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Restrictions of semistable bundles on projective varieties

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Introduction

Let $(X, \mathcal{O}_X(1))$ be an n-dimensional normal projective variety over a field k and let $\mathcal{E}$ be a torsion free semistable $\mathcal{O}_X$-module of rank r in the sense of [11], §1. In this paper we are concerned with the question if for a general hypersurface $H \in |\mathcal{O}_X(d)|$ the restriction of $\mathcal{E}$ to H remains semistable. This problem has been studied in several papers in the last years, see [1, 3, 4, 10, 11, 12, 14, 15].

For example a result of Maruyama [10] says that this is always true for arbitrary d if $r<n$. Without this rank condition such a restriction theorem obviously doesn't hold as the example $\Omega^1_{\mathbb{P}^n}$ shows, since

$$\Omega^1_{\mathbb{P}^n} \mid \mathbb{P}^{n-1} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \oplus \Omega^1_{\mathbb{P}^{n-1}}.$$

We remark that in characteristic 0 this is essentially the only example of rank n on $\mathbb{P}^n$, $n \geq 3$, see (1.15). But when $\Omega^1_{\mathbb{P}^n}$ is restricted to a quadric in $\mathbb{P}^n$ the restriction can be shown to remain semistable.

More generally Mehta and Ramanathan [12] could prove that for a given $\mathcal{E}$ there always exists a number d_0 such that the restriction of $\mathcal{E}$ to a general hypersurface of degree $d \geq d_0$ is semistable. But unfortunately the bound d_0 given there depends on $\mathcal{E}$ itself and not only on invariants of $\mathcal{E}$ such as the rank or the Hilbert polynomial.

In this paper we will give a bound for d_0 in characteristic 0. Namely we show that the restriction theorem holds if

$$\frac{\binom{n+d}{d} - d - 1}{d} > \deg(X) \max\left(\frac{r^2-1}{4}, 1\right),$$

see (1.2). We remark that the inequality is satisfied if d is sufficiently large. The proof of this result is based on a sharpened version of the Grauert–Mülich–Spindler–Maruyama theorem, see (1.4).

The paper is organized as follows. In the first part of §1 we state the main results and show how the restriction theorem can be deduced from the generalized Grauert–Mülich-theorem (1.4). In the second part of §1 we show (1.4) up to a technical proposition, the proof of which is contained in §2. We remark that throughout this paper we work over an algebraically closed field of characteristic 0.

I thank the referee for bringing the papers [17]–[19] to my attention. In particular the reduction of (1.4) to (1.10) is – at least for the case of manifolds – already contained in Hirschowitz [18]. (1.15) has been obtained by Hoppe [19] independently.

§1. The restriction theorem

(1.1) Let $(X, \mathcal{O}_X(1))$ be a n-dimensional normal projective variety over the algebraically closed field k of characteristic 0. We always assume that $\mathcal{O}_X(1)$ is very ample on X. For a nonzero torsion free $\mathcal{O}_X$-module $\mathcal{E}$ we have as usual the number $\mu(\mathcal{E}) := \deg(\mathcal{E})/rk(\mathcal{E})$, where $\deg(\mathcal{E})$ is essentially the intersection number of $c_1(\mathcal{O}_X(1))^{n-1}$ with $c_1(\mathcal{E})$, see [11] for the precise definition and elementary properties. Moreover by $\deg(X)$ we denote the degree of X if X is embedded into a projective space via the linear system $|\mathcal{O}_X(1)|$ or, equivalently, $\deg(X) = \deg(\mathcal{O}_X(1))$.

We recall that $\mathcal{E}$ is called semistable if there is no coherent $\mathcal{O}_X$-submodule $\mathcal{F} \neq 0$ in $\mathcal{E}$ satisfying $\mu(\mathcal{F}) > \mu(\mathcal{E})$. If $\mathcal{E}$ is an arbitrary torsion free coherent $\mathcal{O}_X$-module, then there exists always a canonical filtration

$$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_s = \mathcal{E},$$

called the HN-filtration, which is characterised by the following two properties:

(a) $\mathcal{E}_i/\mathcal{E}_{i-1} \neq 0$ is torsion free and semistable.

(b) $\mu(\mathcal{E}_1/\mathcal{E}_0) > \mu(\mathcal{E}_2/\mathcal{E}_1) > \cdots > \mu(\mathcal{E}_s/\mathcal{E}_{s-1})$,

see [11], (1.13). Since this filtration is canonical it is in particular invariant under automorphisms of $\mathcal{E}$ which we shall heavily use in (2.2) to establish the semistability of a very special sheaf.

We shall also need the notion of relative HN-filtration, see [4], §3 for the case of a curve: Let $f: X \to S$ be a proper flat map of normal k-schemes of finite type with a geometrically connected generic fibre and let $\mathcal{O}_X(1)$ be a line bundle on X which is very ample relative to f. For simplicity we assume that S is irreducible with generic point s_0. For any point $s \in S$ X_s denotes the fibre over s. Suppose that $\mathcal{E}$ is a torsion free sheaf on X and that $0 = \mathcal{E}_0 \subseteq \cdots \subseteq \mathcal{E}_s = \mathcal{E}$ is a filtration such that $\mathcal{E}_{\cdot} \otimes \mathcal{O}_{X_{s_0}}$ is the HN-filtration of the $\mathcal{O}_{X_{s_0}}$-module $\mathcal{E}_{\cdot} \otimes \mathcal{O}_{X_{s_0}}$. Then $\mathcal{E}_{\cdot}$ is called the relative HN-filtration of $\mathscr{E}$ with respect to f. The following facts are more or less well known and can be shown with the usual techniques, see [4], §3 and [10], [9]:

1) There always exists a relative HN-filtration.

2) If $\mathscr{E}_.$ and $\mathscr{E}'_.$ are two relative HN-filtrations, then there exists a neighbourhood U of s_0 in S such that $\mathscr{E}_.$ and $\mathscr{E}'_.$ coincide over U.

3) If $\mathscr{E}_.$ is a relative HN-filtration then for general $s \in S$ $\mathscr{E}_. \otimes \mathcal{O}_{X_s}$ is the (absolute) HN-filtration of $\mathscr{E} \otimes \mathcal{O}_{X_s}$.

Our main result is:

(1.2) THEOREM. *Let X be as in* (1.1) *and let $\mathscr{E}$ be a semistable torsion free $\mathcal{O}_X$-module or rank r and d, c integers, $1 \le c \le n-1$, such that*

$$\frac{\binom{n+d}{d} - cd - 1}{d} > \deg(X) \max\left(\frac{r^2-1}{4}, 1\right).$$

Then for a general complete intersection $T = H_1 \cap \cdots \cap H_c$, $H_i \in |\mathcal{O}_X(d)|$, the restriction $\mathscr{E} \mid Y := \mathscr{E} \otimes \mathcal{O}_Y$ is semistable on Y.

We remark that the inequality above holds if d is sufficiently large.

For the proof of (1.2) we shall need a generalized version of the theorem of Grauert–Mülich–Spindler–Maruyama, which we are going to explain now. We first introduce some notations.

