|<\varepsilon_k=\delta_1'(h)/2k\}$,*

$$|\gamma_n(t)|\le\gamma_n^*(h+\varepsilon_n e_n)+C/n^2. \tag{6.10}$$

Before giving its proof, we use Lemma 6.3 to show that the map

$$l_1^2\ni h\to\{\gamma_n(h)\}\in l^2$$

is real analytic. Fix $h\in l_1^2$ and let

$$V_h=\{t\in\mathbb{C}\otimes l_1^2:\sum k^2|t_k-h_k|^2<(\delta_1'(h))^2/4\}. \tag{6.11}$$

Let

$$t_j^{(N)}=\begin{cases}t_j & j\le N\\ 0 & j>N.\end{cases}$$

By (6.10) and Theorem 3.3, $\{\gamma_n(t^{(N)}):N\ge1\}$ is bounded in $\mathbb{C}\otimes l^2$.[3] Hence it has a weak limit $\gamma_n(t)\in\mathbb{C}\otimes l^2$, still satisfying (6.10). Thus we have a locally bounded map $F$ from a neighborhood of $l_1^2$ in its complexification to $\mathbb{C}\otimes l^2$. When $h$ is real, $F(h)=\{\gamma_n(h)\}$ since $\gamma_n(h^{(N)})$ converges in norm to $\gamma_n(h)$ by Lemma 3.5 and by reflection. To prove analyticity, let $x_0\in V_h$ and let $x\in\mathbb{C}\otimes l_1^2$ be such that $\{x_0+zx:z\in\mathbb{C},|z|<1\}\subset V_h$, and let $y=\{y_n\}\in\mathbb{C}\otimes l^2$. Then by weak convergence

$$\langle F(x_0+zx),y\rangle=\lim_{N\to\infty}\sum_{n=1}^{N}\bar{y}_n\gamma_n((x_0+zx)^N)=\lim_N f_N(z).$$

By Lemma 6.2, (6.10) and Theorem 3.3, $\{f_N(z)\}$ is a bounded sequence of analytic functions on $\{|z|<1\}$. Therefore (6.2) holds and the map is analytic.

*Proof of Lemma* 6.3. We shall use some facts from the proof of Lemma 6.2. By symmetry we may assume $h\in\mathbb{R}_+^N$. Set $k_1=n$ and write $u_1(t,w)$ for the first

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[3] For $n>N$ take $\gamma_n(t^{(N)})=0$.

extension of $\omega_n(\cdot, w)$. The $j$-th extension $u_j(t, w)$ is with respect to the variable $t_{k_j}$. Thus $\{k_1, \ldots, k_N\}$ is a reindexing of $\{1, \ldots, N\}$ with $k_1 = n$.[4] Then $u_j$ is harmonic on

$$W^{(j)} = W^{(j)}(h) = \Omega(h) \setminus \bigcup_{p=1}^{j} \Delta_{k_p}(h)$$

and $u_j = 0$ on $\partial(W^{(j)}) \setminus (T_n \cup \bigcup_{p=1}^{j} \partial\Delta_{k_p}(h))$. Moreover, $u_{j+1} = u_j$ on $\partial(W^{(j+1)}) \setminus \partial\Delta_{k_{j+1}}(h)$ and $u_{j+1}$ is constructed from $u_j$ via Case 3 or Case 4 of the proof of Lemma 6.2.

Let $\alpha_1 = \sup\limits_{|t_n - h_n| < \varepsilon_n} \sup\{|u_1(t, w)| : w \in \partial W^{(1)} \cap (T_n \cup \partial\Delta_n(h))\}$ and

$$\alpha_j = \sup\{|u_j(t, w)| : w \in \partial\Delta_{k_j}(h), |t_{k_l} - h_{k_l}| < \varepsilon_{k_l}, 1 \le l \le j\}.$$

Then

$$u_N(t, z) \le \alpha_1 \omega_n(h + \varepsilon_n e_n, z) + \sum_{j=2}^{N} \alpha_j \omega(W^{(j)}, \partial\Delta_{k_j}, z).$$

If $x \ge x_0(M)$, then

$$\omega(W^{(j)}, \partial\Delta_{k_j}, x + ix) \le C\omega(\Omega(h + \varepsilon_{k_j} e_{k_j}), \Delta_{k_j} \cap \partial\Omega(h + \varepsilon_{k_j} e_{k_j}), x + ix).$$

Therefore, by the Lipschitz estimates (5.4),

$$\begin{aligned} |\gamma_n(t)| &\le \alpha_1 \gamma_n(h + \varepsilon_n e_n) + C \sum_{j=2}^{N} \alpha_j \operatorname{Max}(M^2, k_j) \operatorname{diam}(\Delta_{k_j}) \\ &\le \alpha_1 \gamma_n(h + \varepsilon_n e_n) + CM^2 \sum_{j=2}^{N} \alpha_j, \end{aligned} \tag{6.12}$$

with $C$ independent of $M$ and $n$.

