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# Some topologically locally-flat surfaces in the complex projective plane

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## § 1. Introduction; statement of results

THEOREM 1. *For every integer* $n \geq 6$, *there exists in the homology class* $n[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z})$ *a topologically locally-flatly embedded surface of genus strictly less than that of a nonsingular complex algebraic curve of degree n.*

THEOREM 2. *For every pair* $(m, n)$ *of integers greater than or equal to* 5, *(except possibly* $(5, 5)$*) there is a topologically locally-flatly embedded surface in the 4-disk with boundary a torus link* $O\{m, n\}$ *of type* $(m, n)$ *and genus strictly less than the (classical) genus of* $O\{m, n\}$.

Here, a surface $S$ topologically embedded in a 4-manifold $M$ will be called "topologically locally-flatly embedded" if $S$ has a neighborhood $N$ in $M$ which is homeomorphic to an open 2-disk bundle over $S$ by a homeomorphism carrying $S$ to a section. This is evidently some kind of local homogeneity assumption on the embedding of $S$ in $M$. (For instance, if $S$ is smoothly, or P.L. locally-flatly, embedded in $M$ then it is *a fortiori* topologically locally-flatly embedded. After preparing this paper, the author learned of a new theorem of Akbulut – showing that certain "topologically slice" knots very similar to $\hat{\beta}_6$ in §3, below, definitely are not smoothly slice – which implies that not every topologically locally-flat surface is just a smooth or P.L. locally-flat surface up to a global topological change of coordinates.)

One construction will be used to prove both theorems. It is an instance of a general construction discussed in earlier papers by the author [7, 8, 9]; it now proves the theorems because of a recent result of M. Freedman. The specific construction is given below, following some motivating remarks and a short new (and, I believe, improved) exposition of the general construction.

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*Remark* 1. A conjecture frequently attributed to R. Thom[1] is that no *smoothly* embedded surface in $\mathbb{CP}^2$ can have genus strictly smaller than that of a homologous (smooth) complex algebraic curve. It is well-known [5] that, by being willing to sacrifice local flatness, one can represent every homology class by a piecewise-linearly embedded 2-sphere – for instance, up to orientation, by the complex algebraic curve with affine equation $w = z^n$. But this sphere need not be piecewise-linearly locally-flat – in the example, for $n \geq 3$, there is a singular point at infinity.

The point of Theorem 1 is that by making a global (or at least regional) sacrifice of smoothness, one can salvage a weaker sort of homogeneity of normal structure while "chopping off handles."

*Remark* 2. Theorem 2 is vaguely related to the "problem of Milnor" on the Gordian number (Überschneidungszahl, or unknotting number) of the link of a singularity (cf. [6], [2]). Indeed, $O\{m, n\}$ is such a link, and the problem in this case asks whether the Gordian number $\ddot{u}(O\{m, n\})$ equals $(m-1)(n-1)/2$, which is the classical genus of $O\{m, n\}$ (i.e. the least genus of a surface smoothly embedded in $S^3$ with boundary $O\{m, n\}$). If the answer is affirmative, then any smoothly embedded surface in the 4-disk with boundary $O\{m, n\}$ has genus at least $(m-1)(n-1)/2$. However, even if a smooth surface existed with boundary $O\{m, n\}$ and small genus, no conclusion could be necessarily drawn about $\ddot{u}(O\{m, n\})$; much less for the topologically locally-flat surface of Theorem 2.

*Remark* 3. Here is a sketch of the strategy used to prove both theorems. "By hand" we construct a smooth complex algebraic curve $\Gamma$ of degree 6 in $\mathbb{CP}^2$, and a piecewise-smooth 4-ball $D$ in $\mathbb{C}^2 \subset \mathbb{CP}^2$, such that (i) the transverse intersection $\Gamma \cap \partial D$ is a "topologically slice" knot, i.e., bounds a topologically locally-flatly embedded disk in $D$, while (ii) the smooth surface $\Gamma \cap D$, with the same boundary, has genus 1. Then replacing the surface of genus 1 by the disk, we produce a topologically locally-flatly embedded surface homologous to $\Gamma$ in $\mathbb{CP}^2$, of genus 1 smaller.