(1.3) Let $(X, \mathcal{O}_X(1))$ be as in (1.1). We fix a linear subspace $V \subseteq H^0(X, \mathcal{O}_X(1))$ of dimension $n+1$ such that the induced morphism

$$\varphi : X \to \mathbb{P}(V) = \operatorname{Proj}(S(V))$$

is finite and surjective. We will regard $S^d(V)$ as a subspace of $H^0(X, \mathcal{O}_X(d))$ and so $|S^d(V)|$ is a linear subsystem of $|\mathcal{O}_X(d)|$.

(1.4) THEOREM. *Let $\mathscr{E}$ be semistable and torsion free on X, $Y = H_1 \cap \cdots \cap H_c$, $H_j \in |S^d V|$, a general complete intersection and let $0 = \mathscr{E}_0 \subseteq \mathscr{E}_1 \subseteq \cdots \subseteq \mathscr{E}_s = \mathscr{E} \otimes \mathcal{O}_Y$ be the HN-filtration of $\mathscr{E} \otimes \mathcal{O}_Y$. Then*

$$0 < \mu(\mathscr{E}_i/\mathscr{E}_{i-1}) - \mu(\mathscr{E}_{i+1}/\mathscr{E}_i) \le \frac{d^{c+1}\deg(X)}{\binom{n+d}{d} - cd - 1}.$$

In order to derive (1.2) from (1.4) we must establish two auxiliary lemmata. First we introduce some more notations.

(1.5) Let $(X, \mathcal{O}_X(1))$ and $V \subseteq H^0(X, \mathcal{O}_X(1))$ be as in (1.3). We define $\mathbb{H}_d := \mathbb{P}(S^d(V^\vee))$, where $V^\vee$ denotes the k-dual of V. The closed points of $\mathbb{H}_d$ can be naturally identified with the elements of the linear system $|S^d V|$. Moreover let $\mathbb{F}_{d,X}$ be the subvariety of all $(x, H) \in X \times \mathbb{H}_d$ such that $x \in H$. Then we have the following diagram:

$$\begin{array}{ccc} \mathbb{F}_{d,X} & \xrightarrow{q_X} & \mathbb{H}_d \\ {\scriptstyle p_X}\downarrow & & \\ X & & \end{array}$$

The fibres of p_X are projective spaces of dimension $\dim(\mathbb{H}_d) - 1$ and the fibre of q_X over $H \in \mathbb{H}_d$ is H considered as a closed subspace of X. In the case $X = \mathbb{P}^n$ we shall often write $\mathbb{F}_d$ or $\mathbb{F}$ instead of $\mathbb{F}_{d,X}$ and p resp. q instead of p_X resp. q_X. Moreover in this case $\mathbb{H}_d$ is the space of all hypersurfaces of degree d in $\mathbb{P}^n$.

For arbitrary X we have a natural cartesian diagram

$$\begin{array}{ccc} \mathbb{F}_{d,X} & \longrightarrow & \mathbb{F}_d \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[\varphi]{} & \mathbb{P}^n \end{array}$$

φ being the map of (1.3), and the spaces $\mathbb{H}_d$ associated to X and to $\mathbb{P}^n$ can be naturally identified and so we make no difference in our notation.

By $\mathbb{F}^c_{d,X}$ we shall denote the c-fold fibre product $\mathbb{F}_{d,X} \times_X \cdots \times_X \mathbb{F}_{d,X}$, and the projections are denoted by

$$p_X : \mathbb{F}^c_{d,X} \to X, \qquad q_X : \mathbb{F}^c_{d,X} \to \mathbb{H}^c_d$$

with the convention as above that we omit the subscript X in the case $X = \mathbb{P}^n$.

In the following we shall need the divisor class group of the spaces $\mathbb{F}^c_{d,X}$. If $\mathscr{E}$ is a module on $\mathbb{F}^c_{d,X}$, we set

$$\mathscr{E}(a_1, \ldots, a_c) := \mathscr{E} \otimes \rho_1^*(\mathcal{O}_{\mathbb{H}_d}(a_1)) \otimes \cdots \otimes \rho_c^*(\mathcal{O}_{\mathbb{H}_d}(a_c)).$$

Here $\rho_j : \mathbb{F}^c_{d,X} \to \mathbb{H}_d$ denotes the composition of q_X with the j-th projection of $\mathbb{H}^c_d$.

(1.6) LEMMA. *Suppose $\mathscr{E}$ is a reflexive module on $\mathbb{F}^c_{d,X}$ of rank 1. Then there exists a reflexive $\mathcal{O}_X$-module $\mathscr{L}$ of rank 1 and $a_1, \ldots, a_c \in \mathbb{Z}$ such that $\mathscr{E} \cong p_X^*(\mathscr{L})(a_1, \ldots, a_c)$.*

Proof. Since the fibre of $p_X : \mathbb{F}^c_{d,X} \to X$ is a k-fold product of projective spaces the only invertible sheaves on these fibres are the restrictions of the sheaves $\mathcal{O}_{\mathbb{F}^c_{d,X}}(a_1, \ldots, a_c)$, $a_1, \ldots, a_c \in \mathbb{Z}$. For a generic $x \in X_{reg}$ the restriction of $\mathscr{E}$ to $p_X^{-1}(x)$ is therefore isomorphic to the restriction of $\mathcal{O}_{\mathbb{F}^c_{d,X}}(a_1, \ldots, a_c)$ for some $a_1, \ldots, a_c \in \mathbb{Z}$. Then $\mathscr{E}' = \mathscr{E}(-a_1, \ldots, -a_c)$ is trivial on the generic fibre of p_X and hence $\mathscr{L} := (p_X)_*(\mathscr{E}')$ is a reflexive rank 1-module on X, and it is easily seen, that the canonical mapping $p_X^*(\mathscr{L}) \to \mathscr{E}'$ is an isomorphism, proving the lemma.

(1.7) LEMMA. *Let $\mathscr{E}$ be a torsion free coherent module on $\mathbb{F}^c_{d,X}$ of rank r. Then for a general $H \in \mathbb{H}^c_d$*

$$\mu(\mathscr{E} \mid q_X^{-1}(H)) = d^c\alpha/r,$$

where α is an integer.

Proof. It suffices to show that the number $\mu(\det(\mathscr{E}) \mid q_X^{-1}(H))$ is in $d^c\mathbb{Z}$. Hence we may assume $rk(\mathscr{E}) = 1$ and also that $\mathscr{E}$ is reflexive. By (1.6) $\mathscr{E}$ is isomorphic to $p_X^*(\mathscr{L})(a_1, \ldots, a_c)$ for some reflexive rank 1-module $\mathscr{L}$ on X. Hence $\mu(\mathscr{E} \mid q_X^{-1}(H)) = \mu(\mathscr{L} \mid H_1 \cap \cdots \cap H_c) = \mu(\mathscr{L})d^c$, if $H = (H_1, \ldots, H_c)$, see [11] (1.5). This proves (1.7).

