k-Invariants of knotted 2-spheres.

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k-Invariants of knotted 2-spheres

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§1. Introduction

This paper studies some questions concerning homotopy type invariants of smooth four-dimensional knot complements. Higher-dimensional knot theory diverges sharply from classical knot theory in this respect. It is well known that classical knot complements are aspherical – all higher homotopy groups vanish – so their homotopy types are determined by fundamental groups. This is far from the case in higher dimensions. In fact, Dyer and Vasquez [6] proved that for $S^{n-2} \subset S^n$, $n \geq 4$, $S^n - S^{n-2}$ aspherical implies $\pi_1(S^n - S^{n-2}) \cong \mathbb{Z}$. Well known theorems of Stallings, Levine, Shaneson, Wall now show the knot is trivial, $n \geq 5$. This is now known to be true also for $n = 4$, in the topological category, by Freedman's work. Thus, for knotted 2-spheres in S^4, one must study higher homotopy invariants. A knot complement $S^4 - S^2$ has the homotopy type of a 3-complex, so a natural question is whether the homotopy theory of knot complements in S^4 can be as complicated as that of arbitrary 3-complexes. The main result of this paper indicates that the answer is yes.

The second homotopy group already provides numerous examples of knots in S^4 which fail to be determined by fundamental groups. Using twist-spun knots, discovered by Zeeman [30], Gordon [11] produced examples of knots $K_i \subset S^4$, $i = 1, 2, 3$, whose complements have isomorphic fundamental groups but non-isomorphic second homotopy groups, when viewed as $\mathbb{Z}\pi_1$-modules. The first author generalized this to arbitrarily many knots [22], and then infinitely many [23].

In this work, we study the next higher homotopy invariant, the first k-invariant. Defined by Eilenberg, MacLane and Whitehead [7], [19] for a 3-complex X, this is a class $k(X) \in H^3(\pi_1 X; \pi_2 X)$ that contains more delicate information, beyond π_1 and π_2, as to how the cells of X are glued together. We will define $k(X)$ shortly. For the moment it suffices to know that it is a homotopy

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type invariant. Our main result is

THEOREM 1.1. *Given a positive integer N, there exist N knots in S^4 whose complements have isomorphic π_1 and π_2 (as $\mathbb{Z}\pi_1$-modules), but distinct k-invariants.*

The triple $(\pi_1 X, \pi_2 X, k(X))$ determines the "algebraic 3-type" of X [19]. S. J. Lomonaco, Jr. observed that, with the additional assumption $H_3(\tilde{X})=0$, the algebraic 3-type of a 3-complex determines its homotopy type. In analogy with the classical case, where knot complements are determined by their "algebraic 2-type," i.e. their fundamental groups, knots in S^4 whose exteriors X satisfy $H_3(\tilde{X})=0$ are called *quasi-aspherical*. Not all knots are quasi-aspherical [10], [25]. The knots we construct are quasi-aspherical, so the k-invariant may be regarded as the last obstruction to a homotopy equivalence. These examples answer Question 16 in [17].

The first person to consider k-invariants of knot exteriors in S^4 was Lomonaco [17], who gave a procedure for computing the k-invariant of a 2-knot from a motion picture. In [22], the first author tried to find examples of knots which are distinguished by their k-invariants. Modulo a difficult algebraic problem, involving free modules over $\mathbb{Z}(SL(2,5))$, which arises from the 5-twist spin of the trefoil (see [22, Section 5]), this seems to be impossible in the realm of fibered knots. Accordingly, we look here at non-fibered knots.

Let us describe our examples. Start with two fibered knots in S^4, K_1 and K_2, with fibers the *punctured manifolds* $\mathring{\Sigma}_i \equiv \Sigma_i - \mathring{B}^3$ (Σ_i closed, orientable) and monodromies σ_i, $i=1,2$. The exteriors of these knots are given by $X(K_i)=X_i=\mathring{\Sigma}_i \times_{\sigma_i} S^1 = \mathring{\Sigma}_i \times I/(y,0)\sim(\sigma_i(y),1)$. Letting $\pi_1\mathring{\Sigma}_i = A_i$, we can write $\pi_1 X_i$ as a semi-direct product

$$\pi_1 X_i \equiv H_i = A_i \rtimes \mathbb{Z} = (A_i, x \mid xax^{-1} = \sigma_i(a), \forall a \in A_i),$$

where we write σ_i for the map induced on $\pi_1\mathring{\Sigma}_i$ by σ_i. The connected sum $K=K_1 \# K_2$ is again fibered, with fiber $= \mathring{\Sigma}_1 \natural \mathring{\Sigma}_2$, monodromy $\sigma_1 * \sigma_2$, and exterior $X(K)=X=(\Sigma_i \# \Sigma_2)^0 \times_{\sigma_1 * \sigma_2} S^1$. Letting $A=A_1 * A_2 = \pi_1(\Sigma_1 \# \Sigma_2)$, we have $\pi_1 X \equiv H = H_1 *_{\mathbb{Z}(x)} H_2 = (A_1 \rtimes \mathbb{Z}(x)) *_{\mathbb{Z}(x)} (A_2 \rtimes \mathbb{Z}(x)) = A \rtimes \mathbb{Z}(x)$.

