Radial growth of the derivative of univalent functions.

Autor(en): **Clunie, J.G. / MacGregor, T.H.**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **59 (1984)**

PDF erstellt am: **06.08.2024**

Persistenter Link: https://doi.org/10.5169/seals-45401

Comment. Math. Helvetici 59 (1984) 362–375 0010-2571/84/030362-14$01.50 + 0.20/0

Radial growth of the derivative of univalent functions

J. G. CLUNIE and T. H. MACGREGOR

1. Introduction

Suppose that the function f is analytic and univalent in $U=\{|z|<1\}$. A classical distortion theorem says that

$$|f'(re^{i\theta})|\leqq|f'(0)|\frac{1+r}{(1-r)^3} \tag{1.1}$$

for $0\leqq r<1$. The Koebe function $k(z)=z/(1-z)^2$ shows that (1.1) is sharp with equality occurring at $z=r$. However, for each fixed θ in $(0,2\pi)$, $k'(re^{i\theta})$ is bounded for $0\leqq r<1$.

In general, when suitable "exceptional" values of θ are excluded the growth of $|f'(re^{i\theta})|$ as $r\to1$ is much more restricted than that allowed in (1.1). We are specifically interested in exceptional sets of Lebesgue measure 0. The main result in this direction is due to Seidel and Walsh [7, p. 141]; namely,

$$\lim_{r\to1-}(1-r)^{1/2}f'(re^{i\theta})=0 \tag{1.2}$$

for almost all values of θ.

In [4] Lohwater and Piranian asked whether (1.2) holds when $(1-r)^{1/2}$ is replaced by a function tending to 0 more slowly. We provide a more or less complete answer to this question. In Theorem 2 we prove that

$$\lim_{r\to1-}\frac{\log|f'(re^{i\theta})|}{\left(\log\dfrac{1}{1-r}\right)^{\gamma}}=0 \tag{1.3}$$

for almost all θ, whenever $\gamma>\frac{1}{2}$. A particular consequence of (1.3) is

$$\lim_{r\to1-}(1-r)^{\alpha}f'(re^{i\theta})=0 \tag{1.4}$$

for almost all θ, whenever $\alpha>0$.

In preparation for the proof of Theorem 2 we show that there are positive constants A_λ such that

$$\int_0^{2\pi} |\log |f'(re^{i\theta})||^\lambda \, d\theta < A_\lambda \left(\log \frac{1}{1-r}\right)^{\lambda/2} \tag{1.5}$$

for $\lambda > 0, 0 \leqq r < 1$, where $f(z)$ is assumed to be univalent in U and $f'(0) = 1$. Given U the normalisation $f'(0) = 1$, inequality (1.1) together with the distortion theorem

$$|f'(re^{i\theta})| \geqslant \frac{1-r}{(1+r)^3} \tag{1.6}$$

for $0 \leqq r < 1$, shows that

$$|\log |f'(re^{i\theta})|| \leqq A \log \frac{1}{1-r}$$

for $0 \leqq r < 1$, for a suitable positive constant A, independent of f. Thus, (1.5) may be viewed as an improvement of the "trivial" estimate where the right-hand side of the (1.5) has the exponent λ in place of $\lambda/2$.

We show that the results given in (1.3) and (1.5) are precise in a suitable sense. We also give a fairly complete answer to the question: in (1.3) and (1.5) how necessary is univalency?

As we said above we are concerned with the radial behaviour of $f'(re^{i\theta})$ outside possible exceptional sets of measure 0. One can also consider whether or not (1.2) remains true outside an exceptional set that is "smaller" than just being of measure 0. Lohwater and Piranian [4] have shown that "measure 0" cannot be replaced by "logarithmic capacity 0" at any rate. They give an example of a function f analytic and univalent in U such that

$$\lim_{r \to 1-} (1-r)^{1/2} |f'(re^{i\theta})| = \infty$$

for all θ in a set of positive logarithmic capacity.

Finally, it should perhaps be pointed out that if $f(z)$ is analytic and univalent in U, then $\lim_{r \to 1-} f(re^{i\theta})$ exists finitely for almost all θ, and so there are really no problems corresponding to those dealt with above for $f(z)$ itself.