With the help of (1.4) and (1.7) it is now possible to derive (1.2).

Proof of (1.2). Suppose that $\mathscr{E} \otimes \mathcal{O}_Y$ is not semistable. Let $0 = \mathscr{E}_0 \subseteq \cdots \subseteq \mathscr{E}_s = p_X^*(\mathscr{E})$ be the relative HN-filtration of $p_X^*(\mathscr{E})$ on $\mathbb{F}^c_{d,X}$ with respect to the projection $\mathbb{F}^c_{d,X} \xrightarrow{q_X} \mathbb{H}^c_d$. Then for a general $H = (H_1, \ldots, H_c) \in \mathbb{H}^c_d$ the restriction of $\mathscr{E}_\cdot$ to the fibre $q_X^{-1}(H)$ is the HN-filtration of $p_X^*(\mathscr{E}) \mid q_X^{-1}(H) = \mathscr{E} \mid H_1 \cap \cdots \cap H_c$. By (1.6) $\mu((\mathscr{E}_i/\mathscr{E}_{i-1}) \mid q_X^{-1}(H)) = d^c\alpha_i/r_i$, $\alpha_i \in \mathbb{Z}$, $r_i = rk(\mathscr{E}_i/\mathscr{E}_{i-1})$. From (1.4) we obtain

$$0 < d^c\alpha_i/r_i - d^c\alpha_{i+1}/r_{i+1} \le \frac{d^{c+1}\deg(X)}{\binom{n+d}{d} - cd - 1}.$$

But $r_i + r_{i+1} \le r$, and a simple calculation shows that $d^c\alpha_i/r_i - d^c\alpha_{i+1}/r_{i+1} \neq 0$ implies that this difference is at least $d^c \min(1, 4/(r^2-1))$. Hence we would obtain

$$\min(1, 4/(r^2-1)) \le \frac{d \deg(X)}{\binom{n+d}{d} - cd - 1}$$

contradicting our assumption.

Our next task is to prove (1.4). We have to make a few preparations.

(1.8) NOTATION. If $\mathscr{E}$ is a torsion free $\mathcal{O}_X$-module we define

$\mu_-(\mathscr{E}) := \min\{\mu(Q) : Q$ is a nonzero torsion free quotient of $\mathscr{E}\}$.

If $0 = \mathscr{E}_0 \subseteq \mathscr{E}_1 \subseteq \cdots \subseteq \mathscr{E}_s = \mathscr{E}$ is the HN-filtration of $\mathscr{E}$ then $\mu_-(\mathscr{E}) = \mu(\mathscr{E}_s/\mathscr{E}_{s-1})$. This can be seen as follows: If Q is a quotient of $\mathscr{E}$ with the property $\mu_-(\mathscr{E}) = \mu(Q)$ then Q must be semistable since otherwise Q would have a torsion free quotient $\tilde{Q}$ with $\mu(\tilde{Q}) < \mu(Q)$ contradicting $\mu_-(\mathscr{E}) = \mu(Q)$. Moreover $\mu(Q) \le \mu(\mathscr{E}_s/\mathscr{E}_{s-1})$ and hence $\mu(Q) < \mu(\mathscr{E}_i/\mathscr{E}_{i-1})$, $1 \le i < s$. By the reasoning of [4], (2.2), $\operatorname{Hom}(\mathscr{E}_{s-1}, Q) = 0$ and hence Q is a quotient of $\mathscr{E}_s/\mathscr{E}_{s-1}$, and since $\mathscr{E}_s/\mathscr{E}_{s-1}$ is semistable this yields $\mu(Q) = \mu(\mathscr{E}_s/\mathscr{E}_{s-1})$.

(1.9) LEMMA. *Let $f : (Y, \mathcal{O}_Y(1)) \to (X, \mathcal{O}_X(1))$ be a finite surjective mapping of normal projective k-schemes of degree e such that $\mathcal{O}_Y(1) = f^*(\mathcal{O}_X(1))$. Put $\mathscr{E}_Y := f^*(\mathscr{E})/\mathrm{Torsion}$. Then $\mu_-(\mathscr{E}_Y) = e \cdot \mu_-(\mathscr{E})$.*

Proof. Let $\mathscr{E}_{\cdot}$ be the HN-filtration as above and let $\mathscr{E}'_{\cdot}$ be the filtration of $\mathscr{E}_Y$ given by

$\mathscr{E}'_i := \ker(\mathscr{E}_Y \to (\mathscr{E}/\mathscr{E}_i)_Y)$.

Since $\mathscr{E}_i/\mathscr{E}_{i-1}$ is semistable so is $(\mathscr{E}_i/\mathscr{E}_{i-1})_Y$ by [11], (1.17). $\mathscr{E}'_i/\mathscr{E}'_{i-1}$ and $(\mathscr{E}_i/\mathscr{E}_{i-1})_Y$ are only different in codimension ≥ 2 and so $\mathscr{E}'_i/\mathscr{E}'_{i-1}$ is semistable too. Moreover $\mu(\mathscr{E}'_i/\mathscr{E}'_{i-1}) = e \cdot \mu(\mathscr{E}_i/\mathscr{E}_{i-1})$ and hence we can conclude that $\mathscr{E}'_{\cdot}$ is the HN-filtration of $\mathscr{E}_Y$. From this the assertion immediately follows.

The key step in the proof of (1.4) is

(1.10) PROPOSITION. *Let the notations be as in* (1.5). *Then for a general*

$H \in \mathbb{H}_d^c$

$$\mu_-(\mathcal{T}_{\mathbb{F}^c_{d,X}/X} \mid q_X^{-1}(H)) \geq \frac{-d^{c+1}\deg(X)}{\binom{n+d}{d} - cd - 1}$$

where $\mathcal{T}\cdots$ *denotes the relative tangent sheaf.*

The proof of this proposition will be given in §2. Modulo this proposition we are now able to give the proof of (1.4).

Proof of (1.4). Suppose $\mathscr{E}$ is as in (1.4) and let $0 = \mathscr{E}_0 \subseteq \cdots \subseteq \mathscr{E}_s = p_X^*(\mathscr{E})$ be the relative HN-filtration of $p_X^*(\mathscr{E})$ on $\mathbb{F}^c_{d,X}$ with respect to the projection q_X. Then just as in the proof of [9], (4.6), there must be a nonzero homomorphism from $\mathcal{T}_{\mathbb{F}^c_{d,X}/X}$ to $\mathcal{H}om(\mathscr{E}_i/\mathscr{E}_{i-1}, \mathscr{E}_j/\mathscr{E}_{j-1})$ for some $0<i<j\leq s$, since otherwise one of the sheaves $\mathscr{E}_j$, $j \neq 0, s$, would be isomorphic to $p^*(\mathscr{F}_j)$ for some subsheaf $\mathscr{F}_j$ of $\mathscr{E}$ contradicting the semistability of $\mathscr{E}$, see also (1.11). Restricting this homomorphism to a general fibre of q_X we get a nonzero homomorphism from $\mathcal{T}_{\mathbb{F}^c_{d,X}/X} \mid q^{-1}(H)$ into $\mathcal{H}om((\mathscr{E}_i/\mathscr{E}_{i-1}) \mid q^{-1}(H), (\mathscr{E}_j/\mathscr{E}_{j-1}) \mid q^{-1}(H))$. Since the latter sheaf is semistable by [11], (2.5), see also [12], (6.6), we obtain

$$\mu_-(\mathcal{T}_{\mathbb{F}^c_{d,X}/X} \mid q^{-1}(H)) \leq \mu((\mathscr{E}_j/\mathscr{E}_{j-1}) \mid q^{-1}(H)) - \mu((\mathscr{E}_i/\mathscr{E}_{i-1}) \mid q^{-1}(H)).$$

Now (1.10) gives us the desired estimate for the difference on the right-hand side, q.e.d.