Surgery on K yields $Y=(\Sigma_1 \# \Sigma_2)\times_{\sigma_1 * \sigma_2} S^1$. Form $S^1 \times D^3 \# Y$, and let t represent the generator of $\mathbb{Z} \cong \pi_1(S^1 \times D^3)$. Perform surgery on the curve $txt^{-1}x^{-2}$ in $S^1 \times D^3 \# Y$, and let $\hat{X}$ be the result. We claim $\hat{X}$ is the exterior of a knot $\hat{K}$ in S^4. To see this, add $D^2 \times S^2$ to $\hat{X}$ along $S^1 \times \partial D^3$. This kills t, so the resulting space can be described as $S^4 \# Y$ with surgery on x, or just S^4 (assuming the

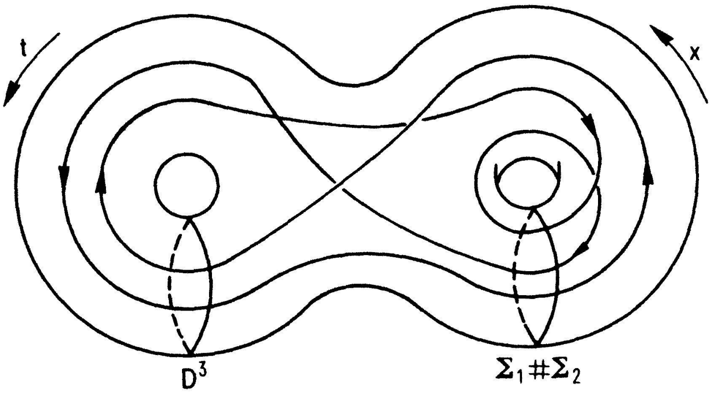


Figure 1

framing for the surgery is chosen correctly). We write

$$\pi_1 = \pi_1\hat{X} = \mathbb{Z}(t) * H/\langle txt^{-1}x^{-2}\rangle \cong G *_{\mathbb{Z}(x)} H, \text{ where } G = (t, x \mid txt^{-1} = x^2)$$
$$\pi_2 = \pi_2\hat{X}.$$

The same construction, replacing K_1 by $-K_1$, yields

$$K' = -K_1 \# K_2$$
$$X(K') = X' = (-\Sigma_1 \# \Sigma_2)^0 \times_{\sigma_1 * \sigma_2} S^1$$
$$Y(K') = (-\Sigma_1 \# \Sigma_2) \times_{\sigma_1 * \sigma_2} S^1$$
$$\hat{X}' = S^1 \times D^3 \# Y' \text{ with surgery on } txt^{-1}x^{-2}, \text{ the exterior of a knot } \hat{K}' \text{ in } S^4.$$

Here $-\Sigma_1$ means Σ_1 with the opposite orientation. The following theorem, which we prove in §7, provides 2 examples for Theorem 1.1.

THEOREM 1.2. *Assume Σ_i is aspherical and admits no orientation reversing homotopy equivalence, $i = 1, 2$. Then the knots $\hat{K}$ and $\hat{K}'$ have isomorphic π_1 and π_2 (as $\mathbb{Z}\pi_1$-modules), but there is no map $f: \hat{X} \to \hat{X}'$ realizing an isomorphism on π_1.*

There are plenty of Σ_i which satisfy the above requirements. We give some examples:

(1) If Σ is a closed, orientable, aspherical Seifert 3-manifold, then Σ admits an orientation reversing homotopy equivalence if, and only if, Σ fibers over S^1 [21]. Thus, no Seifert homology 3-sphere is amphicheiral. The Brieskorn spheres $\Sigma(p, q, r)$, $\{p, q, r\} \neq \{2, 3, 5\}$ are aspherical, and are the (closed) fibers of twist-spun torus knots.

