**Zeitschrift:** Commentarii Mathematici Helvetici

**Band:** 59 (1984)

**Artikel:** Cohomology of classifying spaces of complex Lie groups and related discrete groups.

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**DOI:** https://doi.org/10.5169/seals-45400

Comment. Math. Helvetici 59 (1984) 347–361
0010–2571/84/030347–15$01.50 + 0.20/0


# Cohomology of classifying spaces of complex Lie groups and related discrete groups

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To Armand Borel on the occasion of his 60th birthday

We prove the following theorem, special cases of which have been proved in [4], [6], [8].

THEOREM 1.4. *Let G be a reductive complex Lie group, let p be a prime, and let* $\bar{\mathbb{F}}_p$ *denote the algebraic closure of the prime field* $\mathbb{F}_p$. *Then there exists a map* $BG(\bar{\mathbb{F}}_p) \to BG$ *which induces isomorphisms*

$$H_*(BG(\bar{\mathbb{F}}_p), \mathbb{Z}/n) \to H_*(BG, \mathbb{Z}/n), \qquad (n, p) = 1$$

*where* $G(\bar{\mathbb{F}}_p)$ *is the discrete group of* $\bar{\mathbb{F}}_p$*-rational points of a Chevalley integral group scheme associated to G.*

The anticipation of Theorem 1.4 led the first author to ask whether the identity map $G^\delta \to G$ induces isomorphisms $H_*(BG^\delta, \mathbb{Z}/n) \to H_*(BG, \mathbb{Z}/n)$ for any $n$, where $G^\delta$ denotes the complex Lie group $G$ viewed as a discrete group (cf. [9]). As discussed in Section 2, this conjecture is equivalent to the case of the complex field of what we call the "Generalized Isomorphism Conjecture" (Definition 2.1), and our Theorem 1.4 corresponds to the case of the field $\bar{\mathbb{F}}_p$ of this conjecture (Proposition 2.3). In considering this Generalized Isomorphism Conjecture we prove the generalization to any algebraically closed field of theorems of M. Feshbach and J. Milnor for the complex field. Our proof uses Theorem 1.4 and avoids use of Becker–Gottlieb transfer.

In Section 3 we show for any algebraically closed field $k$ and any linear algebraic group $G_k$ over $k$ that our Generalized Isomorphism Conjecture is

(1) Partially supported by N.S.F.

equivalent to the Finite Subgroup Conjecture which asserts for every non-zero $x \in H^*(BG(k), \mathbb{Z}/n)$ with $(n, \mathrm{char}(k)) = 1$ that there exists some finite subgroup $\pi \subset G(k)$ such that $x$ restricts non-trivially to $H^*(B\pi, \mathbb{Z}/n)$.

The first author gratefully thanks the Max Planck-Institut (Bonn) and the ETH (Zürich) for their warm hospitality.

## 1. Reductive groups over $\bar{\mathbb{F}}_p$

We begin our proof of Theorem 1.4 by recalling the "cohomological Lang fibre square" associated to the Lang map $1/\phi^q : G_k \to G_k$. We refer the reader to [1], [7] for a discussion of the Chevalley integral group scheme associated to a reductive complex Lie group. We refer the reader to [6] for a discussion of etale cohomology $H^*_{\mathrm{et}}(X, \mathbb{Z}/n)$ of a simplicial scheme $X$.

THEOREM 1.1 ([5; Thm. 2.9] or [6; Thm. 12.2]). *Let $G = G(\mathbb{C})^{\mathrm{top}}$ be a reductive complex Lie group, let $G_{\mathbb{Z}}$ be an associated Chevalley integral group scheme, let p be a prime and let $G_{\bar{\mathbb{F}}_p} = G_{\mathbb{Z}} \otimes \bar{\mathbb{F}}_p$. A choice of embedding of the Witt vectors of $\bar{\mathbb{F}}_p$ into $\mathbb{C}$ determines a commutative square in the homotopy category for any prime $l \neq p$ and* p*th power $q = p^d$:*

$$\begin{array}{ccc} BG(\mathbb{F}_q) & \longrightarrow & (\mathbb{Z}/l)_\infty(BG) \\ {\scriptstyle D_q}\downarrow & & \downarrow{\scriptstyle \Delta} \\ (\mathbb{Z}/l)_\infty(BG) & \xrightarrow{1\times\phi^q} & (\mathbb{Z}/l)_\infty(BG^{\times 2}) \end{array} \tag{1.1.1}$$

*with the property that some map on homotopy fibres* $\mathrm{fib}(D_q) \to \mathrm{fib}(\Delta) \simeq (\mathbb{Z}/l)_\infty(BG)$ *associated to* (1.1.1) *induces an isomorphism*

$$H_*(\mathrm{fib}(D_q), \mathbb{Z}/l) \xrightarrow{\sim} H_*(\mathrm{fib}(\Delta), \mathbb{Z}/l)$$

In Theorem 1.1, $G(\mathbb{F}_q)$ is the finite group of $\mathbb{F}_q$-rational points of $G_{\mathbb{Z}}$; $(\mathbb{Z}/l)_\infty(BG)$ is the Bousfield–Kan $\mathbb{Z}/l$-completion of the singular complex of $BG$; $\phi^q$ is associated to the geometric Frobenius $\phi^q : G_{\bar{\mathbb{F}}_p} \to G_{\bar{\mathbb{F}}_p}$; $\Delta$ is induced by the diagonal $G \to G^{\times 2}$; and $G(\mathbb{C})^{\mathrm{top}}$ stands for the group of $\mathbb{C}$-rational points of $G_{\mathbb{Z}}$ with the strong topology.

To apply Theorem 1.1, we require the following corollary of Theorem 1.1 whose proof can be found in the proof of [6; Cor. 12.4]. The map $D_q : BG(\mathbb{F}_q) \to (\mathbb{Z}/l)_\infty(BG)$ considered in (1.2.1) below is the left vertical arrow of (1.1.1); the map $i : BG(\mathbb{F}_q) \to BG(\mathbb{F}_{q'})$ is induced by the inclusion $G(\mathbb{F}_q) \to G(\mathbb{F}_{q'})$.