It is clear that by various expedients (most naively, doing essentially the same surgery in $k$ disjoint balls, on a curve of degree $6k$; or using a more complicated topologically slice knot, which bounds a piece of a curve of degree $5k+1$ that has genus $k$) one can produce as large a gap as desired between the genus of a smooth algebraic curve and that of a homologous topologically locally-flat surface. However, I know of no construction which makes a *proportional* gap bigger than

[1] Professor Thom has remarked (personal communication, November 19, 1982) that the conjecture perhaps more properly belongs to folklore.

10 per cent, which is already achieved by the example of degree 6 (where the genus of the algebraic curve is $\frac{1}{2}(6-1)(6-2)=10$ and one handle is chopped off). In any case, the proportional gap can't be *too* big (whether the topologically locally-flat surface is produced, as here, by "surgery" – rather, amputation – or not); for, as Shmuel Weinberger has kindly pointed out to me, Wall's topological version [10] of the $G$-signature theorem fits into the proof of Hsiang and Szczarba [4] to yield, for topologically locally-flat surfaces in 4-manifolds, exactly the estimates given in [4] for smooth surfaces.

In particular, topologically locally-flat 2-spheres in $\mathbb{CP}^2$ occur in degrees 0, $\pm 1$, $\pm 2$ only (where there are smooth examples).

## §2. A construction of closed braids

Fix an integer $n \geq 2$. For $k=1,\ldots,n-1$, let $\eta_k = \exp(2\pi(k-1)i/(n-1))$ (so $\eta_1 = 1$), and let $J_k = \eta_k[0,1]$ be the line segment in $\mathbb{C}$ from 0 to $\eta_k$. Write $Q_{n-1} = \{\eta_k : k=1,\ldots,n-1\}$. The fundamental group $\pi_i(\mathbb{C}\backslash Q_{n-1}, 0)$ is free of rank $n-1$, with free basis $x_1,\ldots,x_{n-1}$, where $x_k$ is represented by a loop based at 0 and running once counter-clockwise around the boundary of a convex region containing $\eta_k$ and no $\eta_j$, $j \neq k$. This group is, of course, identical to $\pi_1((\mathbb{C}\cup\{\infty\})\backslash(Q_{n-1}\{\infty\}), 0)$. Represent it in the symmetric group on $\{1,\ldots,n\}$ by sending $x_k$ to the transposition $(k \quad k+1)$. Let $X$ be the corresponding $n$-sheeted branched covering space of $\mathbb{C}\cup\{\infty\}$, branched over $Q_{n-1}\cup\{\infty\}$. One readily verifies that $X$ is a 2-sphere, with a single point over $\infty$. Thus the covering map "is" a polynomial of degree $n$, with $n-1$ critical points, and critical values $\eta_k$ $(k=1,\ldots,n-1)$; further requiring the polynomial to be monic and have constant term 0 will specify it completely. We assume this is done, and call the result $p(w) = w^n + \alpha_{n-1}w^{n-1} + \cdots + \alpha_1 w$. Write $\mathbb{C}_w$ for $X - \{\infty\}$, $\mathbb{C}_z$ for the base space $\mathbb{C}$, so $p : \mathbb{C}_w \to \mathbb{C}_z$.

*Remark* 4. Except for $n=2, 3$, I have been unable to find $p(w)$ explicitly. It is *not* in general the elegant $w^n - \alpha w$, where $\alpha = n(1-n)^{1-1/n}$; this (like the even simpler $w^n - nw$, which only differs by rotation and homothety in the base space) corresponds, apparently, to the representation $x_k \to (1 k+1)$. (Of course the construction could be adapted to these polynomials, at the expense of complicating the braid theory a bit.) For $n=2, 3$, the two representations are equivalent.