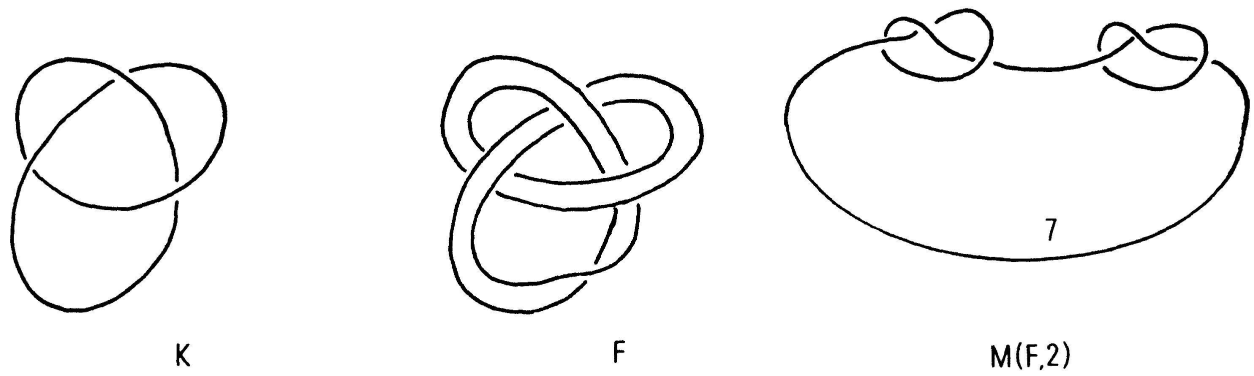


Figure 2

(2) Given a knot $K \subset S^3$, the homology sphere obtained by Dehn surgery of type $1/b$ on K, say M_b, is the (closed) fiber of the b-rolled, 1-twist spin of K, and various generalizations, [15, 24]. If the Arf invariant of K is non-zero, and b is odd, the Rohlin invariant $\mu(M_b)=1$ [9]. Siebenmann [26] showed that, if $\mu(M_b)=1$ and M_b is geometric in the sense of Thurston (Seifert, hyperbolic, or a sum of such along spheres and tori), M_b does not admit orientation reversing homotopy equivalences. In general, one expects M_b to be aspherical and geometric. For instance, if K is a hyperbolic knot, a theorem of Thurston [29] shows that M_b is hyperbolic if $|b|$ is large enough.

(3) This example was suggested by D. Ruberman. Start with a knot K in S^3 drawn with self-linking number l, thicken it to a band, put m right half-twists in the band, where $2l+m=n\equiv 3 \pmod 4$. Push this non-orientable surface F into D^4, and take the double branched cover of D^4 along F, $M(F,2)$ (see Fig. 2). This is a 4-manifold built with one 0-handle and one 2-handle, attached along $K\#K$, with framing n [1].

Then $\partial M(F,2)=\Sigma$ is the double branched cover of S^3 along ∂F. It is also obtained by n surgery along $K\#K$, hence can be expressed as the union of two copies of the exterior of K glued along their boundaries [13], so that Σ is aspherical.

Now $H_1(\Sigma)=\mathbb{Z}_n$, with linking form $\lambda:\mathbb{Z}_n\otimes\mathbb{Z}_n\to\mathbb{Q}/\mathbb{Z}$ given by $\lambda(1,1)=1/n$. If Σ admits an orientation reversing homotopy equivalence inducing multiplication by r on $H_1(\Sigma)$, we find $r^2\equiv -1 \pmod n$, an impossibility. Thus, the 2-twist spin of ∂F yields a knot in S^4 with the required fiber.

We now describe how one finds $k(X)\in H^3(\pi_1;\pi_2)$. Given a 3-complex X, let $\tilde{X}$ be the universal cover of X, and consider the augmented chain complex for $\tilde{X}$, $0\to C_3(\tilde{X})\xrightarrow{\partial_3} C_2(\tilde{X})\xrightarrow{\partial_2} C_1(\tilde{X})\xrightarrow{\partial_1} C_0(\tilde{X})\to\mathbb{Z}\to 0$, where $C_i(\tilde{X})$ is a free $\mathbb{Z}\pi_1$-module, with a generator for each i-cell of X. This fails to be exact at $C_2(\tilde{X})$.

Add a free summand $\bar{C}_3$ and define $\bar{\partial}_3$ on $\bar{C}_3$ so as to kill $\pi_2 X$:

$$\begin{tikzcd}
0 \arrow[r] & \bar{C}_3 \oplus C_3(\tilde{X}) \arrow[r] \arrow[dr, shift left, "\partial_3"] \arrow[dr, shift right, "\bar{\partial}_3"'] \arrow[ddr, "k"'] & C_2(\tilde{X}) \arrow[r, "\partial_2"] & C_1(\tilde{X}) \arrow[r, "\partial_1"] & C_0(\tilde{X}) \arrow[r] & \mathbb{Z} \arrow[r] & 0. \\
 & & \ker \partial_2 \arrow[u, hook] \arrow[d, "p"] & & & & \\
 & & \pi_2 & & & &
\end{tikzcd}$$

The chain complex is now exact at C_2, and the map $k = p\bar{\partial}_3 : \bar{C}_3 \to \pi_2$ determines a well-defined class $[k] \in H^3(\pi_1 X; \pi_2 X)$. An essentially equivalent version of this is the following: view X as a subcomplex of $K(\pi_1 X, 1)$ by adding cells of dimension ≥ 3 to X. Then $[k]$ is the obstruction to extending $\mathrm{id}: X \to X$ to the 3-skeleton of $K(\pi_1 X, 1)$.