2. Theorem 1 and proof

THEOREM 1. *To* $\lambda>0$ *corresponds* $A_\lambda>0$ *such that if* $f(z)$ *is analytic and univalent in* U *and* $f'(0)=1$, *then*

$$\int_0^{2\pi} |\log|f'(re^{i\theta})||^\lambda \, d\theta \leqq A_\lambda\left(\log\frac{1}{1-r}\right)^{\lambda/2} \tag{2.7}$$

for $0\leqq r<1$.

Proof. The case $\lambda=2$ of (2.7) was proved by Flett [2, p. 71]. We first give an inductive proof of (2.7) for $\lambda=2,4,\ldots$. Assume then that λ is an even positive integer and write

$$I_\lambda(r)=\int_0^{2\pi} (\log|f'(re^{i\theta})|)^\lambda \, d\theta. \tag{2.8}$$

Then, by [6, p. 125],

$$\frac{d}{dr}\left[r\frac{dI_\lambda(r)}{dr}\right]=r\lambda(\lambda-1)\int_0^{2\pi}(\log|f'(re^{i\theta})|)^{\lambda-2}\left|\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right|^2 d\theta. \tag{2.9}$$

Since $f(z)$ is analytic and univalent in U,

$$\left|\frac{f''(z)}{f'(z)}\right|\leqq\frac{4}{1-|z|}\ (|z|<1), \tag{2.10}$$

from [6, p. 21], for example. From (2.9) and (2.10) it follows that

$$\frac{d}{dr}\left[r\frac{dI_\lambda(r)}{dr}\right]\leqq\frac{16r\lambda(\lambda-1)}{(1-r)^2}I_{\lambda-2}(r) \tag{2.11}$$

for $0\leqslant r<1$. If we inductively assume that

$$I_{\lambda-2}(r)\leqslant A_{\lambda-2}\left(\log\frac{1}{1-r}\right)^{(\lambda-1)/2},$$

and note that this hypothesis is valid for $\lambda=2$, with $A_0=2\pi$, then (2.11) implies that

$$r\frac{dI_\lambda(r)}{dr}\leqq 16\lambda(\lambda-1)\cdot A_{\lambda-2}\int_0^r \frac{1}{(1-\rho)^2}\left(\log\frac{1}{1-\rho}\right)^{(\lambda-1)/2} d\rho. \tag{2.12}$$

If the integral on the right-hand side of (2.12) is integrated by parts, the integral part may be dropped since

$$\frac{d}{d\rho}\left(\log\frac{1}{1-\rho}\right)^{(\lambda-1)/2} \geqslant 0.$$

In other words,

$$r\frac{dI_\lambda(r)}{dr}\leqq\frac{16\lambda(\lambda-1)}{1-r}A_{\lambda-2}\left(\log\frac{1}{1-r}\right)^{(\lambda-1)/2} \tag{2.13}$$

Suppose that $\frac{1}{2}\leqq r<1$. By integrating (2.13) we find that

$$I_\lambda(r)\leqq\frac{4}{\lambda}16\lambda(\lambda-1)A_{\lambda-2}\left(\log\frac{1}{1-r}\right)^{\lambda/2}+I_\lambda(\tfrac{1}{2}). \tag{2.14}$$

Since, by a well known distortion result,

$$|\log|f'(z)||\leqslant\log\frac{1+|z|}{1-|z|} \text{ in } U,$$

(2.14) implies that (2.7) holds when $0\leqq r<1$. This completes the inductive argument.