Now consider $p^{-1}(J_k)$. This has $n-1$ components, each a simple arc; let $I_k$ be the one containing the critical point with critical value $\eta_k$. Then the endpoints of $I_k$ are two of the preimages of 0, call them $w_k$ and $w_{k+1}$; it is easy to see that they may be numbered so that $I_k \cap I_{k+1} = \{w_{k+1}\}$ for $k = 1, \ldots, n-2$, while $w_1$ belongs only to $I_1$, $w_n$ only to $I_{n-1}$, and $I_k \cap I_l = \emptyset$ if $|k-l|>1$. Let $I = \bigcup_{k=1}^{n-1} I_k$. Then $I$ is a simple arc in $\mathbb{C}_w$.

Next consider the *configuration space* $E_n \backslash \Delta$ of unordered $n$-tuples of distinct points of $\mathbb{C}_w$; that is, form the symmetric product $E_n = \mathbb{C}_w^n/\mathcal{S}_n$, and delete from it the multidiagonal $\Delta$ of $n$-tuples with at least one pair of equal elements. The *$n$-string braid group* is by definition the fundamental group of the configuration space.

Specifically, we will take $p^{-1}(0) \in E_n \backslash \Delta$ as our basepoint. In the usual description of $B_n$, the basepoint is taken to be $\{1, \ldots, n\}$, and for $k = 1, \ldots, n-1$, the loop $l_k : S^1 \to E_n - \Delta : z \to \{1, \ldots, k-1, k+2, \ldots, n\} \cup \{k + \frac{1}{2}(1 \pm z^{\frac{1}{2}})\}$ (where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$) represents an element of $\pi_1(E_n \backslash \Delta, \{1, \ldots, n\})$ called the *standard generator* $\sigma_k$. Here, let $h : \mathbb{C}_w \to \mathbb{C}_w$ be an orientation-preserving homeomorphism with $h(I) = [1, n]$ and $h(w_k) = k$, $k = 1, \ldots, n$. Then $h$ enforces an identification of $\pi_1(E_n \backslash \Delta, \{1, \ldots, n\})$ with $B_n = \pi_1(E_n \backslash \Delta, p^{-1}(0))$, giving a meaning to the standard generators $\sigma_1, \ldots, \sigma_{n-1} \in B_n$.

Finally, note that $p^{-1}$ is well-defined as a continuous map $\mathbb{C}_z \to E_n$, and that by construction $p^{-1} | (\mathbb{C}_z - Q_{n-1})$ has image in $E_n \backslash \Delta$.

PROPOSITION. *The induced homomorphism* $p^{-1} | (\mathbb{C}_z - Q_{n-1})_*$ *from the free group* $\pi_1(\mathbb{C}_z - Q_{n-1}, 0)$ *to* $B_n$ *carries the free generator* $x_k$ *to the standard generator* $\sigma_k$, *for* $k = 1, \ldots, n-1$.

*Proof.* Recall that $x_k$ is represented by a loop which traverses (counterclockwise) the boundary of a convex region – call it $D_k$ – in $\mathbb{C}_z$, and that $\eta_k \in \operatorname{Int} D_k$, $\eta_j \notin D_k$ ($j \neq k$), and $0 \in \partial D_k$ ($k = 1, \ldots, n-1$). As with $I_k \subset D_k$, the preimage $p^{-1}(D_k)$ has $n-1$ components; $n-2$ of them are carried to $D_k$ homeomorphically by $p$, and one – call it $D'_k$ – is a 2-sheeted branched cover of $D_k$ *via* $p$, branched at $w_k \in \operatorname{Int} D'_k$; so $D'_k$ is again homeomorphic to a 2-disk. No component of $p^{-1}(D_k)$ other than $D'_k$ contains any critical point $w_j$ of $p$. The loop in $E_n$, with domain the simple closed curve $\partial D_k$, which takes $z \in \partial D_k$ to $p^{-1}(z) \in E_n$, clearly has image in $E_n \backslash \Delta$. It can easily be homotoped (respecting its basepoint $p^{-1}(0)$), in $E_n \backslash \Delta$, to a path of $n$-tuples each containing the $n-2$ points of $p^{-1}(0)$ not in $D'_k$, together with two points on $\partial D'_k$ which exchange positions (by a counterclockwise "rotation") as the loop is traversed; but such a path clearly represents $\sigma_k$. □

Recall that an oriented (closed) 1-manifold $L$ in the open solid torus $S^1 \times \mathbb{C}$ is a closed braid (on $n$ strings) if $\mathrm{pr}_1 | L : L \to S^1$ is an oriented covering projection (of degree $n$). A braid $\beta \in B_n$ yields a closed braid $\hat{\beta} \subset S^1 \times \mathbb{C}$ (unique up to isotopy respecting $pr_1$) by taking a loop $l: S^1 \to E_n \backslash \Delta$ representing $\beta$ and considering its "graph" (as an $n$-valued complex function) $\mathrm{gr}\, l = \{(z, w) \in S^1 \times \mathbb{C} : w \in l(z)\}$.