COROLLARY 1.2. *Assume the notation of Theorem 1.1. For any $q'=q^e=p^{de}$ and any prime $l\neq p$, there is a natural map of fibration sequences*

$$\begin{array}{ccccc} \mathrm{fib}\,(D_q) & \longrightarrow & BG(\mathbb{F}_q) & \xrightarrow{D_q} & (\mathbb{Z}/l)_\infty(BG) \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle 1} \\ \mathrm{fib}\,(D_{q'}) & \longrightarrow & BG(\mathbb{F}_{q'}) & \xrightarrow{D_{q'}} & (\mathbb{Z}/l)_\infty(BG) \end{array} \tag{1.2.1}$$

*such that the map $j^*: H^*(\mathrm{fib}\,(D_{q'}),\mathbb{Z}/l)\to H^*(\mathrm{fib}\,(D_q),\mathbb{Z}/l)$ can be identified with the map $\theta^*: H^*_{\mathrm{et}}(G_{\bar{\mathbb{F}}_p},\mathbb{Z}/l)\to H^*_{\mathrm{et}}(G_{\bar{\mathbb{F}}_p},\mathbb{Z}/l)$ induced by $\theta=\mu\circ(1\times\phi^q\times\cdots\times\phi^{q'/q}): G_{\bar{\mathbb{F}}_p}\to G_{\bar{\mathbb{F}}_p}$, where $\mu:(G_{\bar{\mathbb{F}}_p})^e\to G_{\bar{\mathbb{F}}_p}$ is the product map.*

We consider the direct limit indexed by $p$th powers of the homological Serre spectral sequences associated to (1.2.1)

$$E^2_{s,t}=H_s\Big(BG,\varinjlim_q H_t(\mathrm{fib}\,(D_q),\mathbb{Z}/l)\Big)\Rightarrow H_{s+t}(BG(\bar{\mathbb{F}}_p),\mathbb{Z}/l) \tag{1.2.2}$$

We conclude that $E^2_{s,t}=0$ for $t\neq 0$ by applying the following lemma.

LEMMA 1.3. *For any $q=p^d$, there exists some $q'=q^e$ with the property that the self-map on the $\mathbb{Z}/l$-dual Hopf-algebra of $H^*_{\mathrm{et}}(G_{\bar{\mathbb{F}}_p},\mathbb{Z}/l)$*

$$j_*: H^*_{\mathrm{et}}(G_{\bar{\mathbb{F}}_p},\mathbb{Z}/l)^\#\simeq H_*(\mathrm{fib}\,(D_q),\mathbb{Z}/l)\to H_*(\mathrm{fib}\,(D_{q'}),\mathbb{Z}/l)\simeq H^*_{\mathrm{et}}(G_{\bar{\mathbb{F}}_p},\mathbb{Z}/l)^\#$$

*induced by $j:\mathrm{fib}\,(D_q)\to\mathrm{fib}\,(D_{q'})$ of (1.2.1) satisfies*

(a) *if $x\in H^*_{\mathrm{et}}(G_{\bar{\mathbb{F}}_p},\mathbb{Z}/l)^\#$ is primitive, then $j_*(x)=0$*

(b) *if $x\in \tilde{H}^*_{\mathrm{et}}(G_{\bar{\mathbb{F}}_p},\mathbb{Z}/l)^\#$ is such that $j_*(x)\neq 0$ whereas $j_*(y)=0$ for all $y$ with homological degree* $\deg(y)$ *satisfying* $0<\deg(y)<\deg(x)$, *then $j_*(x)$ is primitive in $H^*_{\mathrm{et}}(G_{\bar{\mathbb{F}}_p},\mathbb{Z}/l)^\#$.*

*Proof.* We identify $j_*$ with the dual of $\theta^*: H^*_{\mathrm{et}}(G_{\bar{\mathbb{F}}_p},\mathbb{Z}/l)\to H^*_{\mathrm{et}}(G_{\bar{\mathbb{F}}_p},\mathbb{Z}/l)$. If $x\in H^*_{\mathrm{et}}(G_{\bar{\mathbb{F}}_p},\mathbb{Z}/l)^\#$ is primitive, then

$$j_*(x)=x+\phi^q_*(x)+\cdots+\phi^q_*\circ\phi^q_*\circ\cdots\circ\phi^q_*(x)$$

where $\phi^q_*$ is the dual of $\phi^{q^*}: H^*_{\mathrm{et}}(G_{\bar{\mathbb{F}}_p},\mathbb{Z}/l)\to H^*_{\mathrm{et}}(G_{\bar{\mathbb{F}}_p},\mathbb{Z}/l)$. If $\phi^q_*$ has order $m$ as an automorphism of the finite dimensional Hopf-algebra $H^*_{\mathrm{et}}(G_{\bar{\mathbb{F}}_p},\mathbb{Z}/l)^\#$, then $j_*(x)=0$ provided that $x$ is primitive and $q'=q^e$ is such that $e=lm$. To prove (b), assume that $x$ satisfies the conditions of b) and that $e = lm$. Write $\Delta_*(x) = 1 \otimes x + x \otimes 1 + \Sigma x_i \otimes x_j$ in $H^*_{et}(G_{\mathbb{F}_p} \times G_{\mathbb{F}_p}, \mathbb{Z}/l)^{\#}$. Then

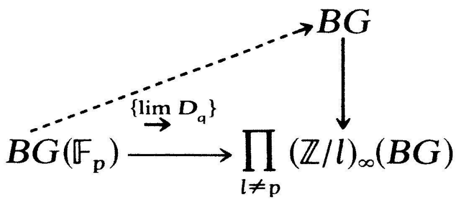
$$\Delta_*(j_*(x)) = j_*(\Delta_*(x)) = 1 \otimes j_*(x) + j_*(x) \otimes 1 + \Sigma j_*(x_i) \otimes j_*(x_j) = 1 \otimes j_*(x) + j_*(x) \otimes 1$$

so that $j_*(x)$ is primitive as asserted.

*Proof of Theorem* 1.4. Clearly it suffices to consider $n = l$ a prime different from $p$. We apply Lemma 1.3 to conclude that for each $q = p^d$ there exists some $q' = q^e$ such that $j_*: \tilde{H}_*(\text{fib}\,(D_q), \mathbb{Z}/l) \to \tilde{H}_*(\text{fib}\,(D_{q'}), \mathbb{Z}/l)$ is the 0-map. Consequently, (1.2.2) collapses to imply that

$$(\varinjlim D_q)_*: H_*(BG(\bar{\mathbb{F}}_p), \mathbb{Z}/l) = \varinjlim H_*(BG(\mathbb{F}_q), \mathbb{Z}/l) \to H_*(BG, \mathbb{Z}/l)$$

is an isomorphism. Because $\tilde{H}_*(BG(\bar{F}_p); \mathbb{Q}) = \tilde{H}_*(BG(\bar{\mathbb{F}}_p), \mathbb{Z}/p) = 0$, Sullivan's arithmetic fibre square technique [11] implies the existence of a (unique) lifting