In the proof above we have used a descent lemma, which was shown in [4]. Since in loc. cit. this lemma is only formulated for vector bundles and since for applications it is important to have it also for arbitrary coherent sheaves (see e.g. [3], (3.2) for such a result) we take here the opportunity to prove more generally

(1.11) PROPOSITION. *Let X, Y be* $\mathbb{Q}$*-schemes and* $f: X \to Y$ *a surjective smooth mapping of finite presentation with geometrically connected fibres. If* $\mathscr{E}$ *is a coherent sheaf on Y and* $\bar{\mathscr{F}} \subseteq f^*(\mathscr{E})$ *is a coherent subsheaf such that*

$$\mathrm{Hom}_{\mathcal{O}_X}(\bar{\mathscr{F}}, \mathfrak{G} \otimes \Omega^1_{X/Y}) = 0, \qquad \mathfrak{G} := f^*(\mathscr{E})/\bar{\mathscr{F}},$$

then there exists a unique coherent subsheaf $\mathscr{F}$ *of* $\mathscr{E}$ *with* $\bar{\mathscr{F}} = f^*(\mathscr{F})$.

Proof. The problem is local in Y. Therefore we may assume that $S=\operatorname{Spec}(A)$ is affine. With a standard argument the assertion is easily reduced to the case that A is a $\mathbb{Q}$-algebra of finite type. By the descent theory of SGA 1, Exp. V, it suffices to show that $p_1^*(\bar{\mathcal{F}})=p_2^*(\bar{\mathcal{F}})$, where p_1 and p_2 denote the projections $X\times_Y X\rightrightarrows X$. Since this condition is invariant under flat base change $Y'\to Y$ we may and we shall assume from now on that Y is an algebraic $\mathbb{C}$-scheme. Consider the canonical relative connection

$$f^*(\mathcal{E})\xrightarrow{\nabla} f^*(\mathcal{E})\otimes\Omega^1_{X|Y}$$

where $\nabla(e\otimes g)=e\otimes dg$ if e is a section of $\mathcal{E}$. The composition

$$\bar{\mathcal{F}}\to f^*(\mathcal{E})\xrightarrow{\nabla} f^*(\mathcal{E})\otimes\Omega^1_{X|Y}\longrightarrow \mathfrak{G}\otimes\Omega^1_{X|Y}$$

is easily seen to be $\mathcal{O}_X$-linear and hence vanishes by assumption. Therefore the connection ∇ induces a connection $\nabla':\bar{\mathcal{F}}\to\bar{\mathcal{F}}\otimes\Omega^1_{X|S}$ which is automatically integrable since ∇ is so.

Now consider the associated analytic situation. It follows from [2], (2.23), that for each point $x\in X^{an}$ there exists a neighbourhood $\mathcal{U}$ such that $\bar{\mathcal{F}}^{an}\mid\mathcal{U}$ is isomorphic to $(f^{an}\mid\mathcal{U})^*(\mathcal{H})$, $\mathcal{H}$ being a coherent subsheaf of $\mathcal{E}^{an}$ in a neighbourhood of $f(x)$. In particular we obtain $(p_1^{an})^*(\bar{\mathcal{F}}^{an})=(p_2^{an})^*(\bar{\mathcal{F}}^{an})$ in a neighbourhood of the diagonal $X^{an}\subseteq(Xx_YX)^{an}$. Since f has connected fibres and since both sheaves are locally induced by subsheaves of $\mathcal{E}$ they must be equal everywhere. It follows, that $p_1^*(\bar{\mathcal{F}})=p_2^*(\bar{\mathcal{F}})$, q.e.d.

Applying this proposition to the morphism $p:\mathbb{F}_1\to\mathbb{P}^n$ of (1.5) we obtain:

(1.12) COROLLARY. *Let $\mathcal{E}$ be a coherent $\mathcal{O}_{\mathbb{P}^n}$-module and $\bar{\mathcal{F}}\subseteq p^*(\mathcal{E})$ a $\mathcal{O}_{\mathbb{F}_1}$-submodule with a torsion free quotient $\mathfrak{G}=p^*(\mathcal{E})/\bar{\mathcal{F}}$. For $H\in\mathbb{H}_1$ let $\bar{\mathcal{F}}_H$ resp. $\mathfrak{G}_H$ denote the restriction of $\bar{\mathcal{F}}$ resp. $\mathfrak{G}$ to the fibre $q^{-1}(H)$. If for a general hyperplane $H\in\mathbb{H}_1$*

$$\operatorname{Hom}(\bar{\mathcal{F}}_H,\mathcal{T}_H(-1)\otimes\mathfrak{G}_H)=0$$

then $\bar{\mathcal{F}}=p^*(\mathcal{F})$ with a coherent $\mathcal{O}_X$-submodule $\mathcal{F}\subseteq\mathcal{E}$.

Proof. In this case we have $\mathcal{T}_{\mathbb{F}_1/\mathbb{P}^n}\mid q^{-1}(H)\cong\Omega^1_H(1)$, see e.g. the example following (2.4).

(1.13) COROLLARY ([3], Prop. (3.2)). *Let $\mathscr{E}$ be a torsion free sheaf of $\mathbb{P}^n$ such that $h^0(\mathscr{E})=0$, $h^0(\mathscr{E}\mid H)\neq 0$ and $h^0(\mathscr{E}\mid H(-1))=0$ for a general hyperplane H of $\mathbb{P}^n$. Then there exists a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_H & \longrightarrow & \mathcal{O}_H\oplus\Omega^1_H(1) & \longrightarrow & \Omega^1_H(1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & \mathcal{O}_H & \longrightarrow & \mathscr{E}_H & \longrightarrow & \mathfrak{G} & \longrightarrow & 0 \end{array}$$

such that $\psi\neq 0$.

This follows easily by applying (1.12) to a subsheaf of the form $\mathcal{O}_{\mathbb{F}_1}(0,t)$ of $p^*(\mathscr{E})$ which exists for some t by assumption.

We end this section with a few remarks concerning (1.3) and (1.4).

(1.14) *Remarks*

(1) The bound for d given in (1.2) is far from being optimal. E.g. the above mentioned result of Maruyama says that the restriction of $\mathscr{E}$ to a hyperplane (i.e. $d=1$) is semistable if $r<n$, whereas our result only gives the bound $\max(r^2-1,4)<4(n-1)/\deg(X)$.