COROLLARY. *If $x_{i(1)}^{\varepsilon(1)} \cdots x_{i(s)}^{\varepsilon(s)}$ is any word in the free group $\pi_1(\mathbb{C}_z - Q_{n-1}, 0)$, and $\gamma: S^1 \to \mathbb{C}_z - Q_{n-1}, \gamma(1) = 0$, is a loop representing it, then the set $\{(z, w) : \gamma(z) = p(w)\}$ is a closed braid $\hat{\beta}$ on $n$ strings in $S^1 \times \mathbb{C}_w$, where $\beta = \sigma_{i(1)}^{\varepsilon(1)} \cdots \sigma_{i(s)}^{\varepsilon(s)} \in B_n$.* □

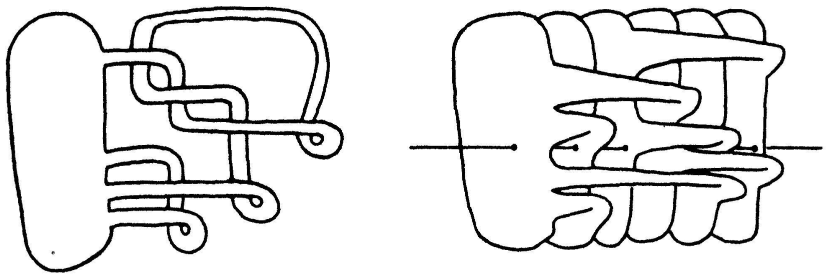
## §3. Freedman's theorem; proofs of theorems 1 & 2

The profound researches of Michael Freedman into the topology of 4-manifolds have recently led him to the following improvement [3a] of a theorem published in [3] (the original theorem applied only to a knot $K$ which was an untwisted double of a knot with Alexander polynomial 1).

FREEDMAN'S THEOREM. *Let $K \subset S^3 = \partial D^4$ be a (smooth) knot with Alexander polynomial $\Delta_K(t)$ identically 1. Then $K$ bounds a topologically locally-flat disk $S \subset D^4$.* □

It is not important for the following proofs to know what an Alexander polynomial is; it is enough to believe that the knot $K$ pictured in Figure 1A, where it is shown as the boundary of a punctured torus in $\mathbb{R}^3$, has $\Delta_K(t) = 1$. (This $K$ is in fact an untwisted double of a trefoil knot; from that fact, or calculating directly from the obvious Seifert matrix of the visible surface, those in the know will see that $\Delta_K(t) = 1$. As readers of [8] will have guessed, this particular $K$ was chosen simply as being about the easiest "quasipositive" knot with corresponding "braided surface" of genus 1 and Alexander polynomial 1.)

Figure 1B shows an isotopic surface, punctured by a line in $\mathbb{R}^3$; the boundary knot is a closed braid in the open solid torus complementary to the line, and is the