This lifting induces the required isomorphisms

$$H_*(BG(\bar{\mathbb{F}}_p), \mathbb{Z}/n) \xrightarrow{\sim} H_*(BG, \mathbb{Z}/n), \qquad (n, p) = 1.$$

*Remark.* We correct an error in the proof given in [7] that $H^*_{et}(BG_k, \mathbb{Z}/n) \simeq H^*_{et}(BG_{\mathbb{C}}, \mathbb{Z}/n)$ for $(n, p) = 1$, an isomorphism implicit in the formulation of Theorem 1.1. The error occurs in the reduction to $G$ semi-simple for a general reductive complex Lie group. Let $G' = [G, G]$, the semi-simple commutator subgroup of $G$ and let $R = \text{rad}\,(G)$, the radical of $G$. Then $G$ is the quotient of $G' \times R$ by the finite central subgroup $H = G' \cap R$. As pointed out by O. Gabber, the associated central subgroup scheme $H_{\mathbb{Z}} \subset G'_{\mathbb{Z}} \times R_{\mathbb{Z}}$ is not etale over spec $\mathbb{Z}$ as claimed in [7]. Write $H = H' \times H''$ with $H'$ a $p$-group and $p \nmid |H''|$. Then $H''_A$ is etale over spec $A$, where $A$ denotes the Witt vectors of $k$, so that the proof given in [7] is valid for $G'' = (G' \times R)/H''$. Because $(n, p) = 1$, $G'' \to G = G''/H'$ induces isomorphisms $H^*_{et}(BG_{\mathbb{C}}, \mathbb{Z}/n) \xrightarrow{\sim} H^*_{et}(BG''_{\mathbb{C}}, \mathbb{Z}/n)$; because $G''_k \to G_k$ is a purely inseparable isogeny, $BG''_k \to BG_k$ induces isomorphisms $H^*_{et}(BG_k, \mathbb{Z}/n) \xrightarrow{\sim} H^*_{et}(BG''_k, \mathbb{Z}/n)$.

If $\pi \subset G$ is an inclusion of discrete groups and if $A$ is a $G$-module, then a class $x \in H^*(B\pi, A)$ is said to be stable for $g \in G$ if the images of $x$ under the two compositions

$$H^*(B\pi, A) \to H^*(B\pi \cap \pi^g, A),$$
$$H^*(B\pi, A) \xrightarrow{c} H^*(B\pi^g, A) \to H^*(B\pi \cap \pi^g, A)$$

are equal, where $c$ is induced by the conjugation isomorphism sending $y \in \pi^g$ to $gyg^{-1} \in \pi$. Similarly, a class $x \in H^*(B\pi, A)$ is said to be stable with respect to $\pi \subset G$ if $x$ is stable for each $g \in G$. The subgroup of stable elements of $H^*(B\pi, A)$ with respect to $\pi \subset G$ will be denoted $H^*(B\pi, A)^S$. If $\pi \subset G$ is an inclusion of topological groups, then $H^*(B\pi, A)^S \subset H^*(B\pi, A)$ for any $\pi_0(G)$-module $A$ is defined similarly.

PROPOSITION 1.5. *Let $G_{\bar{\mathbb{F}}_p}$ be a reductive algebraic group over $\bar{\mathbb{F}}_p$ and let $N_{\bar{\mathbb{F}}_p} \subset G_{\bar{\mathbb{F}}_p}$ be the normalizer of a maximal torus. Then the restriction map*

$$H^*(BG(\bar{\mathbb{F}}_p), \mathbb{Z}/n) \to H^*(BN(\bar{\mathbb{F}}_p), \mathbb{Z}/n)$$

*induces an isomorphism onto those elements stable with respect to $N(\bar{\mathbb{F}}_p) \subset G(\bar{\mathbb{F}}_p)$*

$$H^*(BG(\bar{\mathbb{F}}_p), \mathbb{Z}/n) \xrightarrow{\sim} H^*(BN(\bar{\mathbb{F}}_p), \mathbb{Z}/n)^S, \qquad (n, p) = 1.$$

*Proof.* We recall that $G_{\bar{\mathbb{F}}_p}$ is of the form $G_{\mathbb{Z}} \otimes \bar{\mathbb{F}}_p$ and $N_{\bar{\mathbb{F}}_p} \subset G_{\bar{\mathbb{F}}_p}$ is of the form $(N_{\mathbb{Z}} \subset G_{\mathbb{Z}}) \otimes \bar{\mathbb{F}}_p$, where $G_{\mathbb{Z}}$ is a reductive group scheme over $\mathbb{Z}$. The group $N(\mathbb{F}_q)$ of $\mathbb{F}_q$-rational points of $N_{\mathbb{Z}}$ contains an $l$-Sylow subgroup of $G(\mathbb{F}_q)$ for any prime $l \neq p$ [10], so that the restriction maps

$$H^*(BG(\mathbb{F}_q), \mathbb{Z}/n) \to H^*(BN(\mathbb{F}_q), \mathbb{Z}/n)^S \tag{1.5.1}$$

are isomorphisms for any $p$th power $q$, any integer $n$ not divisible by $p$ [2]. Using the isomorphism $H^*(BN(\bar{\mathbb{F}}_p), \mathbb{Z}/n) \xrightarrow{\sim} \varprojlim H^*(BN(\mathbb{F}_q), \mathbb{Z}/n)$ and the fact that $G(\bar{\mathbb{F}}_p) = \bigcup G(\mathbb{F}_q)$, we conclude that the inverse limit with respect to $q$ of the isomorphisms (1.5.1) is the asserted isomorphism.

## 2. Generalized Isomorphism Conjecture

We recall for a simplicial set $S$ and an algebraically closed field $k$ that the simplicial scheme $S \otimes \operatorname{spec}(k)$ (defined by $(S \otimes \operatorname{spec}(k))_n = \coprod_{S_n} \operatorname{spec}(k)$ with the simplicial structure induced from $S$) has the property that $H^*_{et}(S \otimes \operatorname{spec}(k), \mathbb{Z}/n)$ is naturally isomorphic to $H^*(S, \mathbb{Z}/n)$. If $\rho: D \to G(k)$ denotes a homomorphism of a discrete group $D$ into the group of $k$-rational points of an algebraic group $G_k$, then there is an induced map

$$\rho: D_k = D \otimes \operatorname{spec}(k) \to G_k$$

of goup schemes over $k$, where $D_k$ is regarded as a group scheme in the obvious way. Moreover, $\rho$ induces a morphism of simplicial schemes $B\rho: BD_k \to BG_k$ giving rise to an induced map

$$B\rho^*: H^*_{et}(BG_k, \mathbb{Z}/n) \to H^*(BD, \mathbb{Z}/n)$$

We now introduce the Generalized Isomorphism Conjecture (GIC), incorporated into the following definition.