(2) It would be interesting to have similar restriction theorems in characteristic $p>0$, since this would settle the boundedness problem, see [8], Problem 5. It even seems to be unknown if for a semistable sheaf on $\mathbb{P}^2$ of rank 2 the restriction to a general quadric remains semistable.

(3) (1.4) implies in particular the Grauert–Mülich-theorems of [11], (1.14) and (3.1).

If we apply (1.2) to the special case $d=2$, $c=1$, we get that the restriction of a semistable sheaf on $\mathbb{P}^n$ to a generic quadric remains semistable if $r<\sqrt{n^2+3n-3}$. In the case $r=n$ we can show:

(1.15) PROPOSITION. *Suppose $\mathscr{E}$ is a semistable reflexive sheaf on $\mathbb{P}^n$ of rank $n\geq 3$ such that the restriction of $\mathscr{E}$ to a general hyperplane is not semistable. Then $\mathscr{E}$ is isomorphic to a twist of $\Omega^1_{\mathbb{P}^n}$ or $\mathcal{T}_{\mathbb{P}^n}$.*

Proof. The method of proof of [10], (3.1) shows, that on the general hyperplane H the sheaf $\mathscr{E}_H:=\mathscr{E}\mid H$ must have HN-filtration $\mathcal{O}\subsetneqq\mathscr{F}\subsetneqq\mathscr{E}_H$ such that $rk(\mathscr{F})=1$ or $rk(\mathfrak{G})=1$, $\mathfrak{G}:=\mathscr{E}_H/\mathscr{F}$. Without loss of generality we may assume $rk(\mathscr{F})=1$, $c_1(\mathscr{F})=0$. By (1.13) there must be a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_H & \longrightarrow & \mathcal{O}_H\oplus\Omega^1_H(1) & \longrightarrow & \Omega^1_H(1) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \psi\neq 0} & & \\ 0 & \longrightarrow & \mathcal{O}_H & \longrightarrow & \mathscr{E}_H & \longrightarrow & \mathfrak{G} & \longrightarrow & 0 \end{array}$$

Since $\Omega^1_H(1)$ is stable and $\mu(\mathfrak{G})$ being nonzero is at most $-1/(n-1)$ it follows that ψ must be an isomorphism. Hence $\mathscr{E}_H \cong \mathcal{O}_H \oplus \Omega^1_H(1)$. Now $\mathrm{Ext}^i_{\mathcal{O}_P}(\mathscr{E}, \mathcal{O}_{P^n}(-j)) = 0$ if $i<n$ and $j \gg 0$. Using the obvious Ext-sequences and the vanishing of $\mathrm{Ext}^1_{\mathcal{O}_P}(\mathscr{E}, \mathcal{O}_H(j)) \cong H^1(\mathscr{E}^\vee_H(j))$, $j \leq -1$, we obtain that $\mathrm{Hom}(\mathscr{E}, \mathcal{O}_{P^n}) \to \mathrm{Hom}(\mathscr{E}, \mathcal{O}_H)$ is bijective and hence $\hom(\mathscr{E}, \mathcal{O}_{P^n}) = n+1$. Let $s_0, \ldots, s_n$ be a basis for $\mathrm{Hom}(\mathscr{E}, \mathcal{O}_{P^n})$ and let $\mathscr{E} \xrightarrow{\psi} \mathcal{O}^{n+1}_{P^n}$ be the corresponding map. Since the restriction of ψ to a general hyperplane establishes $\mathscr{E}_H$ as a subbundle of $\mathcal{O}^{n+1}_H$ the same must be true for $\mathscr{E}$ outside a finite set of points. The cokernel of ψ has rank 1 and must be torsion free and hence is isomorphic to $\mathscr{I}_W(1)$ for some ideal $\mathscr{I}_W$ of a 0-dimensional subscheme W of $\mathbb{P}^n$. The morphism $\mathcal{O}^{n+1}_{P^n} \xrightarrow{\pi} \mathscr{I}_W(1)$ is given by linear polynomials $l_0, \ldots, l_n$. If the l_i would be linearly dependent we would have a decomposition $\mathscr{E} \cong \mathscr{T} \oplus \mathcal{O}_{P^n}$ contradicting the semistability of $\mathscr{E}$. Hence the l_i are linearly independent and so $W = \emptyset$ and $\mathscr{E}$ must be isomorphic to $\Omega^1_{P^n}(1)$, q.e.d.

Remark. In the case $n=3$ (1.15) is due to Schneider [14], see also [3], (3.4), for another proof. Recently Hoppe [19] has obtained (1.15) independently.

§2. Proof of Proposition (1.10)

Let V be a $(n+1)$-dimensional vector space over the field k. The group $SL(V)$ acts on $\mathbb{P}^n = \mathbb{P}(V)$ and also on the sheaves $\mathcal{O}_{P^n}(i)$. Moreover there are canonical $SL(V)$-equivariant mappings

$$H^0(\mathcal{O}_{P^n}(r)) \otimes \mathcal{O}_{P^n}(s) \xrightarrow{\delta} H^0(\mathcal{O}_{P^n}(r-1)) \otimes \mathcal{O}_{P^n}(s+1)$$

given by $\delta(v_1 \vee \cdots \vee v_r \otimes \xi) = \sum_i v_1 \vee \cdots \vee \hat{v}_i \vee \cdots \vee v_r \otimes v_i \cdot \xi$ if $v_i \in H^0(\mathcal{O}_{P^n}(1)) \cong V$ and ξ is a section in $\mathcal{O}_{P^n}(s)$. By composition we obtain equivariant mappings

$$\delta^i : H^0(\mathcal{O}_{P^n}(d)) \otimes \mathcal{O}_{P^n} \to H^0(\mathcal{O}_{P^n}(d-i)) \otimes_{P^n}(i).$$

For example $\delta^d : H^0(\mathcal{O}_{P^n}(d)) \otimes \mathcal{O}_{P^n} \to \mathcal{O}_{P^n}(d)$ is up to a constant factor just the multiplication mapping.

(2.1) PROPOSITION. *The only $SL(V)$-invariant submodules of $H^0(\mathcal{O}_{P^n}(d)) \otimes \mathcal{O}_{P^n}$ are the* $\ker(\delta^i)$.