Now, assume only that $\lambda>0$. Let n denote the smallest even integer not less than λ and put $p=n/\lambda$. By Hölder's inequality we have

$$\int_0^{2\pi}|\log|f'(re^{i\theta})||^\lambda\, d\theta\leqq\left(\int_0^{2\pi}|\log|f'(re^{i\theta})||^{p\lambda}\, d\theta\right)^{1/p}(2\pi)^{1-(1/p)}$$

$$\leqq\left\{A_n\left(\log\frac{1}{1-r}\right)^{n/2}\right\}^{\lambda/n}(2\pi)^{1-(1/p)}=A_\lambda\left(\log\frac{1}{1-r}\right)^{\lambda/2}.$$

3. How precise is Theorem 1?

We now show that Theorem 1 is precise in an appropriate sense. If u is a real valued function in U, we write $u^+=\frac{1}{2}(|u|+u)$ and $u^-=\frac{1}{2}(|u|-u)$. We show that there is a function $f(z)$ for which both the positive and the negative contributions of $\log|f(re^{i\theta})|$ to the integral on the left-hand side of (2.7) grow as the right-hand side of (2.7). Although we are primarily interested in the growth of $f'(re^{i\theta})$ to ∞, inequality (2.7) and the statement below concerning the negative contributions give exact information on how $f'(re^{i\theta})$ may tend to 0 as $r\to 1$.

THEOREM 2. *There is a function f which is analytic and univalent in U and satisfies*

$$\int_0^{2\pi}(\log^+|f'(re^{i\theta})|)^\lambda\,d\theta \geqq B_\lambda\left(\log\frac{1}{1-r}\right)^{\lambda/2} \tag{3.15}$$

and

$$\int_0^{2\pi}(\log^-|f'(re^{i\theta})|)^\lambda\,d\theta \geqq B_\lambda\left(\log\frac{1}{1-r}\right)^{\lambda/2}, \tag{3.16}$$

for $0\leqq r<1$, $\lambda>0$ *and where* $B_\lambda>0$.

Proof. Let f denote the function constructed in [1]. There it was proved that f is analytic and univalent in U, satisfies (3.15) for $\lambda=1$ and if $F(\theta)=F(\theta,r)=\log|f'(re^{i\theta})|$, then F has the form

$$F(\theta)=\alpha\sum_{n=1}^{\infty} r^{k^n}\cos(k^n\theta), \tag{3.17}$$

where $\alpha>0$ and $k>3$.

Define $u=\log^+|f'|$. If $\lambda>1$, then Hölder's inequality and the validity of (3.15) when $\lambda=1$ implies that

$$B_1\left(\log\frac{1}{1-r}\right)^{1/2}\leqslant\int_0^{2\pi}u(re^{i\theta})\,d\theta\leqslant(2\pi)^{1-(1/\lambda)}\left(\int_0^{2\pi}(u(re^{i\theta}))^\lambda\,d\theta\right)^{1/\lambda}.$$

Thus, (3.15) holds when $\lambda\geqq 1$.

Since $\int_0^{2\pi}F(\theta)\,d\theta=0$ it follows that

$$\int_0^{2\pi}\log^-|f'(re^{i\theta})|\,d\theta=\int_0^{2\pi}\log^+|f'(re^{i\theta})|\,d\theta$$

and so (3.16) holds when $\lambda = 1$. By the same kind of argument as the preceding it follows that (3.16) is correct whenever $\lambda \geqq 1$.

We next consider the case when $0<\lambda<1$, though the argument we use is actually valid for $0<\lambda<2$. The series in (3.17) is a lacunary trigonometric series and therefore [8, p. 216] there corresponds to each positive β some $C_\beta > 0$ so that

$$\left(\int_0^{2\pi} |F(\theta)|^\beta \, d\theta\right)^{1/\beta} \leqq C_\beta \left(\int_0^{2\pi} |F(\theta)|^2 \, d\theta\right)^{1/2}.$$

If we use Theorem 1 in the case $\lambda = 2$ and (3.15) in the case $\lambda = 1$, we obtain

$$\left(\int_0^{2\pi} (u(re^{i\theta}))^\beta \, d\theta\right)^{1/\beta} \leqq C_\beta \left(\int_0^{2\pi} (\log |f'(re^{i\theta})|)^2 \, d\theta\right)^{1/2}$$

$$\leqq C_\beta \left(A_2 \log \frac{1}{1-r}\right)^{1/2} \leqq \frac{C_\beta A_2^{1/2}}{B_1} \int_0^{2\pi} u(re^{i\theta}) \, d\theta.$$