We have $\alpha_1 \le 2$ by (6.6) and (6.7). Let $\eta > 0$. Then if $\delta_1/\delta_2$ and $\delta_1'/\delta_2'$ are sufficiently small, (6.8), (6.9) and induction give

$$\alpha_j \le (1 + \eta) \sum_{l=1}^{j-1} \alpha_l \omega_j^{(l)}. \tag{6.13}$$

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[4] It will not matter which ordering of $k_j$, $j \ge 2$ is chosen.

where

$$\omega_j^{(1)} = \sup_{z\in\partial\Delta_{k_j}(h)} \omega(W^{(1)}, \partial W^{(1)} \cap (T_n \cup \partial\Delta_n(h)), z)$$

and

$$\omega_j^{(l)} = \sup_{z\in\partial\Delta_k(h)} \omega(W^{(l)}, \partial\Delta_{k_l}(h), z), \qquad 2\le l\le j.$$

We estimate the $\omega_j^{(l)}$.

*Case* 1. $h_{k_j} \le \delta_1/k_j$, $h_k \le \delta_1/k_l$. Let $\tilde W^{(l)} = \{x>0, y>0\}\backslash\Delta_k(h)$ and use the map $z\to z^2$ as in the proof of Theorem 3.3. That gives

$$\omega_j^{(l)} \le \sup_{z\in\partial\Delta_{k_j}} \omega(\tilde W^{(l)}, \partial\Delta_{k_l}, z) \le \frac{C\delta_2^2}{(k_j^2-k_l^2)^2}.$$

*Case* 2. $h_{k_l} > \delta_1/k_l$, $h_{k_j} \le \delta_1/k_j$. Since $k_l h_{k_l} \le M$, the proof of Theorem 3.3 now gives

$$\omega_j^{(l)} \le \frac{C(M)\delta_2}{(k_j^2-k_l^2)^2}$$

because $|z^2 - k_j^2\pi^2| \le C\delta_2$, $z\in\Delta_{k_j}$.

*Case* 3. $h_{k_j} > \delta_1/k_j$, $h_{k_l} \le \delta_1/k_l$. Again using the map $z\to z^2$, we see that

$$\omega_j^{(l)} \le \sup_{\Delta_{k_j}} \omega(\tilde W^{(l)}, \partial\Delta_{k_l}, z) \le \frac{C(M)\delta_2}{(k_j^2-k_l^2)^2}.$$

*Case* 4. $h_{k_j} > \delta_1/k_j$, $h_{k_l} \ge \delta_1/k_l$. Since $k^2 h_k^2 \le M^2$, there are at most $M^4/\delta_1^4$ pairs $(k_j, k_l)$ for which this case applies. Thus there is a constant $B(h, \delta_1)$ such that

$$\sup_{|z-(k_j\pi+ih_{k_j})|} \omega(W^{(l)}(h), T_{k_l} \cup \partial\Delta_{k_l}(h), z) \le \frac{B(h,\delta_1)}{(k_j^2-k_l^2)^2}$$

for all such pairs. But the above harmonic measure vanishes on $T_{k_j}$, so that we have

$$\omega_j^{(l)} \leq \frac{B(h, \delta_1)(\delta_2')^{1/2}}{(k_j^2 - k_l^2)^2}$$

Let $\delta > 0$. Choosing small $\delta_1(M)$ and $\delta_2(M)$ first, and then taking $\delta_1'(h)$ and $\delta_2'(h)$ very small, we obtain

$$\omega_j^l \leq \frac{\delta}{(k_j^2 - k_l^2)^2}. \tag{6.14}$$

in all cases.

By (6.13), (6.14) and induction,

$$\begin{aligned}\sum_{j=2}^{N} \alpha_j \leq 2\delta(1+\eta) \sum_{j=2}^{N} \frac{1}{(k_j^2 - k_1^2)} \\ + 2\delta^2(1+\eta)^2 \sum_{2 \leq j_1 < j_2 \leq N} \frac{1}{(k_{j_1}^2 - k_1^2)^2} \frac{1}{(k_{j_2}^2 - k_{j_1}^2)^2} \\ + 2 \sum_{p=3}^{N} \delta^p (1+\eta)^p \sum_{2 \leq j_1 < j_2 < \cdots < j_p \leq N} \prod_{\alpha=1}^{p} \frac{1}{(k_{j_\alpha}^2 - k_{j_{\alpha-1}})^2}\end{aligned}$$

with $j_0 = 1$. But

$$\sum_{\substack{k \neq n \\ k \geq 1}} \frac{1}{(k^2 - n^2)^2} \leq \frac{A}{n^2}$$

and consequently

$$\sum_{2 \leq j_1 < j_2 < \cdots < j_p} \prod_{\alpha=1}^{p} \frac{1}{(k_{j_\alpha}^2 - k_{j_{\alpha-1}}^2)^2} \leq \frac{A^p}{n^2}.$$