DEFINITION 2.1. Let $k$ be an algebraically closed field and let $n$ be a positive integer invertible in $k$. For any algebraic group $G_k$ over $k$ we say that $G_k$ satisfies the Generalized Isomorphism Conjecture with respect to $n$ (which we abbreviate by $\mathrm{GIC}_n$) if the natural map of group schemes $G(k)_k \to G_k$, $G(k)$ the discrete group of $k$-rational points of $G_k$, induces an isomorphism

$$H^*_{et}(BG_k, \mathbb{Z}/n) \xrightarrow{\sim} H^*(BG(k), \mathbb{Z}/n).$$

We say that $G_k$ satisfies GIC if it satisfies $\mathrm{GIC}_n$ for every $n$ prime to $\operatorname{char}(k)$.

As we proceed to show, the Generalized Isomorphism Conjecture is valid for a connected linear $G_k$ if and only if it is valid for its maximal reductive quotient.

PROPOSITION 2.2. *Let $k$ be an algebraically closed field and $G_k$ a connected linear algebraic group over $k$. Then $G_k$ satisfies* $\mathrm{GIC}_n$ *if and only if the reductive group $G_k/G_k^u$ satisfies* $\mathrm{GIC}_n$, *where $G_k^u$ denotes the unipotent radical of $G_k$.*

*Proof.* The fact that $G^u(k)$ is a successive extension of $k_0$ vector spaces, $k_0$ the prime field of $k$, implies that $G^u(k)$ is acyclic for cohomology with $\mathbb{Z}/n$ coefficients, $n$ invertible in $k$. Consequently, the natural map

$$H^*(BG/G^u(k), \mathbb{Z}/n) \to H^*(BG(k), \mathbb{Z}/n)$$

is an isomorphism. The fact that $G_k \to G_k/G_k^u$ is an affine bundle implies that $G_k \to G_k/G_k^u$ and thus also $BG_k \to BG_k/G_k^u$ induce isomorphisms in etale cohomology with $\mathbb{Z}/n$ coefficients. Thus the proposition follows from the naturality of the map

$$H^*_{\text{et}}(BG_k, \mathbb{Z}/n) \to H^*(BG(k), \mathbb{Z}/n).$$

As we check in Proposition 2.3, the validity of GIC for $G_{\bar{\mathbb{F}}_p}$ is merely a restatement of Theorem 1.4.

PROPOSITION 2.3. *Let p be a prime, and $G_{\bar{\mathbb{F}}_p}$ a connected linear algebraic group over $\bar{\mathbb{F}}_p$. Then $G_{\bar{\mathbb{F}}_p}$ satisfies* GIC.

*Proof.* By Proposition 2.2, it suffices to assume $G_{\bar{\mathbb{F}}_p}$ reductive and it follows that we can assume $G_{\bar{\mathbb{F}}_p} = G_{\mathbb{Z}} \otimes \bar{\mathbb{F}}_p$. Then the map $\varinjlim D_q : BG(\bar{\mathbb{F}}_p) \to (\mathbb{Z}/l)_\infty(BG(\mathbb{C})^{\text{top}})$ occurring in the proof of Theorem 1.4 is induced by the map $G(\bar{\mathbb{F}}_p)_{\bar{\mathbb{F}}_p} \to G_{\bar{\mathbb{F}}_p}$ and a choice of embedding of the Witt vectors of $\bar{\mathbb{F}}_p$ into $\mathbb{C}$ determining isomorphisms $H^*_{\text{et}}(BG_{\bar{\mathbb{F}}_p}, \mathbb{Z}/n) \simeq H^*_{\text{et}}(BG_{\mathbb{C}}, \mathbb{Z}/n) \simeq H^*(BG(\mathbb{C})^{\text{top}}, \mathbb{Z}/n)$. Consequently, the proposition follows directly from Theorem 1.4 and the existence of a commutative triangle

$$\begin{array}{ccc} H^*_{\text{et}}(BG_{\bar{\mathbb{F}}_p}, \mathbb{Z}/n) & \simeq & H^*(BG(\mathbb{C})^{\text{top}}, \mathbb{Z}/n) \\ & \searrow \quad \swarrow & \\ & H^*(BG(\bar{\mathbb{F}}_p), \mathbb{Z}/n) & \end{array}$$

We next show in Proposition 2.4 that the validity of GIC for $G_{\mathbb{C}}$ is equivalent to what J. Milnor calls the isomorphism conjecture for $G(\mathbb{C})^{\text{top}}$ ([9]).

PROPOSITION 2.4. *Let $G_{\mathbb{C}}$ be a complex algebraic group and n a positive integer. Then there exists a natural commutative triangle*

$$\begin{array}{ccc} H^*_{\text{et}}(BG_{\mathbb{C}}, \mathbb{Z}/n) & \simeq & H^*(BG(\mathbb{C})^{\text{top}}, \mathbb{Z}/n) \\ & \searrow \quad \swarrow & \\ & H^*(BG(\mathbb{C})^{\delta}, \mathbb{Z}/n) & \end{array}$$

*in which the isomorphism is given by the classical comparison theorem and the other two maps are induced by $G(\mathbb{C})^{\delta}_{\mathbb{C}} = G(\mathbb{C})_{\mathbb{C}} \to G_{\mathbb{C}}$ and $1 : G(\mathbb{C})^{\delta} \to G(\mathbb{C})^{\text{top}}$. Consequently, $G_{\mathbb{C}}$ satisfies* GIC *if and only if the identity map induces isomorphisms $H_*(BG(\mathbb{C})^{\delta}, \mathbb{Z}/n) \xrightarrow{\sim} H_*(BG(\mathbb{C})^{\text{top}}, \mathbb{Z}/n)$.*

*Proof.* The isomorphism $H^*_{\text{et}}(BG_{\mathbb{C}}, \mathbb{Z}/n) \simeq H^*(BG(\mathbb{C})^{\text{top}}, \mathbb{Z}/n)$ is induced by maps $(BG_{\mathbb{C}})_{\text{et}} \leftarrow (BG_{\mathbb{C}})_{\text{s.et}} \to \text{Sing}\,(BG(\mathbb{C})^{\text{top}})$ ([6], 8.4). We readily verify that the corresponding maps for $BG(\mathbb{C})_{\mathbb{C}}$

$$(BG(\mathbb{C})_{\mathbb{C}})_{et} \leftarrow (BG(\mathbb{C})_{\mathbb{C}})_{s.et} \rightarrow BG(\mathbb{C})^{\delta}$$

are weak equivalences. Thus, the proposition follows from the naturality of these maps with respect to $BG(\mathbb{C})_{\mathbb{C}} \rightarrow BG_{\mathbb{C}}$.