Therefore, there is a positive number D_β so that

$$\int_0^{2\pi} u(re^{i\theta}) \, d\theta \geqq D_\beta \left(\int_0^{2\pi} (u(re^{i\theta}))^\beta \, d\theta\right)^{1/\beta}. \tag{3.18}$$

If $0<\lambda<2$ and $\beta = 2-\lambda$, then (3.18) implies that

$$\int_0^{2\pi} u(re^{i\theta}) \, d\theta = \int_0^{2\pi} (u(re^{i\theta}))^{\lambda/2} \cdot (u(re^{i\theta}))^{1-(\lambda/2)} \, d\theta$$

$$\leqq \left(\int_0^{2\pi} (u(re^{i\theta}))^\lambda \, d\theta\right)^{1/2} \cdot \left(\int_0^{2\pi} (u(re^{i\theta}))^{2-\lambda} \, d\theta\right)^{1/2}$$

$$\leqq \left(\int_0^{2\pi} (u(re^{i\theta}))^\lambda \, d\theta\right)^{1/2} \cdot \frac{1}{D_\beta^\beta} \left(\int_0^{2\pi} u(re^{i\theta}) \, d\theta\right)^{(2-\lambda)/2}.$$

Therefore

$$\int_0^{2\pi} (u(re^{i\theta}))^\lambda \, d\theta \geqq E_\lambda \left(\int_0^{2\pi} u(re^{i\theta}) \, d\theta\right)^\lambda, \tag{3.19}$$

where $0<\lambda<2$ and E_λ is positive and depends only on λ. Since (3.15) is valid when $\lambda = 1$, (3.19) implies that (3.15) holds whenever $0<\lambda<2$.

A proof of (3.16) for the case $0<\lambda<2$ may be given in a similar way based on the fact that (3.16) holds when $\lambda=1$.

4. The main theorem

THEOREM 3. *If $f(z)$ is analytic and univalent in U and if $\gamma>\frac{1}{2}$, then*

$$\lim_{r\to 1-}\frac{\log|f'(re^{i\theta})|}{\left(\log\dfrac{1}{1-r}\right)^{\gamma}}=0 \tag{4.20}$$

for almost all θ.

Proof. Suppose that $\alpha>0$, β is a positive integer, and that $0<r<1$. We define

$$G(r,\theta)=\left(\log\frac{1}{1-r}\right)^{-\alpha}(\log|f'(re^{i\theta}|)^{\beta}$$

and

$$H(r,\theta)=\frac{\partial G(r,\theta)}{\partial r}.$$

Because of (2.10),

$$\left|\frac{\partial}{\partial r}\log|f'(re^{i\theta})|\right|=\left|\mathrm{Re}\left\{\frac{f''(re^{i\theta})}{f'(re^{i\theta})}e^{i\theta}\right\}\right|\leqq\frac{4}{1-r}.$$

Therefore,

$$|H(r,\theta)|\leqq\frac{4\beta}{1-r}\left(\log\frac{1}{1-r}\right)^{-\alpha}|\log|f'(re^{i\theta})||^{\beta-1}$$
$$+\frac{\alpha}{1-r}\left(\log\frac{1}{1-r}\right)^{-\alpha-1}|\log|f'(re^{i\theta})||^{\beta}$$

and so Theorem 1 implies that

$$\int_0^{2\pi}|H(r,\theta)|\,d\theta\leqq\frac{4\beta A_{\beta-1}}{1-r}\left(\log\frac{1}{1-r}\right)^{-\alpha+(\beta-1)/2}$$
$$+\frac{\alpha A_{\beta}}{1-r}\left(\log\frac{1}{1-r}\right)^{-\alpha-1+(\beta/2)}. \tag{4.21}$$