Hence if $A\delta < 1/2$, we have

$$\sum_{j=2}^{N} \alpha_j \leq C\delta / n^2 .$$

independent of $\mathbb{N}$. With (6.12), that proves Lemma 6.3. $\square$

The Jacobian of $\gamma(h)$ is the linear operator $J(h): l_1^2 \to l^2$ represented, with respect to the basis $\{e_n\}$, by the infinite matrix

$$A(h) = \{\partial\gamma_k/\partial h_n\}_{k,n\geq 1}.$$

Because $h \to \gamma(h)$ is real analytic, $J(h)$ is bounded for each $h \in l_1^2$.

LEMMA 6.4. *For each* $h \in l_1^2, J(h)$ *is one-to-one and onto, and hence invertible.*

By the Inverse Function Theorem, Lemma 6.4 implies that $h \to \gamma(h)$ is bianalytic.

*Proof.* Fix $h \in l_1^2$ and let $A_N(h)$ be the finite square matrix

$$\{\partial\gamma_k/\partial h_n\}_{1\leq k,n\leq N}$$

The proof of Theorem 2, and a reflection if $h_j < 0$, show that $A_N(h)$ is invertible and that

$$\|A_N(h)x\|_{l^2} \geq \frac{c_1(h)}{N^{1/2}} \|x\|_{l_1}. \tag{6.15}$$

where $c_1(h)$ does not depend on $N$.

Also, by (5.2) and (5.4) and by the fact that $\gamma_n$ is an odd function of $h_n$, we have

$$c_2(h) \leq \frac{1}{h} \partial\gamma_n/\partial h_n \leq 1/c_2(h) \tag{6.16}$$

for some positive constant $c_2(h)$.

We now estimate the off diagonal entries $\partial\gamma_k/\partial h_n$, $k \neq n$ for Max $(k, n) > N$ and we choose $N = N(h)$ so that $|jh_j| \leq 1$ if $j \geq N$. Because we will be bounding $|\partial\gamma_k/\partial h_n|$, we assume $h_j \geq 0$. Let

$$\Gamma_n = \{z \in \Omega(h) : |z - (n\pi + ih_n)| = 1\},$$

and

$$\alpha_{k,n} \neq \sup_{z\in\Gamma_n} \omega(\Omega(h), T_k, z).$$

The argument used to prove (5.4) shows that

$$|\partial\gamma_k/\partial h_n| \le C\alpha_{k,n}\gamma_n^*(h).$$

Using the map $z \to z^2$, we get the majorization

$$\alpha_{k,n} \le \frac{c_3(h)}{(k^2-n^2)^2}$$

where $c_3(h)$ depends only in $\sup_j |jh_j|$, and hence

$$|\partial\gamma_k/\partial h_n| \le \frac{c_4(h)}{(k^2-n^2)^2}\gamma_n^*(h). \tag{6.17}$$

if $\operatorname{Max}(k, n) > N$, with $c_4(h)$ independent of $N$.

Let $x \in l_1^2$ and write $A(h)(x)$

$$\begin{pmatrix} A_N(h) & B_N(h) \\ C_N(h) & (D_N+R_N)(h) \end{pmatrix}\begin{pmatrix} x_N \\ x_N' \end{pmatrix}$$

where $x_N = \sum_1^N x_j e_j$, $x_N' = \sum_{N+1}^\infty x_j e_j$, $B_N$ has $N$ rows, $C_N$ has $N$ columns, $D_N$ is the diagonal matrix $\{\partial\gamma_n/\partial h_n\}_{n\ge N}$ and $R_N = \{\partial\gamma_k/\partial h_n\}_{n\ne k,n,k\ge N}$. Then

$$\|B_N(h)x_N'\|_{l^2}^2 = \sum_{k=1}^N \left(\sum_{n=N+1} \frac{\partial\gamma_k}{\partial h_n} x_n\right)^2 \le \sum_{k=1}^N \left\{\sum_{n=N+1}^\infty \left(\frac{1}{n^2}\frac{\partial\gamma_k}{\partial h_n}\right)^2\right\}\|x_N'\|_{l_1^2}$$