A B

Fig. 1

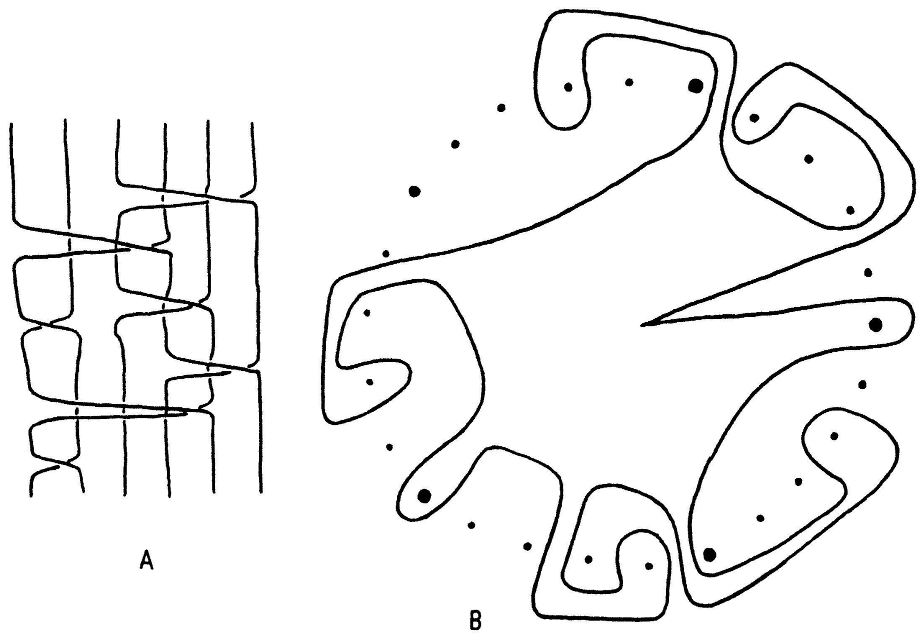



Fig. 2

same type of closed braid as in Figure 2A. (The surface is less explicit but still visible.) Call the pictured braid $\beta_6 \in B_6$. If we abbreviate $aba^{-1}$ by ${}^ab$, and $\sigma_k$ by $k$ $(k=1,\ldots,5)$, then we may write $\beta_6 = {}^{34}5 \cdot {}^{12}3 \cdot {}^{3}4 \cdot 1 \cdot {}^{4}5 \cdot {}^{123}4 \cdot 1$. (The raised dots are for clarity only.)

Fix integers $n \geq 2$, $m \geq 1$. Let $p: \mathbb{C}_w \to \mathbb{C}_z$ be the $n^{\text{th}}$ degree polynomial of §2; let $f(z, w) = p(w) - z^m$; and let $\Gamma_\varepsilon(m, n) = \{(z, w) \in \mathbb{C}^2 : f(z, w) = \varepsilon\}$. Then $\mathrm{pr}_1 \mid \Gamma_0(m, n) : \Gamma_0(m, n) \to \mathbb{C}_z$ is an $n$-sheeted branched covering branched over $Q_{n-1}^{1/m} = \{\xi : \xi^m \in Q_{n-1}\} = \{\exp[2\pi i(k-1)/m(n-1)] : k = 1, \ldots, m(n-1)\}$.

Let $\gamma: S^1 \to \mathbb{C}_z - Q_{n-1}^{1/m}$ be a loop with $\gamma(1) = 0$. Then in $S^1 \times \mathbb{C}_w$ the set $\{(z, w) : (\gamma(z), w) \in \Gamma_0(m, n)\}$ is a closed $n$-string braid, and it is easy to see which one it is: compose $\gamma$ with $z \to z^m$ to obtain $\gamma^m : S^1 \to \mathbb{C}_z - Q_{n-1}$, $\gamma^m(1) = 0$; then look at the element of $B_n$ corresponding to the class of $\gamma^m$ *via* the proposition of §2 and its corollary.

In particular, if $R$ is a closed region homeomorphic to a disk in $\mathbb{C}_z$, with $0 \in \partial R$, $Q_{n-1}^{1/m} \cap \partial R = \varnothing$, then we may take $\gamma$ to be a (counterclockwise) parametrization of $\partial R$; we find that $L = \{(z, w) : z \in \partial R, (z, w) \in \Gamma_0(m, n)\}$ is a closed braid in $\partial R \times \mathbb{C}_w$. Being compact, $L$ lies in some closed solid torus $\partial R \times D$, $D \subset \mathbb{C}_w$ a closed disk; finally, then, $L$ lies in the 3-sphere (with corners) $\partial(R \times D)$. In fact (by, say, the maximum principle), $L = \Gamma_0(m, n) \cap \partial(R \times D)$, that is, $L$ is the complete boundary of $\Gamma_0(m, n) \cap R \times D$. Also, it is easy to calculate the Euler characteristic of the surface $\Gamma_0(m, n) \cap R \times D$, for it is the branched cover of $R$ branched over $Q_{n-1}^{1/m} \cap R$.