Proof. In a first step we show that the $SL(V)$-modules $S^j(\Omega^1_{P^n})$ are irreducible, i.e. they have no $SL(V)$-invariant submodules other than 0 and $S^j(\Omega^1_{P^n})$. This is probably known, but in lack for a reference we give the simple proof: If $\mathfrak{G} \subseteq S^i(\Omega^1_{\mathbb{P}^n})$ is a $SL(V)$-invariant submodule, then $\mathfrak{G}$ is necessarily a subbundle, since $SL(V)$ acts transitively on $\mathbb{P}^n$. We look at the fibre of $\mathfrak{G}$ in a closed point $x \in \mathbb{P}^n$. In order to do this we choose a splitting $V = W \oplus k$, where W is the 1-codimensional subspace of V corresponding to x. Then x has an affine neighbourhood isomorphic to $W^\vee$, and so $\mathfrak{G}/m_x\mathfrak{G}$ is a $SL(W)$-invariant subspace of $S^i(\Omega^1_{\mathbb{P}^n}/m_x\Omega^1_{\mathbb{P}^n}) \cong S^iW$, and since S^iW is known to be an irreducible representation of $SL(W)$ we obtain that $\mathfrak{G}/m_x\mathfrak{G} = 0$ or $\mathfrak{G}/m_x\mathfrak{G} = S^i(\Omega^1_{\mathbb{P}^n}/m_x\Omega^1_{\mathbb{P}^n})$. Since $SL(V)$ acts transitively this implies $\mathfrak{G} = 0$ or $\mathfrak{G} = S^i(\Omega^1_{\mathbb{P}^n})$.

Now in the second step in the proof of (2.1) we show that $\ker(\delta^i)/\ker(\delta^{i-1}) \cong S^{d-i-1}(\Omega^1_{\mathbb{P}^n})(d)$. For this we consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(\delta^{i-1}) & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \otimes \mathcal{O}_{\mathbb{P}^n} & \xrightarrow{\delta^{i-1}} & H^0(\mathcal{O}_{\mathbb{P}^n}(d-i+1)) \otimes \mathcal{O}_{\mathbb{P}^n}(i-1) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle\delta} & & \\
0 & \longrightarrow & \ker(\delta^{i}) & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \otimes \mathcal{O}_{\mathbb{P}^n} & \xrightarrow{\delta^{i}} & H^0(\mathcal{O}_{\mathbb{P}^n}(d-i)) \otimes \mathcal{O}_{\mathbb{P}^n}(i) & \longrightarrow & 0
\end{array}$$

It follows that $\ker(\delta^i)/\ker(\delta^{i-1})$ is isomorphic to $\ker(\delta)$. The canonical inclusion $\Omega^1_{\mathbb{P}^n}(1) \subseteq V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^n}$ induces an inclusion

$$S^{d-i+1}(\Omega^1_{\mathbb{P}^n}(1))(i-1) \overset{\alpha}{\hookrightarrow} S^{d-i+1}(V) \otimes \mathcal{O}_{\mathbb{P}^n}(i-1)$$
$$\cong H^0(\mathcal{O}_{\mathbb{P}^n}(d-i+1) \otimes \mathcal{O}_{\mathbb{P}^n}(i-1))$$

and it is not difficult to see that $\mathrm{Im}(\alpha) = \ker(\delta)$, proving our second step.

In the third step we prove the proposition by induction on d. Suppose that $\mathfrak{G} \neq 0$ is an arbitrary $SL(V)$-invariant submodule of $H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \otimes \mathcal{O}_{\mathbb{P}^n}$. Then $\bar{\mathfrak{G}} = \mathfrak{G}/\mathfrak{G} \cap \ker(\delta)$ is a $SL(V)$-invariant submodule of $H^0(\mathcal{O}_{\mathbb{P}^n}(d-1)) \otimes \mathcal{O}_{\mathbb{P}^n}(1)$ and by induction hypothesis $\bar{\mathfrak{G}}$ is isomorphic to $\ker(\delta^i) \subseteq H^0(\mathcal{O}_{\mathbb{P}^n}(d-1)) \otimes \mathcal{O}_{\mathbb{P}^n}(1)$ for some i. Since $\ker(\delta) = S^d(\Omega^1_{\mathbb{P}^n})(d)$ is irreducible else $\mathfrak{G} \cap \ker(\delta) = \ker(\delta)$ or $\mathfrak{G} \cap \ker(\delta) = 0$. In the first case

$$\mathfrak{G} = \ker(\delta^{i+1}) \subseteq H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \otimes \mathcal{O}_{\mathbb{P}^n}$$

and we are done.

In the case $\mathfrak{G} \cap \ker(\delta) = 0$ we may replace $\mathfrak{G}$ by $\mathfrak{G} \cap \ker(\delta^2)$. Then $\mathfrak{G} \cong \bar{\mathfrak{G}} \cong S^{d-1}(\Omega^1_{\mathbb{P}^n})(d)$, and we shall show that this is impossible. Indeed, that $S^{d-1}(\Omega^1_{\mathbb{P}^n})(d)$ is a submodule of $H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \otimes \mathcal{O}_{\mathbb{P}^n}$ would mean dually that $S^{d-1}(\mathcal{T}_{\mathbb{P}^n})(-d)$ is a quotient of a trivial bundle and hence generated by global sections. But it is well known and follows from the exact sequence

$$0 \to S^{d-2}(V^\vee)\otimes\mathcal{O}_{\mathbb{P}^n}(-2)\to S^{d-1}(V^\vee)\otimes_{\mathbb{P}^n}(-1)\to S^{d-1}(\mathcal{T}_{\mathbb{P}^n})(-d)\to 0$$

that this sheaf has no sections at all.

(2.2) COROLLARY. *Let $\mathcal{F}$ be the kernel of the mapping $\delta^d: H^0(\mathcal{O}_{\mathbb{P}^n}(d))\otimes \mathcal{O}_{\mathbb{P}^n}\to\mathcal{O}_{\mathbb{P}^n}(d)$. Then $\mathcal{F}$ is semistable on $\mathbb{P}^n$.*

Proof. Let $0=\mathcal{F}_0\subseteq\cdots\subseteq\mathcal{F}_r=\mathcal{F}$ be the HN-filtration of $\mathcal{F}$. Since this is a canonical filtration it must be invariant under the action of $SL(V)$. This means that, if $\mathcal{F}$ is not semistable, there would be an $SL(V)$-invariant subsheaf $\mathfrak{G}\subseteq\mathcal{F}$ such that $\mu(\mathfrak{G})>\mu(\mathcal{F})$. By (2.1) $\mathfrak{G}$ is one of the modules $\ker(\delta^i)$, $1\le i\le d-1$. But a simple calculation shows that $\mu(\ker(\delta^i))<\mu(\mathcal{F})$ which would give a contradiction.