where $\|x_N'\|_{l_1} = \left(\sum_{n\ge N+1} n^2 x_n^2\right)^{1/2}$. So by (6.17) and Theorem 3.3,

$$\|B_N(h)x_N'\|_{l^2} \le \frac{c_5(h)}{N^2}\|x_N'\|_{l_1^2}.$$

And for the same reasons,

$$\|C_N(h)x_N\|_{l^2}^2 = \sum_{k=N+1}^\infty \left(\sum_{n=1}^N \frac{\partial\gamma_k}{\partial h_n} x_n\right)^2 \le \sum_{k=N+1}^\infty \left\{\sum_{n=1}^N \frac{1}{n^2}\left(\frac{\partial\gamma_k}{\partial h_n}\right)^2\right\}\|x_N\|_{l_1^2}^2,$$

so that

$$\|C_N(h)x_N\|_{l^2} \le \frac{c_5(h)}{N^2}\|x_N\|_{l_1^2},$$

and

$$\|R_N(h)x_N'\|_{l^2} \le \frac{c_5(h)}{N^2}\|x_N'\|_{l_1^2}.$$

Write

$$J_N = \begin{pmatrix} A_N(h) & 0 \\ 0 & D_N(h) \end{pmatrix}, \qquad S_N = \begin{pmatrix} 0 & B_N(h) \\ C_N(h) & R_N(h) \end{pmatrix}.$$

Then $\|S_N\| \le c_5(h)/N^2$, while (6.15) and (6.16) show $J_N$ is invertible and $\|J_N^{-1}\| \le c_6(h)N^{1/2}$, where the norms are those of $B(l_1^2, l^2)$ and $B(l_1^2, l_1^2)$ respectively. Hence we see that

$$A(h) = J_N + S_N = J_N(I + J_N^{-1}S_N)$$

in invertible, by taking $N$ large. □

That finishes the proof of part (a) of Theorem 6.1. The proof of part (b) is much like the arguments behind Lemma 6.2 and Lemma 6.3, and we only outline it.

Let $u_n(h, w) = \omega(\Omega(h),\ 0 \le \operatorname{Re} z \le n\pi,\ w)$ and $v_n(h, w) = u_n(h, w) - \omega_n(h, w)$. Then for $h \in \mathbb{R}_+^N$, $N < \infty$

$$\lambda_{2n}(h) = \lim_{x \to \infty} 2\pi x^2 u_n(h, x + ix)$$

$$\lambda_{2n-1}(h) = \lim_{x \to \infty} 2\pi x^2 v_n(h, x + ix).$$

Take $0 < \delta_1' < \delta_2' < \delta_1 < \delta_2$, $\varepsilon_k = \delta_1'/2k$ as before and let $W_N(t)$ be the domain defined in Lemma 6.2.

LEMMA 6.5. *The functions $u_n(h, w)$ and $v_n(h, w)$ on $\mathbb{R}_+^N$ extend to be analytic in*

$$\{t \in \mathbb{C}^N : |\operatorname{Im} t_k| < \varepsilon_k,\ 1 \le k \le N\} \cap \{\operatorname{Re} t_n > -\varepsilon_n\}$$

*and harmonic in $w \in W_N(t)$. Moreover $u_n(t, w)$ and $v_n(t, w)$ are even functions of*

$t_k$, $k \neq n$. *As functions of* $t_n$ *they satisfy*

$$u_n(t, w) = v_n(-t, w), \qquad |t_n| < \varepsilon_n, \tag{6.18}$$

*Consequently*

$$U_n(t, w) = \begin{cases} u_n(t, w), & \operatorname{Re} t_n > 0 \\ v_n(-t, w), & \operatorname{Re} t_n < 0 \end{cases}$$

*defines a function analytic on* $\{t \in \mathbb{C}^N : |\operatorname{Im} t_k| < \varepsilon_k,\ 1 \leq k \leq N\}$.

By (6.18) and (6.1), $\mu_n(h) = \lim_{x \to \infty} 2\pi x^2 U_n(h, x + ix)$ is real analytic on $\mathbb{R}^N$.

*Proof*: Write $F_k = \{(k - 1/2)\pi \leq x \leq (k + 1/2)\pi,\ y = 0\}$ and

$$\sigma_k(h, w) = \omega(\Omega(h), F_k, w), \qquad 1 \leq k \leq n - 1.$$
$$\sigma_0(h, w) = \omega(\Omega(h), 0 < x < \pi/2, w)$$
$$\sigma_n(h, w) = \omega(\Omega(h), (n - 1/2)\pi < x < n\pi, w).$$

Then $u_n = \sum_{k=0}^n \sigma_k + \sum_{k=1}^n \omega_k(h, w)$ and $v_n = u_n - \omega_n(h, w)$, and it is enough to extend each $\sigma_k$ analytically.