The usefulness of the k-invariant is shown by the following theorem of MacLane and Whitehead [19]: let X and X' be 3-complexes. A map $\alpha: \pi_1 X \to \pi_1 X'$ and an α-map $\beta: \pi_2 X \to \pi_2 X'$ are geometrically realizable (induced by $f: X \to X'$) if, and only if, $\alpha^*[k'] = \beta_*[k] \in H^3(\pi_1 X; (\pi_2 X')_\alpha)$. *If this holds, we say α and β preserve k-invariants.*

Given X, X' with $\pi_1 X \cong \pi_1 X'$, $\pi_2 X \underset{\pi_1}{\cong} \pi_2 X'$, the k-invariant represents the obstruction to realizing these maps geometrically. If $H_3(\tilde{X}) = H_3(\tilde{X}') = 0$, the Hurewicz and Whitehead theorems show that a map $X \to X'$ realizing these isomorphisms is a homotopy equivalence.

We can regard Theorem 1.2 as a 4-dimensional analogue of the following special case of a theorem of C. B. Thomas [28]: If Σ_i^3 is aspherical and does not admit an orientation reversing homotopy equivalence, $i = 1, 2$, then $(-\Sigma_1) \# \Sigma_2 \not\simeq \Sigma_1 \# \Sigma_2$. Thus, we begin in §2 with a proof of this via k-invariants. The calculations here serve as a good warmup for the later sections. In §3 we compute π_2 of the knot exteriors. Our method, while elementary, seems useful in its own right, and can be used to compute π_2 of a large class of knots, including spun knots. §4 describes a cell complex for $\hat{X}$, computes π_2 in a slightly different fashion, and determines $k(\hat{X})$, $k(\hat{X}')$ on the cochain level. In §5 we compute enough of $H^3(\pi_1; \pi_2)$ to locate k-invariants. In §6 we investigate automorphisms of π_1, and use this in §7 to complete the proof of Theorem 1.2. Finally, §8 contains the straightforward generalization from two examples to arbitrarily many.

We will always regard π_2 as a *left* $\mathbb{Z}\pi_1$-module. If $\mathbb{Z}\pi$ is a left $\mathbb{Z}\pi$-module, a map $\beta: \mathbb{Z}\pi \to \mathbb{Z}\pi$ is just *right* multiplication by $\beta(1)$. More generally, vectors in $(\mathbb{Z}\pi)^n$ are row vectors and matrices with entries in $\mathbb{Z}\pi$ act on the right. An excellent reference for cohomology of groups is [5].

§2. k-invariants for $\Sigma_1 \# \Sigma_2$

Let Σ_i, $i = 1, 2$, be closed, orientable, aspherical 3-manifolds, $\pi_1\Sigma_i \cong A_i$.

First consider $\mathring{\Sigma}_i$. Since $\tilde{\Sigma}_i$ is contractible (presumably $\mathbb{R}^3$), we easily see that $\pi_2(\mathring{\Sigma}_i) \cong \mathbb{Z}A_i$, naturally generated by the boundary sphere. We add $\bar{C}_3 = \mathbb{Z}A_i$ to make $C_*(\tilde{\mathring{\Sigma}}_i)$ exact:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}A_i \arrow[r] \arrow[dr, "k_i"'] & C_2(\tilde{\mathring{\Sigma}}_i) \arrow[r, "\partial_2^i"] & C_1(\tilde{\mathring{\Sigma}}_i) \arrow[r, "\partial_1^i"] & \mathbb{Z}A_i \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\
& & \ker \partial_2^i = \pi_2(\mathring{\Sigma}_i) \cong \mathbb{Z}A_i. \arrow[u, hook] & & & &
\end{tikzcd}$$

Thus, k_i is represented by $\mathbb{Z}A_i \xrightarrow{\text{id}} \mathbb{Z}A_i$, the natural generator of $H^3(A_i; \mathbb{Z}A_i) \cong \mathbb{Z}$. Of course, this map depends on the choice of isomorphism $\pi_2 \cong \mathbb{Z}A_i$. Another choice would give a k-invariant $\tilde{k}$, with $k = \beta\tilde{k}$ for some $\beta \in \mathrm{Aut}_{\mathbb{Z}A_i}(\pi_2)$. This complex is a resolution. In fact, it is precisely $C_*(\tilde{\Sigma}_i)$, with the map $\mathbb{Z}A_i \to \ker \partial_2^i$ corresponding to gluing a 3-ball into $\mathring{\Sigma}_i$. Notice that the augmentation map $\mathbb{Z}A_i \to \mathbb{Z}$ induces an isomorphism $H^3(A_i, \mathbb{Z}A_i) \cong H^3(\Sigma_i; \mathbb{Z})$, under which k_i corresponds to the orientation class of Σ_i.

Now consider $(\Sigma_1 \# \Sigma_2)^0 \simeq \mathring{\Sigma}_1 \vee \mathring{\Sigma}_2$, $A = A_1 * A_2$.

LEMMA 2.1. $C_*(\widetilde{(\Sigma_1 \# \Sigma_2)^0}) \cong \bigoplus_{i=1}^{2} \mathbb{Z}A \otimes_{\mathbb{Z}A_i} C_*(\tilde{\mathring{\Sigma}}_i)$.