THEOREM 1. *For every integer $n \geq 6$, there exists in the homology class $n[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z})$ a topologically locally-flatly embedded surface of genus strictly less than that of a nonsingular complex algebraic curve of degree n.*

*Proof.* In Figure 2B is sketched a simple closed curve in $\mathbb{C}_z \setminus Q_5^{1/5}$ which gives the braid $\beta_6$. (The 25th roots of 1 are indicated by dots, the 5th roots among them by larger dots; 0 is the basepoint.) Let $R$ be the region it bounds. Then (for a suitably large disk $D \subset \mathbb{C}_w$) the surface $\Gamma_0(5,6) \cap R \times D$ has Euler characteristic $-1$ and a connected boundary (of type $\hat{\beta}_6$), so it is of genus 1. (It is essentially the surface of Figure 1A, "pushed in.") Now, $\Gamma_0(5,6)$ is nonsingular in $\mathbb{C}^2$, but has a singular point at infinity in $\mathbb{C}P^2$; but for sufficiently small $\varepsilon \neq 0$, $\Gamma_\varepsilon(5,6)$ will be nonsingular when completed in $\mathbb{C}P^2$, while $\Gamma_\varepsilon(5,6) \cap R \times D$ will still be a punctured torus with boundary in $\partial(R \times D)$ of type $\hat{\beta}_6$. The homology class of the completion of $\Gamma_\varepsilon(5,6)$ is of course $6[\mathbb{C}P^1]$.

By Freedman's Theorem, the smooth surface $S' = \Gamma_\varepsilon(5,6) \cap R \times D$, of genus 1, shares its boundary with a topologically locally-flatly embedded disk $S$ in $R \times D$. Replace $S'$ by $S$ on the completion of $\Gamma_\varepsilon(5,6)$; the resulting surface is still in the homology class $6[\mathbb{C}P^1]$, is topologically locally flat, and has genus 1 smaller than the genus of $\Gamma_\varepsilon(5,6)$. The theorem is thus proved for $n = 6$.

For larger $n$, one may apply the same technique, starting with the braid $\beta_n = \beta_6\sigma_6 \cdots \sigma_{n-1} \in B_n$ and taking the appropriate simple closed curve in $\mathbb{C} \setminus Q_{n-1}^{1/5}$; for $\hat{\beta}_n$ is of the same knot type as $\hat{\beta}_6$ (and 5 replications of $Q_{n-1}$ still suffice to write the whole word properly). □

THEOREM 2. *For every pair $(m, n)$ of integers greater than or equal to 5, (except possibly $(5,5)$) there is a topologically locally-flatly embedded surface in the 4-disk with boundary a torus link $O\{m, n\}$ of type $(m, n)$ and genus strictly less than the (classical) genus of $O\{m, n\}$.*

*Proof.* Follow the proof of Theorem 1 up to the final paragraph.

Without loss of generality, we may assume $n \geq m \geq 5$ and $n \geq 6$. Then we may apply the same technique as above, starting with $\beta_n$ and taking the simple closed curve to lie in $\mathbb{C} \setminus Q_{n-1}^{1/m}$; again, $\hat{\beta}_n$ is the correct knot type, and extra replications of $Q_{n-1}$ do no harm. So $\Gamma_0(m, n)$ can have a handle surgered away inside $\mathbb{C}^2$, in the topologically locally flat sense. But for $r_1, r_2$ sufficiently large, the intersection of $\Gamma_0(m, n)$ with the boundary of the bidisk $\{(z, w) : |z| \leq r_1, |w| \leq r_2\}$ is a link of type $O\{m, n\}$ (in fact it is the closure of the $m^{\text{th}}$ power of the $n$-string braid $\sigma_1\sigma_2 \cdots \sigma_{n-1}$), and the intersection of $\Gamma_0(m, n)$ with the whole bidisk has genus $(m-1)(n-1)/2$, the classical genus of $O\{m, n\}$ (by direct calculation). □

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