The following theorem gives a partial description of the map $H^*_{et}(BG_k, \mathbb{Z}/n) \rightarrow H^*(BG(k), \mathbb{Z}/n)$ occurring in GIC.

THEOREM 2.5. *Let k be an algebraically closed field, n a positive integer invertible in k, $G_k$ a connected linear algebraic group over k, and $N_k \subset G_k$ the normalizer of a maximal torus in $G_k$. Then* $\mathrm{GIC}_n$ *holds for $N_k$, and the composition*

$$H^*_{et}(BG_k, \mathbb{Z}/n) \rightarrow H^*_{et}(BN_k, \mathbb{Z}/n) \xrightarrow{\sim} H^*(BN(k), \mathbb{Z}/n)$$

*is an injection with image the stable elements $H^*(BN(k), \mathbb{Z}/n)^S$ with respect to $N(k) \subset G(k)$.*

*Proof.* As argued in the proof of Proposition 2.2, we may assume $G_k$ reductive by replacing $G_k$ by $G_k/G_k^u$ (leaving unchanged $N_k$ and the subgroup of stable elements of $H^*(BN(k), \mathbb{Z}/n)$). Using the map of fibration sequences

$$\begin{array}{ccccc} B\mathbb{Z}^r & \rightarrow & B(\mathbb{C}^r)^{\delta} & \rightarrow & BT(\mathbb{C})^{\delta} \\ \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow \\ B\mathbb{Z}^r & \rightarrow & B(\mathbb{C}^r)^{top} & \rightarrow & BT(\mathbb{C})^{top} \end{array}$$

together with the $\mathbb{Z}/n$ acyclicity of $B(\mathbb{C}^r)^{\delta}$ and the contractibility of $B(\mathbb{C}^r)^{top}$, we conclude the natural isomorphism for a maximal torus $T_{\mathbb{Z}}$ of $G_{\mathbb{Z}}$

$$H^*(BT(\mathbb{C})^{top}, \mathbb{Z}/n) \xrightarrow{\sim} H^*(BT(\mathbb{C})^{\delta}, \mathbb{Z}/n)$$

Employing the map of fibration sequences

$$\begin{array}{ccccc} BT(\mathbb{C})^{\delta} & \rightarrow & BN(\mathbb{C})^{\delta} & \rightarrow & BW \\ \downarrow & & \downarrow & & \downarrow \\ BT(\mathbb{C})^{top} & \rightarrow & BN(\mathbb{C})^{top} & \rightarrow & BW \end{array}$$

we conclude the natural isomorphism (compare also [9])

$$H^*(BN(\mathbb{C})^{top}, \mathbb{Z}/n) \xrightarrow{\sim} H^*(BN(\mathbb{C})^{\delta}, \mathbb{Z}/n) \qquad (2.5.1)$$

Moreover, by Proposition 2.4, we can reinterpret (2.5.1) as the isomorphism

$$H^*_{et}(BN_{\mathbb{C}}, \mathbb{Z}/n) \xrightarrow{\sim} H^*(BN(\mathbb{C})^{\delta}, \mathbb{Z}/n) \qquad (2.5.2)$$

Choose a prime $p$ not dividing $n$ and let $R \subset \mathbb{C}$ denote the strict Henselization at $p$ of $\mathbb{Z}_{(p)} = \{m/n; p \nmid n\}$ (provided with an embedding into the complex field). Thus, $R$ has residue field $\bar{\mathbb{F}}_p$ and quotient field contained in $\bar{\mathbb{Q}}$. The maps $\bar{\mathbb{F}}_p \leftarrow R \rightarrow \bar{\mathbb{Q}} \rightarrow \mathbb{C}$ determine the following commutative diagram of schemes

$$\begin{array}{ccccccccccccc}
N(\bar{\mathbb{F}}_p)_{\bar{\mathbb{F}}_p} & \leftarrow & N(R)_{\bar{\mathbb{F}}_p} & \rightarrow & N(R)_R & \leftarrow & N(R)_{\bar{\mathbb{Q}}} & \rightarrow & N(\bar{\mathbb{Q}})_{\bar{\mathbb{Q}}} & \leftarrow & N(\bar{\mathbb{Q}})_{\mathbb{C}} & \rightarrow & N(\mathbb{C})_{\mathbb{C}} \\
\downarrow & & & & \downarrow & & & & \downarrow & & & & \downarrow \\
N_{\bar{\mathbb{F}}_p} & & \longrightarrow & & N_R & & \longleftarrow & & N_{\bar{\mathbb{Q}}} & & \longleftarrow & & N_{\mathbb{C}}
\end{array} \qquad (2.5.3)$$

By Hensel's Lemma, the kernel of the surjection $R^* \rightarrow \bar{\mathbb{F}}_p^*$ is uniquely $n$-divisible, as are the cokernels of the injections $R^* \rightarrow \bar{\mathbb{Q}}^*$ and $\bar{\mathbb{Q}}^* \rightarrow \mathbb{C}^*$. Consequently, the maps $BT(\bar{\mathbb{F}}_p) \leftarrow BT(R) \rightarrow BT(\bar{\mathbb{Q}}) \rightarrow BT(\mathbb{C})^\delta$ each induce isomorphisms in $\mathbb{Z}/n$ cohomology and, since $(\operatorname{spec} R)_{\mathrm{et}}$ is contractible the maps on classifying spaces associated to the upper horizontal maps of (2.5.3) all induce isomorphism in $\mathbb{Z}/n$ cohomology. We obtain therefore a commutative diagram in cohomology

$$\begin{array}{ccccccc}
H^*(BN(\bar{\mathbb{F}}_p), \mathbb{Z}/n) & \simeq & H^*(BN(R), \mathbb{Z}/n) & \simeq & H^*(BN(\bar{\mathbb{Q}}), \mathbb{Z}/n) & \simeq & H^*(BN(\mathbb{C})^\delta, \mathbb{Z}/n) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
H^*_{\mathrm{et}}(BN_{\bar{\mathbb{F}}_p}, \mathbb{Z}/n) & \leftarrow & H^*_{\mathrm{et}}(BN_R, \mathbb{Z}/n) & \rightarrow & H^*_{\mathrm{et}}(BN_{\bar{\mathbb{Q}}}, \mathbb{Z}/n) & \rightarrow & H^*_{\mathrm{et}}(BN_{\mathbb{C}}, \mathbb{Z}/n)
\end{array} \qquad (2.5.4)$$