Proof. Consider the universal cover $\widetilde{(\Sigma_1 \# \Sigma_2)^0} \xrightarrow{p} (\Sigma_1 \# \Sigma_2)^0$. Covering space theory shows that $p^{-1}(\mathring{\Sigma}_i)$ consists of disjoint copies of $\tilde{\mathring{\Sigma}}_i$, indexed by the cosets A/A_i. As a left $\mathbb{Z}A$-module, these copies are permuted transitively, and the stabilizer of the copy corresponding to the identity coset is A_i. Therefore $C_*(p^{-1}(\mathring{\Sigma}_i)) \cong \mathbb{Z}A \otimes_{\mathbb{Z}A_i} C_*(\tilde{\mathring{\Sigma}}_i)$ [5, page 67–68]. ■

The lemma shows that $\ker \partial_2 = \bigoplus_{i=1}^{2} (\mathbb{Z}A \otimes_{\mathbb{Z}A_i} \ker \partial_2^i) \cong \bigoplus_{i=1}^{2} (\mathbb{Z}A \oplus_{\mathbb{Z}A_i} \mathbb{Z}A_i)$, with natural generators given by the two boundary spheres. We make the complex exact at C_2 by adding $\bar{C}_3 = \bigoplus_{i=1}^{2} (\mathbb{Z}A \otimes_{\mathbb{Z}A_i} \mathbb{Z}A_i)$ and mapping via $k = \text{id} : \bar{C}_3 \to \ker \partial_2$:

$$\begin{tikzcd}
(\mathbb{Z}A \otimes_{\mathbb{Z}A_1} \mathbb{Z}A_1) \oplus (\mathbb{Z}A \otimes_{\mathbb{Z}A_2} \mathbb{Z}A_2) \arrow[r] \arrow[dr, "k = \text{id}"'] & \mathbb{Z}A \otimes_{\mathbb{Z}A_1} C_2(\tilde{\mathring{\Sigma}}_1) \oplus \mathbb{Z}A \otimes_{\mathbb{Z}A_2} C_2(\tilde{\mathring{\Sigma}}_2) \arrow[r, "\partial_2"] & . \\
& \ker \partial_2 = (\mathbb{Z}A \otimes_{\mathbb{Z}A_1} \mathbb{Z}A_1) \oplus (\mathbb{Z}A \otimes_{\mathbb{Z}A_2} \mathbb{Z}A_2) \arrow[u, hook] &
\end{tikzcd}$$

The following well known lemma will be useful here and later. Let $B = \pi_1 M^n$, where M is a closed, orientable, aspherical n-manifold. Suppose $B \subset A$.

LEMMA 2.2.

$$H^k(B;\mathbb{Z}A)=\begin{cases}\mathbb{Z}\otimes_{\mathbb{Z}B}\mathbb{Z}A; & k=n\\ 0; & k\neq n.\ \blacksquare\end{cases}$$

It is useful to interpret this via cosets. Since $\mathbb{Z}A=\oplus_{B\backslash A}\mathbb{Z}B$ as left $\mathbb{Z}B$-modules, $H^k(B;\mathbb{Z}A)=H^k(B;\oplus_{B\backslash A}\mathbb{Z}B)=\oplus_{B\backslash A}H^k(B;\mathbb{Z}B)=H^k(B;\mathbb{Z}B)\otimes_{\mathbb{Z}B}\mathbb{Z}A$. Here we use that B has a finite $K(B,1)$ so that cohomology commutes with direct sums. The right $\mathbb{Z}A$-module structure of $H^n(B;\mathbb{Z}A)\cong\mathbb{Z}\otimes_{\mathbb{Z}B}\mathbb{Z}A\cong\oplus_{B\backslash A}\mathbb{Z}$ is the natural permutation action of A on the cosets $B\backslash A$.

Using Lemma 2.2, we find

$$\begin{aligned}H^3(\pi_1(\Sigma_1\#\Sigma_2)^0;\ \pi_2(\Sigma_1\#\Sigma_2)^0)&=H^3(A_1*A_2;\ (\mathbb{Z}A)^2)\\ &\cong\bigoplus_{i=1}^{2}H^3(A_i,(\mathbb{Z}A)^2)\cong\bigoplus_{i=1}^{2}(\mathbb{Z}\otimes_{\mathbb{Z}A_i}\mathbb{Z}A)^2.\end{aligned}$$

It should be clear that under these identifications, the k-invariant corresponds to $(1\otimes_{\mathbb{Z}A_1}1,0,0,1\otimes_{\mathbb{Z}A_2}1)$. This is seen explicitly by restricting the complex to the resolution $0\to\mathbb{Z}A_i\to C_2(\tilde{\mathring{\Sigma}}_i)\to\dots$, restricting k to this piece, and using the discussion for $\mathring{\Sigma}_i$.