If $k \neq j$ the extension of $\sigma_k$ with respect to $h_j$ proceeds as in Case 3 or Case 4 of the proof of Lemma 6.2, and if $h_k \geq b_k \geq \delta_1/k$, then so does the extension of $\sigma_k$ with respect to $h_k$. If $h_k < b_k$, and if $k \neq n$, we repeat Case 1 of that proof, except that we start with the function $v_\sigma(t, z) = \omega(\Delta_t, \mathbb{R} \cap \partial\Delta_t, z)$.[5] Thus for $k \neq n$ $\sigma_k(h, w)$ has an extension analytic in $\{t \in \mathbb{C}^N : |\operatorname{Im} t_k| \leq \varepsilon_k,\ 1 \leq k \leq N\}$. Note that $\sigma_k(t, w)$ is an even function $t_j$, $j \neq k$, and that if we continue $\sigma_k + \omega_k$ through $h_k = 0$, we start with the sum $v_\sigma(t, z) + v(t, z)$. Instead of (6.5) we have

$$v_\sigma(t, z) + v(t, z) = \int_{-1}^{1} K(x, w_t(z))\, dx,$$

which, since $t \to w_t(z)$ is even, is an even function of $t$. It follows that for $k \neq n$, $\sigma_k(t, w) + \omega_k(t, w)$, and consequently $u_n(t, w)$ and $v_n(t, w)$, are even functions of $t_k$.

---

[5] The contribution to $\sigma_k$ from $F_k \backslash \Delta_t$ can be extended in $h_k$ using Case 3 or Case 4 of Lemma 6.2.

To continue $\sigma_n(h, w)$ through $h_n = 0$ we use Case 1, but we start with

$$v'_\sigma(t, z) = \int_{-1}^{-t/r_n} K(x, w_t(z))\, dx. \tag{6.19}$$

Then the proof of Lemma 6.2 yields analytic extensions of $u_n$ and $v_n$ to $\{\mathrm{Re}\, t_n > -\varepsilon_n\} \cap \{|\mathrm{Im}\, t_k| < \varepsilon_k,\ 1 \le k \le N\}$. Since the other terms are even in $t_n$, (6.18) holds if and only if

$$\sigma_n(t_n, w) + \omega_n(t_n, w) = \sigma_n(-t_n, w).$$

But, expanding $K(x, w_t(z))$ in powers of $x$ and $w_t(z)$ and using (6.5) and (6.19), we obtain $v'_\sigma(t, z) + v(t, z) = v'_\sigma(-t, z)$, which implies (6.18).

The extension to $N$ variables is exactly the same as in the proof of Lemma 6.2. □

Lemma 6.5 shows that $\mu_n(h)$ is a real analytic function of finitely many variables. To complete the proof of Part (b) of Theorem 6.1, we must show

$$\tilde{\mu}_n(h) = \mu_n(h) - c(h) - n^2\pi^2$$

is a real anlytic map from $l_1^2$ into $l^2$. Recall from the proof of Theorem 3.4 that $c(h) = \sum_{k=1}^{\infty} a_k(h)$, where

$$a_k(h) = \lim_{N \to \infty} \lim_{x \to \infty} 2\pi x^2 \int_{T_k} \frac{2}{\pi} \arg \zeta\, d\omega^{(N)}(x + ix, \zeta);$$

$h \in (l_1^2)^+$. With small changes, the proofs of Lemma 6.2 and Lemma 6.3 show that whenever $h \in l_1^2$ and $V_h$ is defined by (6.11), $a_k(h)$ has a continuation $a_k(t)$ analytic on $V_h$ and

$$\sup_{t \in V_h} |a_k(t)| \le \text{Const.}\, (|h_k| + \varepsilon_k)(\gamma_k^*(h + \varepsilon_k c_k) + c/k^2). \tag{6.20}$$

Consequently the series defining $c(h)$ converges uniformly on $V_h$, and so $c(h)$ is a real analytic function in $l_1^2$, and $\tilde{\mu}_n(h)$ is real analytic in the first $N$ variables. Now define

$$\mu_n^*(h) = \sup_{N > 0} \sup_{t \in V_h} |\tilde{\mu}_n(t^{(N)})|.$$

LEMMA 6.6. *If* $h \in l_1^2$, *then*

$$\sum_{n=1}^{\infty} |\mu_n^*(h)|^2 < \infty.$$

By an argument like the one immediately after the statement of Lemma 6.3, Lemma 6.6 implies that $\tilde{\mu}_n$ is a real analytic map from $l_1^2$ into $l^2$, which is statement (b) of Theorem 6.1.