The lower horizontal maps are induced by the appropriate base changes and are therefore isomorphisms. By (2.5.2), we conclude the natural isomorphisms

$$H^*_{\mathrm{et}}(BN_{\bar{\mathbb{F}}_p}, \mathbb{Z}/n) \xrightarrow{\sim} H^*(BN(\bar{\mathbb{F}}_p), \mathbb{Z}/n), \quad \text{and} \quad H^*_{\mathrm{et}}(BN_{\bar{\mathbb{Q}}}, \mathbb{Z}/n) \xrightarrow{\sim} H^*(BN(\bar{\mathbb{Q}}), \mathbb{Z}/n) \qquad (2.5.5)$$

Let $L/K$ be an extension of algebraically closed fields. This extension induces isomorphisms $H^*(BT(L), \mathbb{Z}/n) \rightarrow H^*(BT(K), \mathbb{Z}/n)$ because $L^*/K^*$ is uniquely divisible, and thus also isomorphisms

$$H^*(BN(L), \mathbb{Z}/n) \xrightarrow{\sim} H^*(BN(K), \mathbb{Z}/n) \qquad (2.5.6)$$

Etale cohomology base change theorems imply that $N_L \rightarrow N_K$ induces isomorphisms $H^*_{\mathrm{et}}(BN_K, \mathbb{Z}/n) \rightarrow H^*_{\mathrm{et}}(BN_L, \mathbb{Z}/n)$. Therefore, (2.5.5) implies the natural isomorphism

$$H^*_{\mathrm{et}}(BN_k, \mathbb{Z}/n) \xrightarrow{\sim} H^*(BN(k), \mathbb{Z}/n)$$

for any algebraically closed field containing $1/n$; thus $N_k$ satisfies $\mathrm{GIC}_n$.

The natural map

$$H^*(BN(\bar{\mathbb{F}}_p), \mathbb{Z}/n)^S \to H^*(BN(R), \mathbb{Z}/n)^S \tag{2.5.7}$$

is injective by (2.5.4); we will show that this map is actually an isomorphism. Suppose $x \in H^*(BN(R), \mathbb{Z}/n)$ is stable for $g \in G(R)$. Then the corresponding element in $H^*(BN(\bar{\mathbb{F}}_p), \mathbb{Z}/n)$ is stable for the image $\bar{g} \in G(\bar{\mathbb{F}}_p)$ of $g$, since $H^*(BN(\bar{\mathbb{F}}_p) \cap N(\bar{\mathbb{F}}_p)^{\bar{g}}, \mathbb{Z}/n) \xrightarrow{\sim} H^*(BN(R) \cap N(R)^g, \mathbb{Z}/n)$ (the kernel of $N(R) \cap N(R)^g \to N(\bar{\mathbb{F}}_p) \cap N(\bar{\mathbb{F}}_p)^{\bar{g}}$ is uniquely $n$-divisible). Observe that since $R$ is Henselian, the map $G(R) \to G(\bar{\mathbb{F}}_p)$ is surjective, and we conclude the surjectivity of the map (2.5.7).

From the diagram (2.5.4) and the morphism of group schemes $N_{\mathbb{Z}} \to G_{\mathbb{Z}}$ we obtain then the following diagram

$$\begin{array}{ccccc} H^*(BN(\bar{\mathbb{F}}_p), \mathbb{Z}/n)^S & \xrightarrow{\sim} & H^*(BN(R), \mathbb{Z}/n)^S & \leftarrow & H^*(BN(\bar{\mathbb{Q}}), \mathbb{Z}/n)^S \\ \uparrow & & \uparrow & & \uparrow \\ H^*_{et}(BG_{\bar{\mathbb{F}}_p}, \mathbb{Z}/n) & \xleftarrow{\sim} & H^*_{et}(BG_R, \mathbb{Z}/n) & \xrightarrow{\sim} & H^*_{et}(BG_{\bar{\mathbb{Q}}}, \mathbb{Z}/n) \end{array} \tag{2.5.8}$$

Theorem 1.4 and Proposition 1.5 imply the isomorphism

$$H^*_{et}(BG_{\bar{\mathbb{F}}_p}, \mathbb{Z}/n) \xrightarrow{\sim} H^*(BN(\bar{\mathbb{F}}_p), \mathbb{Z}/n)^S \tag{2.5.9}$$

for $p$ not dividing $n$. Therefore, all vertical arrows in (2.5.8) are isomorphisms. In particular, we have

$$H^*_{et}(BG_{\bar{\mathbb{Q}}}, \mathbb{Z}/n) \xrightarrow{\sim} H^*(BN(\bar{\mathbb{Q}}), \mathbb{Z}/n)^S \tag{2.5.10}$$

Let now $k$ be an arbitrary algebraically closed field and $k_0 \subset k$ the algebraic closure of the prime field. Applying (2.5.9) or (2.5.10), the base change isomorphisms in etale cohomology give rise to the commutative diagram

$$\begin{array}{ccc} H^*_{et}(BG_{k_0}, \mathbb{Z}/n) & \xrightarrow{\sim} & H^*_{et}(BG_k, \mathbb{Z}/n) \\ \downarrow \cong & & \downarrow \\ H^*_{et}(BN(k_0), \mathbb{Z}/n)^S & \hookleftarrow & H^*(BN(k), \mathbb{Z}/n)^S \end{array}$$

The injectivity of the bottom arrow follows from (2.5.6). We conclude therefore the asserted isomorphism $H^*_{et}(BG_k, \mathbb{Z}/n) \xrightarrow{\sim} H^*(BN(k), \mathbb{Z}/n)^S$ for any algebraically closed field $k$ containing $1/n$.

If we specialize Theorem 2.5 to the case $k = \mathbb{C}$, we obtain the following result proved (for any generalized cohomology theory) by M. Feshbach [3].