Now consider $\Sigma_1\#\Sigma_2$. We add a 3-cell to $(\Sigma_1\#\Sigma_2)^0$ so as to kill the sum of the natural generators of $\ker\partial_2^1$ and $\ker\partial_2^2$, giving $\partial_3\colon C_3=\mathbb{Z}A\xrightarrow{(1,1)}\mathbb{Z}A\oplus\mathbb{Z}A=\ker\partial_2$. We make $C_*(\widetilde{\Sigma_1\#\Sigma_2})$ exact at C_2 by adding $\bar{C}_3=\mathbb{Z}A$, and defining $\mathbb{Z}A=\bar{C}_3\longrightarrow\ker\partial_2=\mathbb{Z}A\oplus\mathbb{Z}A$ by $\bar\partial_3(1)=(1,0)$.

$$\begin{tikzcd}
0 \arrow[r] & \bar{C}_3\oplus C_3 \arrow[r] \arrow[dr, "(\bar\partial_3,\partial_3)"'] & \mathbb{Z}A\otimes_{\mathbb{Z}A_1}C_2(\tilde{\mathring{\Sigma}}_1)\oplus\mathbb{Z}A\otimes_{\mathbb{Z}A_2}C_2(\tilde{\mathring{\Sigma}}_2) \arrow[r, "\partial_2"] & {} \\
& & \ker\partial_2=(\mathbb{Z}A\otimes_{\mathbb{Z}A_1}\mathbb{Z}A_1)\oplus(\mathbb{Z}A\otimes_{\mathbb{Z}A_2}\mathbb{Z}A_2) \arrow[u, phantom, "\cup"] \arrow[d] & \\
& & \mathbb{Z}A\oplus\mathbb{Z}A/\operatorname{im}\partial_3\cong\mathbb{Z}A & \\
& & (a,b)\longmapsto a-b. &
\end{tikzcd}$$

Thus, $k(\Sigma_1\#\Sigma_2)$ is represented by $\bar{C}_3=\mathbb{Z}A\xrightarrow{\text{id}}\mathbb{Z}A\cong\pi_2$.

As above, we have $H^3(\pi_1(\Sigma_1\#\Sigma_2);\pi_2(\Sigma_1\#\Sigma_2))=H^3(A_1*A_2;\mathbb{Z}A)\cong(\mathbb{Z}\otimes_{\mathbb{Z}A_1}\mathbb{Z}A)\oplus(\mathbb{Z}\otimes_{\mathbb{Z}A_2}\mathbb{Z}A)$. To locate k, first consider A_1. The natural inclusion $C_*(\tilde{\mathring{\Sigma}}_1)\hookrightarrow\mathbb{Z}A\otimes_{\mathbb{Z}A_1}C_*(\tilde{\mathring{\Sigma}}_1)$ extends to a chain map $C_*(\tilde{\Sigma}_1)\to C_*(\widetilde{\Sigma_1\#\Sigma_2})\oplus\bar{C}_3$ by defining $C_3(\tilde{\Sigma}_1)=\mathbb{Z}A_1\hookrightarrow\mathbb{Z}A=\bar{C}_3$ via the natural inclusion. Thus, k restricts to $k_1\colon\mathbb{Z}A_1\hookrightarrow\mathbb{Z}A$, also the natural inclusion. For A_2, we extend the natural inclusion $C_*(\tilde{\tilde{\Sigma}}_2) \hookrightarrow \mathbb{Z}A \otimes_{\mathbb{Z}A_2} C_*(\tilde{\tilde{\Sigma}}_2)$ by defining $C_3(\tilde{\Sigma}_2) = \mathbb{Z}A_2 \xrightarrow{(-1,1)} \bar{C}_3 \oplus C_3 = \mathbb{Z}A \oplus \mathbb{Z}A$, so that k restricts to $k_2 : \mathbb{Z}A_2 \overset{-1}{\hookrightarrow} \mathbb{Z}A$. Thus, under the above identifications

$$H^3(\pi_1(\Sigma_1 \# \Sigma_2); \pi_2(\Sigma_1 \# \Sigma_2)) \xrightarrow{\cong} (\mathbb{Z} \otimes_{\mathbb{Z}A_1} \mathbb{Z}A) \oplus (\mathbb{Z} \otimes_{\mathbb{Z}A_2} \mathbb{Z}A)$$
$$k \rightarrow (1 \otimes 1, -1 \otimes 1).$$

Now consider $(-\Sigma_1) \# \Sigma_2$. The same discussion applies, modified as follows:

$$\partial_3 : C_3 = \mathbb{Z}A \xrightarrow{(-1,1)} \mathbb{Z}A \oplus \mathbb{Z}A = \ker \partial_2$$
$$\ker \partial_2 / \operatorname{im} \partial_3 = \mathbb{Z}A \oplus \mathbb{Z}A / \operatorname{im} \partial_3 \cong \mathbb{Z}A$$
$$(a, b) \mapsto a + b$$

$$C_3(\tilde{\Sigma}_2) = \mathbb{Z}A_2 \xrightarrow{(1,1)} \mathbb{Z}A \oplus \mathbb{Z}A = \bar{C}_3 \oplus C_3.$$