Suppose that $r_0 = 1 - 1/e$ and $r_0 \leqq r < 1$. Then $\log(1/1-r) \geqq 1$ and so for real p, q and $p \geqq q$ we have $(\log(1/1-r))^p \geqq (\log(1/1-r))^q$. Therefore, if $r_0 \leq r < 1$, (4.21) may be written

$$\int_0^{2\pi} |H(r, \theta)|\, d\theta \leq \frac{A}{1-r} \frac{1}{\left(\log \frac{1}{1-r}\right)^{\alpha-(\beta/2)+1/2}}, \tag{4.22}$$

where A is a suitable positive number. Since the integral

$$\int_{r_0}^1 \frac{1}{1-r} \frac{1}{\left(\log \frac{1}{1-r}\right)^{\delta}}\, dr$$

converges whenever $\delta > 1$, (4.22) implies that if $\alpha > (\beta+1)/2$, then

$$\int_{r_0}^1 \left\{\int_0^{2\pi} |H(r, \theta)|\, d\theta\right\} dr < \infty. \tag{4.23}$$

By the Tonelli-Hobson theorem, (4.23) implies that

$$\int_0^{2\pi} \left\{\int_{r_0}^1 |H(r, \theta)|\, dr\right\} d\theta < \infty. \tag{4.24}$$

In particular, (4.24) implies that

$$\int_{r_0}^1 |H(r, \theta)|\, dr < \infty \tag{4.25}$$

for almost all θ in $[0, 2\pi]$. Therefore, if β is a positive integer and $\alpha > (\beta+1)/2$ then there is a set Θ contained in $[0, 2\pi]$ and having measure 2π for which (4.25) holds whenever $\theta \in \Theta$.

Suppose that β is a positive integer, $\alpha > (\beta+1)/2$ and choose α' so that $\alpha > \alpha' > (\beta+1)/2$. The pair β, α' determine a set Θ as described above. If $r_0 < r < 1$ and $\theta \in \Theta$ then

$$\begin{aligned} |G(r, \theta) - G(r_0, \theta)| &\leqq \int_{r_0}^{r} \left|\frac{\partial G(\rho, \theta)}{\partial \rho}\right| d\rho \\ &= \int_{r_0}^{r} |H(\rho, \theta)|\, d\rho \leqq \int_{r_0}^{1} |H(\rho, \theta)|\, d\rho = D < \infty. \end{aligned}$$

Writing $E=D+|G(r_0, \theta)|$ we find that

$$\frac{|\log|f'(re^{i\theta})||^{\beta}}{\left(\log\dfrac{1}{1-r}\right)^{\alpha}} \leqq E\cdot\frac{1}{\left(\log\dfrac{1}{1-r}\right)^{\alpha-\alpha'}}.$$

This shows that

$$\lim_{r\to 1-}\frac{|\log|f'(re^{i\theta})||^{\beta}}{\left(\log\dfrac{1}{1-r}\right)^{\alpha}}=0 \tag{4.26}$$

whenever $\theta\in\Theta$. Since (4.26) is the same as

$$\lim_{r\to 1-}\frac{\log|f'(re^{i\theta})|}{\left(\log\dfrac{1}{1-r}\right)^{\alpha/\beta}}=0 \tag{4.27}$$

we see that (4.27) holds for almost all θ if β is a positive integer and $\alpha>(\beta+1)/2$. The inequality $\alpha>(\beta+1)/2$ is the same as $\alpha/\beta>\frac{1}{2}+(1/2\beta)$ and so the conditions on α and β allow α/β to take on any value $\gamma>\frac{1}{2}$. This completes the proof of Theorem 3.

5. How precise is Theorem 3?

The next theorem shows that the condition $\gamma>\frac{1}{2}$ in Theorem 3 is necessary in so far as it cannot be replaced by $\gamma=\frac{1}{2}$.

THEOREM 4. *There is a function f analytic and univalent in U such that*

$$\limsup_{r\to 1-}\frac{|\log|f'(re^{i\theta})||}{\left(\log\dfrac{1}{1-r}\right)^{1/2}}>0 \tag{5.28}$$

for almost all θ.

Remark. In the above statement 'almost all' is essential. See § 10.2 of [6].