From now on we take over the notations of (1.1), (1.3), (1.5). Since $q^{-1}(H)\subseteq\mathbb{P}^n$ resp. $p^{-1}(x)\subseteq\mathbb{H}_d=\mathbb{P}(S^d(V^\vee))$ are subspaces of degree d resp. 1, the ideal sheaf of $\mathbb{F}_d$ in the structure sheaf of $\mathbb{P}^n\times\mathbb{H}_d$ is isomorphic to $\rho_1^*(\mathcal{O}_{\mathbb{P}^n}(-d))\otimes\rho_2^*(\mathcal{O}_{\mathbb{H}_d}(-1))$, the ρ_i being the projections. Hence we obtain two exact sequences ($\mathbb{F}:=\mathbb{F}_d$):

(1) $0\to\mathcal{T}_{\mathbb{F}}\to p^*(\mathcal{T}_{\mathbb{P}^n})\otimes q^*(\mathcal{T}_{\mathbb{H}_d})\to p^*(\mathcal{O}_{\mathbb{P}^n}(d))(1)\to 0.$

(2) $0\to\mathcal{T}_{\mathbb{F}/\mathbb{P}^n}\to q^*(\mathcal{T}_{\mathbb{H}_d})\to p^*(\mathcal{O}_{\mathbb{P}^n}(d))(1)\to 0.$

(2.3) LEMMA. *$p_*(\mathcal{T}_{\mathbb{F}/\mathbb{P}^n}(-1))$ is isomorphic to the kernel $\mathcal{F}$ of the multiplication mapping*

$$\delta^d: H^0(\mathcal{O}_{\mathbb{P}^n}(d))\otimes\mathcal{O}_{\mathbb{P}^n}\to\mathcal{O}_{\mathbb{P}^n}(d).$$

Proof. By the exact sequence (2) $p_*(\mathcal{T}_{\mathbb{F}/\mathbb{P}^n}(-1))$ is isomorphic to the kernel of the mapping

$$p_*q^*(\mathcal{T}_{\mathbb{H}_d}(-1))\xhookrightarrow{\alpha} p_*(p^*(\mathcal{O}_{\mathbb{P}^n}(d)))=\mathcal{O}_{\mathbb{P}^n}(d).$$

Moreover on $\mathbb{H}_d$ one has an exact sequence

$$0\to\mathcal{O}_{\mathbb{H}_d}(-1)\to S^dV\otimes_{\mathbb{C}}\mathcal{O}_{\mathbb{H}_d}\to\mathcal{T}_{\mathbb{H}_d}(-1)\to 0$$

and applying p_*q^* we get an isomorphism between $S^dV\otimes\mathcal{O}_{\mathbb{P}^n}$ and

$p_*q^*(\mathcal{T}_{\mathbb{H}_d}(-1))$. That via this isomorphism the mapping α can be identified with the multiplication mapping follows easily from the fact that α is $SL(V)$-invariant.

(2.4) PROPOSITION. *Let $\mathcal{F}$ be as above. Then there is an exact sequence on* $\mathbb{F}:=\mathbb{F}_d$

$$0 \to \mathcal{O}_{\mathbb{F}}(-1) \to p^*(\mathcal{F}) \to \mathcal{T}_{\mathbb{F}/\mathbb{P}^n}(-1) \to 0,$$

whose restriction to $q^{-1}(H)$, $H \in \mathbb{H}_d$, is splitting.

Proof. The canonical homomorphism of $\mathcal{O}_{\mathbb{F}}$-modules $p^*p_*(\mathcal{T}_{\mathbb{F}/\mathbb{P}^n}(-1) \xrightarrow{\beta} \mathcal{T}_{\mathbb{F}/\mathbb{P}^n}(-1)$ is surjective as is easily seen by restricting this mapping to the fibres of p. By the preceeding lemma the first sheaf is isomorphic to $p^*(\mathcal{F})$. Moreover $rk(\ker \beta) = 1$, and since $\det(p^*(\mathcal{F})) = p^*(\mathcal{O}_{\mathbb{P}^n}(-d))$ and $\det(\mathcal{T}_{\mathbb{F}/\mathbb{P}^n}(-1)) = p^*(\mathcal{O}_{\mathbb{P}^n}(-d))(+1)$ the kernel of β must be $\mathcal{O}_{\mathbb{F}}(-1)$.

Example. Suppose $X = \mathbb{P}^n$ and $\mathbb{F} = \mathbb{F}_1$, i.e. $d = 1$. Then $\mathcal{F}$ is simply $\Omega^1_{\mathbb{P}^n}(1)$ and $\mathcal{T}_{\mathbb{F}/\mathbb{P}^n} \mid H$ being the quotient $\Omega^1_{\mathbb{P}^n}(1) \otimes \mathcal{O}_H/\mathcal{O}_H$ must be isomorphic to $\Omega^1_H(1)$.

(2.5) COROLLARY. *Let $\mathbb{F}$ be as above. Then there is an exact sequence on* $\mathbb{F}^c := \mathbb{F}^c_d$

$$0 \to \mathcal{O}^c_{\mathbb{F}^c} \to \bigoplus_{j=1}^{c} p^*(\mathcal{F}) \otimes \mathcal{L}_j \to \mathcal{T}_{\mathbb{F}^c/\mathbb{P}^n} \to 0$$

with $\mathcal{L}_j := \mathcal{O}_{\mathbb{F}^c}(0, \ldots, -1, \ldots, 0)$, the -1 being at the jth place.

Proof. Since $\mathbb{F}^c$ is the c-fold fibre product of $\mathbb{F}_d$ over $\mathbb{P}^n$, the sheaf $\mathcal{T}_{\mathbb{F}^c/\mathbb{P}^n}$ is isomorphic to $\bigoplus \rho_j^*(\mathcal{T}_{\mathbb{F}/\mathbb{P}^n})$ if ρ_j denotes the j-th projection from $\mathbb{F}^c$ onto $\mathbb{F}$. Hence the assertion follows immediately from (2.4).

(2.6) LEMMA. *Let $Y = H_1 \cap \cdots \cap H_c$ be a complete intersection of hypersurfaces of degree d. Then $(\bigwedge^p \mathcal{F}^\vee) \otimes \mathcal{O}_Y(s)$ has no sections if*

$$s < -(p+c)d \Big/ \left[\binom{n+d}{d} - 1\right].$$

Proof. Since $\mathcal{F}$ is semistable by (2.2) so is $\bigwedge^q \mathcal{F}^\vee$, see [11], (2.6), hence in particular $(\bigwedge^q \mathcal{F}^\vee)(s)$ has no sections if $s < q\mu(\mathcal{F}) = -qd/(N-1)$, $N := \binom{n+d}{d}$. Let $\varphi : \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}^N_{\mathbb{P}^n}(d)$ be the homomorphism dual to the multiplication mapping in

(2.3). The Cokoszulcomplex of φ gives an exact sequence

$$\mathcal{L}^\cdot : \cdots \to \left(\bigwedge^{p-1} \mathcal{O}^N\right)((p-1)d) \xrightarrow{\alpha_{p-1}} \left(\bigwedge^{p} \mathcal{O}^N\right)(pd) \xrightarrow{\alpha_p} \cdots$$

such that $\ker(\alpha_{p+1}) \cong (\bigwedge^p \mathcal{F}^\vee)(pd)$.

We also need the Koszulcomplex $\mathcal{K}_\cdot(f_1, \ldots, f_c)$, where f_j is a homogeneous equation of degree d for H_j, i.e. $\mathcal{K}_j := \mathcal{K}_j(f_1, \ldots, f_c) = \bigwedge^j(\mathcal{O}(-d)^c)$. We consider the double complex

$$C^{ij} := H^0(\mathcal{L}^i \otimes \mathcal{K}_{-j}(-pd+s)),$$

with $s < -(p+c)d/(N-1)$. If we form the cohomology with respect to the first variable (i.e. j is fixed), we write $'H^i(C^{\cdot\cdot})$, and similarly we write $''H^j(C^{\cdot\cdot})$ for the cohomology in the other direction.