COROLLARY 2.6. *Let $G_{\mathbb{C}}$ be a connected linear complex algebraic group and $n$ a positive integer. Then the restrcition map induces a natural isomorphism*

$$H^*(BG(\mathbb{C})^{top}, \mathbb{Z}/n) \xrightarrow{\sim} H^*(BN(\mathbb{C})^{top}, \mathbb{Z}/n)^S$$

*Proof.* As argued in the proof of Proposition 2.4, the isomorphism given by the classical comparison theorem $H^*_{et}(BG_{\mathbb{C}}, \mathbb{Z}/n) \cong H^*(BG(\mathbb{C})^{top}, \mathbb{Z}/n)$ fits in a commutative square

$$\begin{array}{ccc} H^*_{et}(BG_{\mathbb{C}}, \mathbb{Z}/n) & \cong & H^*(BG(\mathbb{C})^{top}, \mathbb{Z}/n) \\ \downarrow & & \downarrow \\ H^*(BN(\mathbb{C})^{\delta}, \mathbb{Z}/n) & \leftarrow & H^*(BN(\mathbb{C})^{top}, \mathbb{Z}/n) \end{array} \tag{2.6.1}$$

The naturality of the isomorphism (2.5.1) implies that for $g \in G(\mathbb{C})$, $H^*(B(N(\mathbb{C}) \cap N(\mathbb{C})^g)^{top}, \mathbb{Z}/n) \to H^*(B(N(\mathbb{C}) \cap N(\mathbb{C})^g)^{\delta}, \mathbb{Z}/n)$ is an isomorphism. Therefore, we obtain from (2.5.1) the isomorphism $H^*(BN(\mathbb{C})^{top}, \mathbb{Z}/n)^S \xrightarrow{\sim} H^*(BN(\mathbb{C})^{\delta}, \mathbb{Z}/n)^S$. Fitting this isomorphism and Theorem 2.5 into (2.6.1), we conclude the corollary.

The following corollary of Theorem 2.5 sharpens a theorem of J. Milnor in the special case $k = \mathbb{C}$, [9].

COROLLARY 2.7. *Assume the notation of Theorem 2.5. Then the following triangle commutes*

$$\begin{array}{ccc} H^*_{et}(BG_k, \mathbb{Z}/n) & \xrightarrow{\alpha} & H^*(BG(k), \mathbb{Z}/n) \\ & {\scriptstyle \cong}\searrow^{\gamma} \quad \swarrow^{\beta} & \\ & H^*(BN(k), \mathbb{Z}/n)^S & \end{array}$$

*in which the horizontal map is that of the* GIC, *the left slant map is the isomorphism of Theorem 2.5 and the right slant map is the restriction homomorphism. In particular*

$$H^*_{et}(BG_k, \mathbb{Z}/n) \hookrightarrow H^*(BG(k), \mathbb{Z}/n)$$

*is split injective and the splitting is given by the algebra map* $\gamma^{-1}\beta$. *Furthermore*

$$H^*(BG(k), \mathbb{Z}/n) \longrightarrow H^*(BN(k), \mathbb{Z}/n)^S$$

*is split surjective, with splitting the algebra map* $\alpha\gamma^{-1}$.

*Proof.* The commutativity of the triangle follows from the naturality of the map $H^*_{et}(BG_k, \mathbb{Z}/n) \to H^*(BG(k), \mathbb{Z}/n)$.

Finally, we obtain the following equivalent form of the GIC. The proof is immediate from Corollary 2.7.

COROLLARY 2.8. *Assume the notation of Theorem* 2.5. *Then* $G_k$ *satisfies* $\mathrm{GIC}_n$ *if and only if the restriction map*

$$H^*(BG(k), \mathbb{Z}/n) \to H^*(BN(k), \mathbb{Z}/n)^S$$

*is an isomorphism, if and only if the restriction map*

$$H^*(BG(k), \mathbb{Z}/n) \to H^*(BN(k), \mathbb{Z}/n)$$

*is injective.*

## 3. Finite Subgroup Conjecture

As with the Generalized Isomorphism Conjecture, we incorporate the Finite Subgroup Conjecture (FSC) in a definition.

DEFINITION 3.1. Let $k$ be an algebraically closed field and let $n$ be a positive integer invertible in $k$. For any algebraic group $G_k$ over $k$ we say that $G_k$ satisfies the Finite Subgroup Conjecture with respect to $n$ (which we abbreviate by $\mathrm{FSC}_n$) if for any non-zero $x \in H^*(BG(k), \mathbb{Z}/n)$ there exists some finite subgroup $\pi \subset G(k)$ such that $x$ restricts non-trivially to $H^*(B\pi, \mathbb{Z}/n)$. We say $G_k$ satisfies FSC, if $G_k$ satisfies $\mathrm{FSC}_n$ for every $n$ prime to char $(k)$.

THEOREM 3.2. *Let* $k$ *be an algebraically closed field and let* $G_k$ *be a connected linear algebraic group over* $k$. *Then* $G_k$ *satisfies* $\mathrm{GIC}_n$ *if and only if it satisfies* $\mathrm{FSC}_n$.

*Proof.* If $G_k$ satisfies $\mathrm{GIC}_n$, then $H^*(BG(k), \mathbb{Z}/n) \to H^*(BN(k), \mathbb{Z}/n)$ is injective by Corollary 2.8. In case char $(k) = p > 0$ we choose an embedding $\bar{\mathbb{F}}_p \subset k$, which will induce isomorphisms

$$H^*(BN(k), \mathbb{Z}/n) \xrightarrow{\sim} H^*(BN(\bar{\mathbb{F}}_p), \mathbb{Z}/n) \xrightarrow{\sim} \varprojlim H^*(BN(\mathbb{F}_q), \mathbb{Z}/n).$$

Thus, the mod $n$ cohomology of $G(k)$ is detected by the family of finite subgroups $N(\mathbb{F}_q) \subset G(k)$. On the other hand, if char $(k)=0$, we first choose a prime $p$ which doesn't divide $n$ and which is prime to the order of the Weyl group $W$ of $G_k$. Then we choose an embedding of the strict Henselization $R$ of $\mathbb{Z}_{(p)}$ into $k$, giving rise to maps

$$\bar{\mathbb{F}}_p \leftarrow R \rightarrow k$$

which induce isomorphisms (cf. proof of Theorem 2.5)

$$H^*(BN(\bar{\mathbb{F}}_p), \mathbb{Z}/n) \xrightarrow{\sim} H^*(BN(R), \mathbb{Z}/n) \xleftarrow{\sim} H^*(BN(k), \mathbb{Z}/n)$$

Because $R^* \rightarrow \bar{\mathbb{F}}_p^*$ admits a (unique) splitting with uniquely $|W|$-divisible cokernel, this splitting $\bar{\mathbb{F}}_p^* \rightarrow R^*$ induces a $W$-equivariant map $T(\bar{\mathbb{F}}_p) \rightarrow T(R)$ inducing the inverse $H^*(BW, T(\bar{\mathbb{F}}_p)) \xrightarrow{\sim} H^*(BW, T(R))$ to the reduction isomorphism $H^*(BW, T(R)) \xrightarrow{\sim} H^*(BW, T(\bar{\mathbb{F}}_p))$. In particular, the reduction map $N(R) \rightarrow N(\bar{\mathbb{F}}_p)$ (interpreted as a map of extensions of $W$ whose classes are related by the reduction isomorphism $H^2(BW, T(R)) \xrightarrow{\sim} H^2(BW, T(\bar{\mathbb{F}}_p))$) admits a splitting $N(\bar{\mathbb{F}}_p) \rightarrow N(R)$ which induces an isomorphism

$$H^*(BN(R), \mathbb{Z}/n) \xrightarrow{\sim} H^*(BN(\bar{\mathbb{F}}_p), \mathbb{Z}/n)$$

The composite map $N(\bar{\mathbb{F}}_p) \rightarrow N(R) \rightarrow N(k) \rightarrow G(k)$ detects the mod $n$ cohomology of $G(k)$ and, since the mod $n$ cohomology of $N(\bar{\mathbb{F}}_p)$ is detected by the finite subgroups $N(\mathbb{F}_q) \subset N(\bar{\mathbb{F}}_p)$, we conclude that $G(k)$ satisfies $\mathrm{FSC}_n$.