Proof. Let f be the function used for the argument in Theorem 2. Suppose that $\Theta \subseteq [0, 2\pi]$ and $|\Theta|$, the measure of Θ, is positive. Because $\log|f'(re^{i\theta})| = F(\theta)$ is a lacunary trigonometric series when $0<r<1$, the argument given in [8, pp. 119–122] shows that if we drop a finite number of terms from the beginning of the series to form a truncated series $T(\theta)$, then

$$\int_\Theta |T(\theta)|^2 \, d\theta \geqq A \int_0^{2\pi} |T(\theta)|^2 \, d\theta \tag{5.29}$$

The positive number A depends only on Θ (and the value of k in (3.17)) and the number of terms that may have to be dropped is independent of r. Hence, there is a positive number B, depending only on Θ, so that

$$\int_\Theta (\log|f'(re^{i\theta})|)^2 \, d\theta \geqq B \int_0^{2\pi} (\log|f'(re^{i\theta})|)^2 \, d\theta. \tag{5.30}$$

Theorem 2 (with $\lambda = 2$) and (5.30) imply that

$$\int_\Theta (\log|f'(re^{i\theta})|)^2 \, d\theta \geqq C \log\frac{1}{1-r}, \tag{5.31}$$

where $C>0$ and C depends only on Θ.

Assume that Theorem 4 is false and let Φ with $|\Phi|>0$ be the set of θ such that

$$\frac{\log|f'(re^{i\theta})|}{\left(\log\dfrac{1}{1-r}\right)^{1/2}} \to 0 \ (r \to 1-).$$

We recall from (2.10) that

$$\left|\frac{\partial}{\partial r}\log|f'(re^{i\theta})|\right| \leqq \frac{4}{1-r}.$$

We also deduce from Egorov's theorem applied to the sequence

$$f_n(\theta) = \frac{\log\left|f'\left(\left(1-\dfrac{1}{2^n}\right)e^{i\theta}\right)\right|}{(\log 2^n)^{1/2}}$$

defined on Φ that there is a subset $\Theta \subseteq \Phi$ with $|\Theta|>0$ such that

$$\frac{\log|f'(re^{i\theta})|}{\left(\log\frac{1}{1-r}\right)^{1/2}} \to 0 \quad (r\to 1-)$$

uniformly on Θ. Hence it follows that

$$\frac{\int_\Theta (\log|f'(re^{i\theta})|)^2\,d\theta}{\log\frac{1}{1-r}} \to 0 \quad (r\to 1-)$$

and this contradicts (5.31). This proves Theorem 4.

6. Some consequences of the main theorem

The first of our two corollaries relates the result of Theorem 3 to that of Seidel and Walsh.

COROLLARY 1. *Suppose that f is analytic and univalent in U. If $\gamma>\frac{1}{2}$, then*

$$\lim_{r\to 1-}\left[\exp\left(-\left(\log\frac{1}{1-r}\right)^\gamma\right)\cdot f'(re^{i\theta})\right]=0 \tag{6.32}$$

for almost all θ. Also, if $\alpha>0$, then

$$\lim_{r\to 1-}(1-r)^\alpha f'(re^{i\theta})=0 \tag{6.33}$$

for almost all θ.

Proof. Both (6.32) and (6.33) can be deduced immediately from (4.20).

Our next result says that in Theorem 3, radial limit can be replaced by Stolz and angle limit.

COROLLARY 2. *Let f be analytic and univalent in U. If $\gamma > \frac{1}{2}$, then for almost all θ, if Δ_ζ is a Stolz angle with vertex $\zeta = e^{i\theta}$,*

$$\lim_{\substack{z \to \zeta \\ z \in \Delta_\zeta}} \frac{\log |f'(z)|}{\left(\log \dfrac{1}{1-|z|}\right)^\gamma} = 0.$$

Proof. If $\zeta = e^{i\phi}$ let Δ_ζ be a Stolz angle with vertex ζ. There is a positive K depending on Δ_ζ, such that for $z \in \Delta_\zeta$ we have

$$\frac{|\zeta - z|}{1-|z|} \leqq K. \tag{6.34}$$

If $z = re^{i\theta} \in \Delta_\zeta$, then

$$\begin{aligned}\log |f'(z)| - \log |f'(re^{i\phi})| &= \operatorname{Re}\,[\log f'(z) - \log f'(re^{i\phi})] \\ &= \operatorname{Re} \int_L \frac{f''(\omega)}{f'(\omega)}\, d\omega,\end{aligned}$$

where L is the segment $[re^{i\phi}, z]$. From (2.10) and (6.34) we obtain

$$\begin{aligned}|\log |f'(z)|\,| &\leqq |\log |f'(re^{i\phi})|\,| + \frac{4}{1-r} |re^{i\phi} - z| \\ &\leqq |\log |f'(re^{i\phi})|\,| + \frac{4}{1-r} (|re^{i\phi} - e^{i\phi}| + |\zeta - z|) \\ &\leqq |\log |f'(re^{i\phi})|\,| + 4(K+1).\end{aligned}$$