Since $f_1, \ldots, f_c$ is a regular sequence in $k[X_0, \ldots, X_n]$, the complex $C^{i\cdot}$ is exact except at zero, and $''H^0(C^{i\cdot}) = \Gamma(Y, \mathcal{L}^i \otimes \mathcal{O}_Y(-pd+s))$. It easily follows that $H^{ij} := H^i(''H^j(C^{\cdot\cdot})) = 0$ if $j = 0$, $i \leq p$ or if $j < 0$, and that

$$H^{0,p+1} = \ker\left(\Gamma\left(Y, \left(\bigwedge^{p+1} \mathcal{O}_Y^N\right)(s+d)\right) \to \Gamma\left(Y, \left(\bigwedge^{p+2} \mathcal{O}_Y^N\right)(s+2d)\right)\right)$$

$$= \Gamma\left(Y, \left(\bigwedge^{p} \mathcal{F}^\vee\right) \otimes \mathcal{O}_Y(s)\right).$$

Hence if $C^\cdot$ is the total complex associated to $C^{\cdot\cdot}$, we obtain, that $H^i(C^\cdot) = 0$, $i < p+1$, and that

$$H^{p+1}(C^\cdot) = \Gamma\left(Y, \bigwedge^{p} \mathcal{F}^\vee \otimes \mathcal{O}_Y(s)\right). \tag{1}$$

Now we examine the other spectral sequence

$$E_1^{ij} = {}'H^i(C^{\cdot j}) \Rightarrow H^{i+j}(C^\cdot).$$

Since $\mathcal{L}^i \otimes \mathcal{K}_{-j}(-pd+s)$ is a direct sum of copies of $\mathcal{O}(id+jd-pd+s)$ we have $C^{ij} = 0$ if $i+j \leq p$, and

$$'H^i(C^{\cdot j}) = \Gamma\left(Y, \bigwedge^{i-1} \mathcal{F}^\vee \otimes \mathcal{K}_{-j}(-jd+s)\right) \quad \text{if} \quad i+j = p+1.$$

But $\bigwedge^{i-1}\mathcal{F}^\vee\otimes\mathcal{K}_{-j}(-jd+s)$ is a direct sum of copies of $\bigwedge^{i-1}(\mathcal{F}^\vee)(s)$ and so $H^i(C^{\cdot j})$ vanishes for $i+j=p+1$, since $\bigwedge^{i-1}\mathcal{F}^\vee(s)$ has no sections because $s<-(p+c)d/(N-1)\leq-(i-1)d/(N-1)$. Therefore we obtain $H^{p+1}(C^\cdot)=0$, and together with (1) this yields the assertion.

(2.7) COROLLARY. *Let* $Y=H_1\cap\cdots\cap H_c$ *be a generic complete intersection. Then*

$$\mu_-(\mathcal{F}\otimes\mathcal{O}_Y)\geq -d^{c+1}/(N-cd-1),\quad N:=\binom{n+d}{d}.$$

Proof. Assume that Q is a quotient of $p^*(\mathcal{F})$ on $\mathbb{F}_d^c$ such that $\mu(Q\mid q^{-1}(H))=\mu_-:=\mu_-(\mathcal{F}\otimes\mathcal{O}_Y)$ if $H=(H_1,\ldots,H_c)\in\mathbb{H}_d^c$ is in general position. Put $p:=rk(Q)$. Then $\bigwedge^p p^*(\mathcal{F}^\vee)\otimes(\bigwedge^p Q)^{\vee\vee}$ has a nonzero section, and since $(\bigwedge^p Q)^{\vee\vee}\mid q^{-1}(H)\cong\mathcal{O}_Y(s)$, $s:=p\cdot\mu_-/d^c$, we get that $(\bigwedge^p\mathcal{F}^\vee)\otimes\mathcal{O}_Y(s)$ has a section. By (2.6) this is only possible if

$$p\mu_-/d^c=s\geq-(p+c)d/(N-1). \tag{*}$$

In order to obtain a lower bound for p we make the following consideration. Since $\mu(\mathcal{F})<0$ we also have $\mu_-(\mathcal{F})<0$ and hence $s=p\mu_-/d^c\leq-1$, and the inequality (*) yields $p\geq(N-cd-1)/d$. Combining this with the inequality (*) we obtain the desired result.

(2.8) *Remark.* The restriction of $\mathcal{F}$ to $Y=H_1\cap\cdots\cap H_c$ has c canonical sections given by the equations of $H_1,\ldots,H_c$. It is natural to ask if $\mathcal{F}\otimes\mathcal{O}_Y$ modulo these sections is even semistable. If this would be true the estimation of (2.7) could be improved, namely we would get $\mu_-(\mathcal{F}\otimes\mathcal{O}_Y)=-d^{c+1}/(N-c-1)$. In the same way one could also improve the bounds in (1.2) and (1.4).

We are now able to give the *proof* of Prop. (1.10):

First case: $X=\mathbb{P}^n$. Let $H=(H_1,\ldots,H_c)\in\mathbb{H}_d^c$ be choosen in general position, $Y:=H_1\cap\cdots\cap H_c$. By (2.5) $\mathcal{T}_{\mathbb{F}^c/\mathbb{P}^n}\mid q^{-1}(H)$ is isomorphic to a direct sum of copies of $(\mathcal{F}\otimes\mathcal{O}_Y)/\mathcal{O}_Y$. Hence

$$\mu_-(\mathcal{T}_{\mathbb{F}^c/\mathbb{P}^n}\mid q^{-1}(H))\geq\mu_-(\mathcal{F}\otimes\mathcal{O}_Y)$$

and the assertion follows from (2.7).

In the general case let $H=(H_1,\ldots,H_c)\in\mathbb{H}_d^c$ be general hypersurfaces of X and $Y:=H_1\cap\cdots\cap H_c$. We denote by $H_1',\ldots,H_c'$ the corresponding hypersurfaces of $\mathbb{P}^n$ such that $H_i=\varphi^{-1}(H_i')$ φ being as in (1.3). Put $Y':=H_1'\cap\cdots\cap H_c'$.

Then $Y \xrightarrow{\varphi} Y'$ is a finite morphism of degree $\deg(X)$. Moreover $\mathcal{T}_{\mathbb{F}^c_{d,X}/X} \cong \Phi^*(\mathcal{T}_{\mathbb{F}^c_d/\mathbb{P}^n})$ if $\Phi: \mathbb{F}^c_{d,X} \to \mathbb{F}^c_d$ is the canonical morphism, see (1.5). In particular $\mathcal{T}_{\mathbb{F}^c_{d,X}/X} \otimes \mathcal{O}_Y \cong \varphi^*(\mathcal{T}_{\mathbb{F}^c_d/\mathbb{P}^n} \otimes \mathcal{O}_{Y'})$ and so our assertion follows from (1.7) and the special case just proven.

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