*Proof of Lemma* 6.6. By (6.10) and (6.1) we may assume $h_n \geq 0$, so that

$$\tilde{\mu}_n(h) = \lambda_{2n}(h) - c(h) - n^2\pi^2.$$

By (3.8) and the proof of Lemma 6.3,

$$\sup_{N>0} \sup_{t \in V_h} |\lambda_{2n}(t^{(N)}) - \lambda_{2n}(t^{(n)})| \in l^2,$$

and by (6.20), $\sup_{N>0} |c(t^{(N)}) - c(t^{(n)})| \in l^2$.

In the proof of Theorem 3.4 we obtained the decomposition.

$$\lambda_{2n}(h^{(n)}) - n^2\pi^2 - c(h^{(n)}) = B_n + C_n + D_n,$$

given by (3.9), (3.10) and (3.11). Using (6.10) and the estimates on $B_n$ and $C_n$ from Section 3, we get

$$\sup_{N>0} \sup_{t \in V_h} |B_n(t) + C_n(t)| \in l^2,$$

and (6.20) includes such a bound for $D_n$. That proves Lemma 6.6. □

## 7. A final remark

In this section we want to give a different approach to the analysis of the mapping

$$\{\text{band spectra}\} \to \{\text{gap lengths}\} \subset l^2.$$

In fact it was the following line of reasoning that led us to conjecture Theorem 2.

Let $\mu_n(q), n \geq 1$ and $\nu_n(q), n \geq 1$ be the Dirichlet and Neumann spectra respectively of $q \in L^2_{\mathbb{R}}[0, 1]$. We have the real analytic map

$$
\begin{aligned}
&q \in \mathscr{E} = \{q \in L^2_{\mathbb{R}}[0, 1] \mid q(x) = q(1-x), 0 \le x \le 1, \text{ and } \lambda_0(q) = 0\} \\
&\downarrow \\
&(\mu_n(q) - \nu_n(q), n \ge 1) \in l^2
\end{aligned}
$$

The numbers $\mu_n(q) - \nu_n(q)$, $n \ge 1$ are the signed gap lengths of $q \in E_0$, because $\mu_n$ and $\nu_n$ lie at the ends of the $n$th gap when $q$ is even. $\mathscr{E} \subset E$, the even subspace of $L^2_{\mathbb{R}}[0, 1]$, is a real analytic hypersurface since $(\partial/\partial g)\lambda_0 = f_0^2$ never vanishes. The aim is to try to use a covering argument to verify that the map is one-to-one.

Let $\mathscr{E}_N \subset \mathscr{E}$ be the subspace of all $q \in \mathscr{E}$ with $\mu_n(q) = \nu_n(q)$ for all $n > N$. The gradients[6]

$$\frac{\partial}{\partial q}(\mu_n - \nu_n) = g_n^2(x, q) - h_n^2(x, q), \qquad n > N,$$

are normal vectors to $\mathscr{E}_N$ at $q$. They are independent in the sense that no one of them is in the closed linear span of all the others. It is easy to check this independence by verifying the orthogonality relations

$$\int_0^1 (g_n^2 - h_n^2)(g_n h_n)' \, dx \neq 0$$

and

$$\int_0^1 (g_m^2 - h_m^2)(g_n h_n)' \, dx = 0$$

for all $m \neq n$. A simple Fredholm argument now shows that $\mathscr{E}_N$ is an $N$ dimensional real analytic submanifold of $\mathscr{E}$.

Consider the restricted map

$$
\begin{aligned}
&q \in \mathscr{E}_N \\
&\downarrow \\
&(\mu_n(q) - \nu_n(q), n \le N) \in \mathbb{R}^N.
\end{aligned}
$$

The fiber of this map over $0 \in \mathbb{R}^N$ consists of just one point, namely $q = 0$. This is a

---

[6] $h_n$ is the normalized eigenfunction corresponding to $\nu_n$.

classical result of Borg [see 1]. Next we observe that the identity from [7]

$$q(t)=\lambda_0+\sum_{n\geq 1}\lambda_{2n}+\lambda_{2n-1}-2\mu_n(t),$$

where $\mu_n(t)=\mu_n(T_tq)$ and $T_tq(x)=q(x+t)$, gives us a bound on the supremum of $|q|$ in terms of the gaps lengths. Precisely, if $q\in\mathscr{E}_n$

$$\begin{aligned}\sup_{0\leq t\leq 1}|q(t)| &\leq \sum_{n=1}|\lambda_{2n}-\lambda_{2n-1}|\\ &=\sum_{n=1}^{N}\gamma_n(q).\end{aligned}$$

It follows that the fibers are compact since they are closed and bounded by the above estimate. Finally, the invertibility of the Jacobians proved in Section 5 can be used here to show that the map is a local homeomorphism. Therefore the map is globally one to one by a covering argument.