Conversely, assume $G_k$ satisfies $\mathrm{FSC}_n$. By Corollary 2.8 it suffices to prove that the restriction map $H^*(BG(k), \mathbb{Z}/n) \rightarrow H^*(BN(k), \mathbb{Z}/n)$ is injective in order to conclude that $G_k$ satisfies $\mathrm{GIC}_n$. To prove this injectivity, it clearly suffices to assume $n = l^d$ for some prime $l$ invertible in $k$. Let $x \in H^*(BG(k), \mathbb{Z}/l^d)$ be a non-zero element and choose a finite subgroup $\pi \subset G(k)$ such that $x$ restricts non-trivially to $H^*(B\pi, \mathbb{Z}/l^d)$. Replacing $\pi$ by an $l$-Sylow subgroup, we may assume that $\pi$ is an $l$-group. Such an $l$-group $\pi \subset G(k)$ consists entirely of semi-simple elements and thus normalizes some maximal torus of $G(k)$ (cf. [10], 5.17). Thus, $\pi$ is conjugate to a subgroup of $N(k)$ so that the restriction of $x$ to $H^*(BN(k), \mathbb{Z}/n)$ is non-trivial.

As an easy corollary we conclude the following.

COROLLARY 3.3. *Assume the notation of Theorem* 3.2 *and let* $k = \bigcup k_\alpha$ *where each* $k_\alpha$ *is algebraically closed. Then* $G_k$ *satisfies* $\mathrm{GIC}_n$ *if and only if each* $G_{k_\alpha}$ *satisfies* $\mathrm{GIC}_n$.

*Proof.* Suppose that $G_k$ satisfies $\mathrm{GIC}_n$ and let $k_\alpha \subset k$ be a fixed algebraically closed subfield. Then $k = \bigcup A_\beta$ where each $A_\beta$ is a finitely generated $k_\alpha$-algebra and thus admits a $k_\alpha$-algebra map $A_\beta \to k_\alpha$. Therefore it follows that the natural map

$$H_*(BG(k_\alpha), \mathbb{Z}/n) \to H_*(BG(k), \mathbb{Z}/n) = \varinjlim_{\beta} H_*(BG(A_\beta), \mathbb{Z}/n)$$

is injective. This implies that the restriction map

$$H^*(BG(k), \mathbb{Z}/n) \to H^*(BG(k_\alpha), \mathbb{Z}/n)$$

is surjective. Using (2.5.6) and Corollary 2.8 we conclude that $H^*(BG(k_\alpha), \mathbb{Z}/n) \to H^*(BN(k_\alpha), \mathbb{Z}/n)$ is injective and thus $\mathrm{GIC}_n$ holds for $G_{k_\alpha}$.

If each $G_{k_\alpha}$ satisfies $\mathrm{GIC}_n$ and therefore $\mathrm{FSC}_n$, we see that $G_k$ satisfies $\mathrm{FSC}_n$, since

$$H^*(BG(k), \mathbb{Z}/n) \xrightarrow{\sim} \varprojlim_{\alpha} H^*(BG(k_\alpha), \mathbb{Z}/n)$$

Therefore, $G_k$ satisfies $\mathrm{GIC}_n$ by Theorem 3.2.

Corollary 3.3 may be used to show that it suffices to prove $\mathrm{GIC}_n$ for one "sufficiently large" field of each characteristic in order to show that $\mathrm{GIC}_n$ holds for all fields.

COROLLARY 3.4. *Assume the notation of Theorem* 3.2 *and let $k$ be an algebraically closed field of infinite transcendence degree over its prime subfield. If* $G_k$ *satisfies* $\mathrm{GIC}_n$ *then* $G_L$ *satisfies* $\mathrm{GIC}_n$ *for every algebraically closed field $L$ with* char $(L) =$ char $(k)$.

*Proof.* Write $L = \bigcup L_\alpha$ where each $L_\alpha$ is algebraically closed and of finite transcendence degree over the prime subfield. Then every $L_\alpha$ admits an embedding into $k$ and thus $G_{L_\alpha}$ satisfies $\mathrm{GIC}_n$ by Corollary 3.3. Since $L = \bigcup L_\alpha$, we conclude that $G_L$ satisfies $\mathrm{GIC}_n$ by Corollary 3.3.

## REFERENCES

[1] A. BOREL, *Properties and linear representations of Chevalley groups.* Lecture Notes in Math. *131*, Springer (1970), 1–50.

[2] H. CARTAN and S. EILENBERG, *Homological Algebra.* Princeton University Press, 1956.

[3] M. FESHBACH, *Some general theorems on the cohomology of classifying spaces of compact Lie groups.* Trans. A.M.S. 264, no. *1* (1981), 49–58.

[4] E. FRIEDLANDER, *Unstable K-theories of the algebraic closure of a finite field.* Comment. Math. Helvetici *50* (1975), 145–154.

[5] ——, *Computations of K-theories of finite fields.* Topology *15* (1976), 89–109.

[6] ——, *Etale Homotopy of Simplicial Schemes.* Annals of Math. Studies 104, Princeton University Press, 1982.

[7] —— and B. PARSHALL, *Etale cohomology of reductive groups.* Lecture Notes in Math. 854, Springer (1981), 127–140.

[8] S. N. KLEINERMAN, *The cohomology of Chevalley groups of exceptional Lie type.* Memoirs Amer. Math. Soc. *39* (268), 1982.

[9] J. MILNOR, *On the homology of Lie groups made discrete.* Comment. Math. Helvetici *58* (1983), 72–85.

[10] T. A. SPRINGER and R. STEINBERG, *Conjugacy classes.* Lecture Notes in Math. 131, Springer (1970), 167–266.

[11] D. SULLIVAN, *Genetics of homotopy theory and the Adams Conjecture.* Annals of Math. *100* (1974), 1–79.

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Received September 8, 1983