The result of Theorem 3 together with the above gives Corollary 2.

Remarks. (1) From the proof one sees that a Stolz angle at ζ can be replaced by a larger domain in Ω that is tangential to $\{|z| = 1\}$ at ζ.

(2) In (6.32) and (6.33) one can replace radial limit by Stolz angle limit or limit within a domain of the kind referred to in (1).

7. Concluding remarks

Without being too discursive we shall make a number of remarks and observations which anticipate some questions that naturally arise from our results.

In Theorem 3 the numerator of (4.20) is

$$\log |f'(re^{i\theta})| = \operatorname{Re}[\log f'(re^{i\theta})]$$

and so one can ask if the theorem remains true when $\log |f'(re^{i\theta})|$ is replaced by $|\log f'(re^{i\theta})|$. The answer is "yes" since the proof we give will still be valid provided that the result corresponding to (2.14) still holds; and, given the form of the proof of (2.14), this is the case by Riesz's result on conjugate functions [8, p. 147].

As far as widening the class of functions in the hypotheses of Theorem 3 one must in our presentation take (2.10) into account. Hence within our context perhaps the largest "natural" class consists of functions analytic in U which are locally univalent and strongly finitely valent [5]. In this connection the example given in §3.4 of [3] is instructive.

When one considers results like those of Theorem 1 and Theorem 3 more generally, then clearly some restrictions must be placed on the functions considered. It would therefore seem appropriate to consider functions analytic in U which are locally univalent and of bounded characteristic. It is easy to show in this case that one has (2.7) with the index $\lambda/2$ on the right-hand side replaced by λ and one has (4.20) with $\lambda > \frac{1}{2}$ replaced by $\lambda > 1$. That these results are essentially best possible can be seen by considering functions $f(z)$ with

$$f'(z) = \exp\left(c \sum_{n=1}^{\infty} \log \lambda_n z^{\lambda_n}\right) \quad (|z| < 1),$$

where C is small and positive and (λ_n) is very gappy. In this case $f \in H^{\infty}$, but for the results we are dealing with one expects "bounded characteristic" and "bounded" to be more or less equivalent.

We are grateful to the referee for the careful way he read our manuscript and for a number of valuable observations.

REFERENCES

[1] J. Clunie, *On schlicht functions*, Ann. Math. *69* (1959), 511–519.

[2] T. M. Flett, *Some remarks on schlicht functions and harmonic functions of uniformly bounded variation*, Quart. J. Math. (Oxford) *6* (1955), 59–72.

[3] W. K. Hayman, *Multivalent functions*, C.U.P., Cambridge 1958.

[4] A. J. Lohwater and G. Piranian, *On the derivative of a univalent function*, Proc. Amer. Math. Soc. *4* (1953), 591–594.

[5] CH. POMMERENKE, *Linear-invariante Familien analytischer Funktionen, I.*, Math. Ann. *155* (1964), 108–154.
[6] CH. POMMERENKE, *Univalent functions*, Vandenhoeck and Ruprecht (1975), Göttingen.
[7] W. SEIDEL and J. L. WALSH, *On the derivative of functions analytic in the unit circle and their radii of univalence and p-valence*, Trans. Amer. Math. Soc. *52* (1942), 128–216.
[8] A. ZYGMUND, *Trigonometrical Series*, 2nd Edition, Chelsea (1952), New York.

Faculty of Mathematics,
The Open University,
Milton Keynes, U.K.

Department of Mathematics and Statistics,
State University of New York at Albany,
Albany, New York, U.S.A.

Received September 10, 1983