## Appendix

Let $E_0$ be the subspace of all even functions in $L^2_{\mathbb{R}}[0,1]$ with mean zero, i.e., $\int_0^1 q\,dx=0$, and $S$ the Hilbert manifold of all strictly increasing real sequences $\sigma_n$, $n\geq 1$, of the form

$$\sigma_n=n^2\pi^2+\tilde{\sigma}_n$$

where $\sum_{n\geq 1}\tilde{\sigma}_n^2<\infty$. The manifold structure on $S$ is induced from $l^2$ by the correspondence between $\sigma$ and $\tilde{\sigma}$. The purpose of this Appendix is to sketch, for the convenience of the reader, part of the proof of

THEOREM 4.2. *The map*

$$q\rightarrow\mu(q)=(\mu_1(q),\mu_2(q),\ldots)$$

*is an analytic isomorphic between $E_0$ and $S$.*

We are going to follow the presentation given in [4] which contains a full proof. We will limit ourselves to verifying that the map is onto.

*Proof.* The first step is to compute the Jacobian of our map at $q=0$. That is, the linear transformation

$$E_0 \simeq (T(E_0))_{q=0} \ni v \to (d_0\mu_n(v),\, n\geq 1) \in T(S)_{(n^2\pi^2, n\geq 1)} \simeq l^2,$$

where $d_0(\mu_n)(v)$ denotes the directional derivative of $\mu_n$ at $q=0$ in the direction $v$; i.e.

$$d_0\mu_n(v) = \frac{d}{d\varepsilon}\,\mu_n(\varepsilon v)|_{\varepsilon=0}.$$

Let $g_n(x, q)$, $n\geq 1$, be the normalized Dirichlet eigenfunction, with $g_n'(0, q) > 0$, corresponding to $\mu_n(q)$. Denoting $(d/d\varepsilon)g_n(x, \varepsilon q)|_{\varepsilon=0}$ and $(d/d\varepsilon)\mu_n(\varepsilon q)|_{\varepsilon=0}$ by $\dot{g}_n$ and $\dot{\mu}_n$ we have

$$-g_n'' + vg_n = \dot{\mu}_n g_n + \mu_n \dot{g}_n$$

Taking the inner product of both sides of the equation with $g_n(x, 0)$ we obtain

$$(-\dot{g}_n'', g_n) + (v, g_n^2) = \dot{\mu}_n(g_n, g_n) + \mu_n(\dot{g}_n, g_n).$$

But

$$(-\dot{g}_n'', g_n) = (\dot{g}_n, -g_n'') = \mu_n(0)(\dot{g}_n, g_n)$$

so that

$$\dot{\mu}_n = (v, g_n^2).$$

Therefore,

$$\begin{aligned} d_0\mu_n(v) &= \int_0^1 v(x) 2\sin^2 n\pi x\, dx \\ &= -\int_0^1 v(x)\cos 2\pi n x\, dx. \end{aligned}$$

since $g_n(x, 0) = \sqrt{2}\sin n\pi x$ and $\int_0^1 v(x)\, dx = 0$.

It follows from elementary Fourier theory that the Jacobian has a bounded inverse. Therefore, a neighborhood of 0 in $E_0$ is mapped onto a neighborhood of $(n^2\pi^2, n\geq 1)$ in $S$ by the Inverse Function Theorem.

Of course, the preceding argument is formal in the sense that we have not checked that $\mu(q)$ is actually a differentiable function of $q$ in the topologies on $E_0$ and $S$. However, it can be shown that $\mu$ is actually real analytic. Briefly, each individual eigenvalue $\mu_n(q)$, $n \geq 1$, is, by the Implicit Function Theorem, a real analytic function of $q$, because

$$y_2(1, \mu_n(q), q) = 0$$

and

$$\frac{\partial}{\partial\lambda} y_2(1, \lambda, q)|_{\lambda=\mu_n(q)} \neq 0.$$

Here $y_2(x, \lambda, q)$ is the solution of $-y'' + gy = \lambda y$ with initial data $y(0) = 0$, $y'(0) = 1$. To go on and show that the map

$$q \in E_0 \rightarrow \mu(q) \in S$$

is real analytic one simply notices that the estimate

$$\mu_n(q) = n^2\pi^2 - (\cos 2\pi nx, q) + 0\left(\frac{1}{n}\right)$$

holds uniformly on a complex neighborhood of every point in $E_0$. It then follows that the map is locally bounded and hence real analytic, by the uniform boundedness principle.