Therefore, the k-invariant here is given by

$$H^3(\pi_1(-\Sigma_1 \# \Sigma_2); \pi_2(-\Sigma_1 \# \Sigma_2)) \xrightarrow{\cong} (\mathbb{Z} \otimes_{\mathbb{Z}A_1} \mathbb{Z}A) \oplus (\mathbb{Z} \otimes_{\mathbb{Z}A_2} \mathbb{Z}A)$$
$$k \mapsto (1 \otimes 1, 1 \otimes 1).$$

We now prove a special case of Thomas' theorem [28]:

PROPOSITION 2.3. *Let Σ_1 and Σ_2 be closed, oriented, aspherical 3-manifolds admitting no orientation reversing homotopy equivalences. Then $(-\Sigma_1) \# \Sigma_2 \not\simeq \Sigma_1 \# \Sigma_2$.*

Proof. Suppose we are given $\alpha : \pi_1(-\Sigma_1 \# \Sigma_2) \xrightarrow{\cong} \pi_1(\Sigma_1 \# \Sigma_2)$ and an α-map $\beta : \pi_2 \rightarrow \pi_2$ which preserve k-invariants. By Bloomberg's theorem [4], $\alpha = (f, g) \in \operatorname{Aut} A_1 \oplus \operatorname{Aut} A_2$, up to an inner automorphism. Both f^{-1} and g^{-1} are realizable by homotopy equivalences, and since both preserve orientation, we can realize (f^{-1}, g^{-1}) by a homotopy equivalence of $\Sigma_1 \# \Sigma_2$. Composing this with the map realized by (α, β), we can assume $\alpha = \text{identity}$. Therefore, $\beta_*(k') = k$. The map $\beta : \mathbb{Z}A \rightarrow \mathbb{Z}A$ is right multiplication by $\beta(1)$, so that $(1 \otimes \beta(1), 1 \otimes \beta(1)) = (1 \otimes 1, -1 \otimes 1)$. If $\beta(1) = \sum n_g g$, this gives, in terms of cosets,

$$\left(\sum n_g A_1 g, \sum n_g A_2 g\right) = (A_1 \cdot 1, -A_2 \cdot 1).$$

Let $g \in A_1 * A_2$ be a word having maximal length among those with $n_g \neq 0$. Since $\beta \neq 1$, $g \neq 1$, so $g = a_1 h$, with $1 \neq a_1 \in A_1$, say. Since $n_g A_2 g \neq 0$ and $A_2 g \neq A_2$, there exists g' with $n_{g'} \neq 0$, $A_2 g' = A_2 g$. But then $g' = a_2 g$, so that length $g' >$ length g, a contradiction. ■

Notice that we proved a stronger result than stated in the proposition. There is no map $(-\Sigma_1) \# \Sigma_2 \to \Sigma_1 \# \Sigma_2$ inducing an isomorphism on π_1. In fact, much more is known. Thomas' theorem says that an orientation-preserving homotopy equivalence between closed, orientable 3-manifolds exists only if the prime summands pair off as oriented manifolds. Swarup [27] proved that a map $f : M \to N$ between connected sums of closed, aspherical n-manifolds, $n \geq 3$, that induces an isomorphism on fundamental groups is a homotopy equivalence.

§3. Computation of π_2

Recall that $\hat{X}$ is obtained by surgery on $txt^{-1}x^{-2}$ in $S^1 \times D^3 \# Y$. Let M be the cover of $S^1 \times D^3 \# Y$ corresponding to the kernel of $\mathbb{Z} * H \to (\mathbb{Z} * H/\langle txt^{-1}x^{-2}\rangle = \pi)$. If we perform equivariant surgery on the lifts of $txt^{-1}x^{-2}$ in M, we obtain $\hat{M} = \tilde{\hat{X}}$. Since $\pi_1 Y$ injects into π, sitting over Y in M we see copies of the universal cover $\tilde{Y}$, indexed by the cosets π/H. Similarly for $S^1 \times D^3$. A schematic picture, together with two lifts of the surgery curve, is shown below.

Now do surgery. Let $M = M_0 \cup_{\coprod S^1 \times S^2} (\coprod S^2 \times D^3)$, $\hat{M} = M_0 \cup_{\coprod S^1 \times S^1} (\coprod D^2 \times S^2)$, with copies of $S^1 \times S^2$, $S^1 \times D^3$, $D^2 \times S^2$ indexed by π. The Mayer–Vietoris sequences corresponding to these decompositions yield

$$0 \to \oplus H_3(S^1 \times S^2) \to H_3(M_0) \to H_3(M)$$
$$\xrightarrow{\varphi} \oplus H_2(S^1 \times S^2) \to H_2(M_0) \to H_2(M) \to 0$$

and

$$0 \to \oplus H_3(S^1 \times S^2) \to H_3(M_0) \to H_3(\hat{M}) \to \oplus H_2(S^1 \times S^2)$$
$$\to \oplus H_2(D^2 \times S^2) \oplus H_2(M_0) \to H_2(\hat{M}) \to \oplus H_1(S^1 \times S^2) \xrightarrow{\psi} H_1(M_0) \to 0.$$