We have seen that an open neighborhood of the sequence $(n^2\pi^2, n \geq 1)$ is covered by $\mu$. To see that all of $S$ is covered we construct flows on $E_0$.

Let $q \in E_0$, and set

$$\eta_n(x, t, q) = y_1(x, \mu_n + t) + \frac{y_1(1, \mu_n) - y_1(1, \mu_n + t)}{y_2(1, \mu_n + t)} y_2(x, \mu_n + t),$$

$n \geq 1$, for all $t$ such that

$$\mu_{n-1}(q) \leq \mu_n(q) + t < \mu_{n+1}(q),$$

(take $\mu_0(q) = -\infty$) and set[7]

$$\omega_n(x, t, q) = [\eta_n, \dot{y}_2(\cdot, \mu_n)].$$

---

[7] $[f, g] = fg' - f'g$.

An important property of $\omega_n$ is its strict positivity, i.e. $\omega_n(x, t, q) > 0$. The argument is by contradiction: suppose $\omega_n$ vanishes at a point $(x, t)$ with $(x, \mu_n + t) \in [0, 1]x\ (\mu_n, \mu_{n+1})$—the case of a root in $[0, 1]x(\mu_{n-1}, \mu_n)$ can be handled in the same way.

Let $\bar{\lambda}$ be the smallest point in $(\mu_n, \mu_{n+1})$ such that for some $\bar{x}$, $\omega_n(\bar{x}, \bar{\lambda}) = 0$. It is easy to see that $\omega_n(\cdot, \bar{\lambda})$ has a local minimum at $\bar{x}$, i.e.,

$$\omega_n'(\bar{x}, \bar{\lambda}) = 0.$$

But

$$\omega_n'(\bar{x}, \bar{\lambda}) = (\bar{\lambda} - \mu_n) g_n(\bar{x}) \eta_n(\bar{x}, \bar{\lambda})$$

Thus, either $g_n(\bar{x})$ or $\eta_n(\bar{x}, \bar{\lambda})$ vanish, since $\bar{\lambda} \neq \mu_n$. But the roots of $g_n$ and $\eta_n(\cdot, \bar{\lambda})$ being all simple,

$$0 = \omega_n(\bar{x}, \bar{\lambda}) = g_n(\bar{x}) \eta_n'(\bar{x}, \bar{\lambda}) - g_n'(\bar{x}) \eta_n(\bar{x}, \bar{\lambda})$$

implies that both $g_n(\bar{x})$ and $\eta_n(\bar{x}, \bar{\lambda})$ vanish. This further implies that

$$u_n'(x, \bar{\lambda}) = c(x - \bar{x})^2 + 0(|x - \bar{x}|^2)$$

with $c \neq 0$ by Taylor's rule. But this contradicts the fact that $\omega_n'(\cdot, \bar{\lambda})$ has to change sign at $\bar{x}$.

Thus, $\omega_n$ has no roots in $[0, 1] \times (\mu_{n-1}\ \mu_{n+1})$.

It is now possible to define the flow

$$\Psi_n^t(q) = q(x) - 2 \frac{d^2}{dx^2} \log \omega_n(x, t, q)$$

for $\mu_{n-1}(q) < \mu_n(q) + t < \mu_{n+1}(q)$. By direct calculation it can be checked that

(1) $\Psi_n^t(q) \in E_0$

(2) $\mu_m(\Psi_n^t(q) = \mu_m(q) + \delta_{mn} t$

Even though it is not too hard to verify the last statement we will not do so here it would take us too long

We are ready to prove that $\mu$ maps $E_0$ onto $S$. For any sequence $\sigma \in S$, define the modified sequence $\sigma^N$ by

$$\sigma_m^N = \begin{cases} m^2\pi^2, & m \leq N \\ \sigma_m, & m > N \end{cases}$$

If $N$ is chosen large enough $\sigma^N$ will be inside the neighborhood of $(n^2\pi^2, n \geq 1)$ given to us above by the Inverse Function Theorem. Therefore,

$$\sigma_n^N = \sigma_n(q), \qquad n \geq 1,$$

For some $q \in E_0$ near $q = 0$. Now use the flows $\Psi_j^t$, $(cj \leq N)$ to move $j^2\pi^2$ to $\sigma_j$. However, care must be taken to avoid crossing of eigenvalues. To be safe, first shift $\pi^2, \ldots, N^2\pi^2$ to the far left, i.e., all below $\sigma_1$, and then move them into the desired positions beginning with $\mu_N$.

Added in proof: The approach in Section 7 can be carried through to yield a simpler proof of the analyticity in Section 6. See [4].

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