Now $H_3(M) \cong \mathbb{Z}\pi$, generated by the lifts of the "connector" S^3. Also, notice that we can take the surgery curve disjoint from all 2-cycles of $S^1 \times D^3 \# Y$, so there is

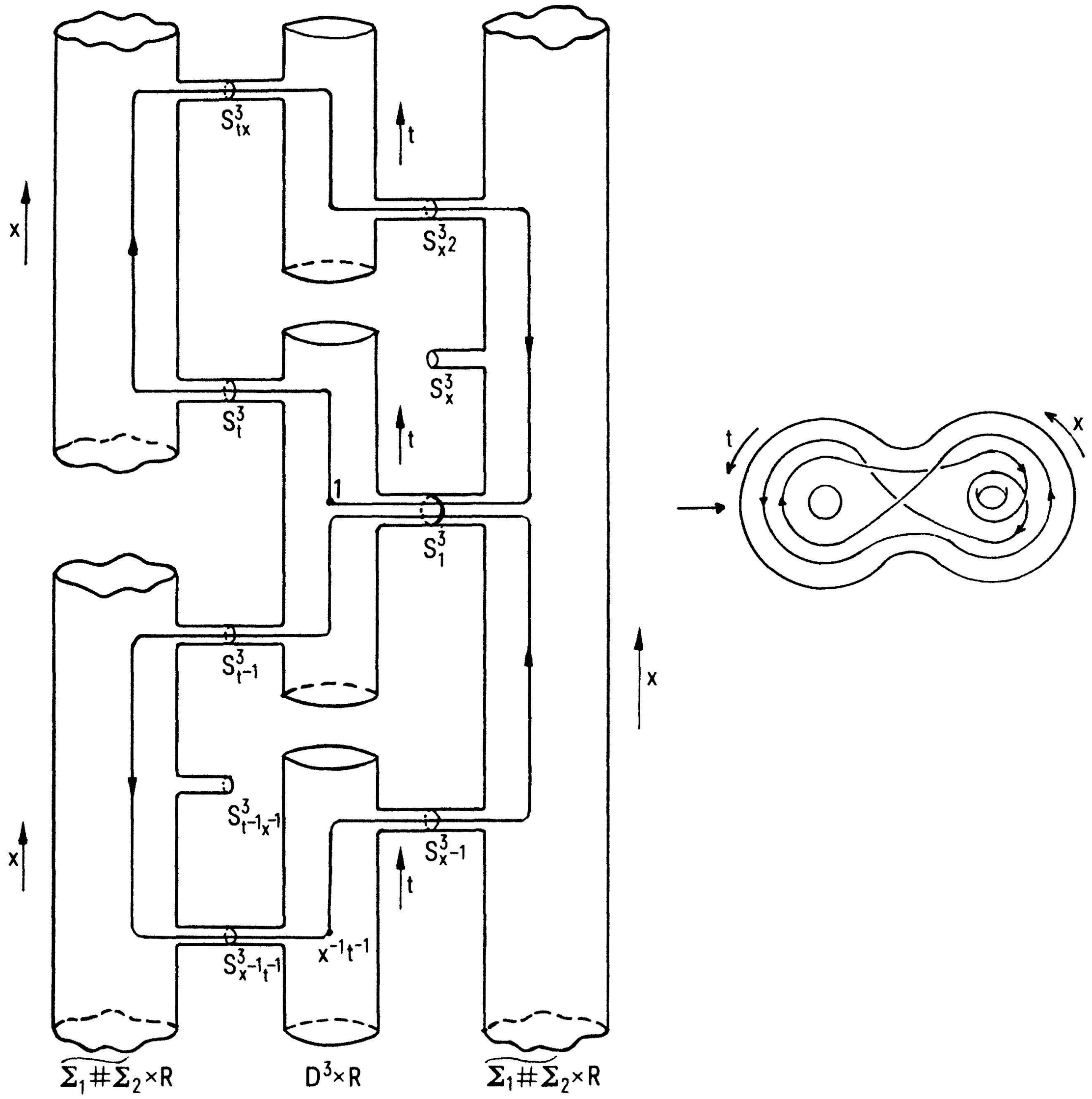


Figure 3

a natural splitting $H_2(M_0) \leftrightarrows H_2(M)$. These sequences simplify to give

$$
\begin{tikzcd}
 & & H_3(\hat{M}) = \ker(\mathbb{Z}\pi \xrightarrow{\varphi} \mathbb{Z}\pi) & & \\
 & & 0 \arrow[d] & & \\
0 \arrow[r] & \operatorname{coker}\varphi \arrow[r] & H_2(M_0) \arrow[r] \arrow[d] & H_2(M) \arrow[r] \arrow[l, bend right] & 0 \\
 & & H_2(\hat{M}) \arrow[d] & & \\
 & & \ker\psi \arrow[d] & & \\
 & & 0. & &
\end{tikzcd}
$$

An argument similar to that of Lemma 2.1 shows that $H_2(M)=\mathbb{Z}\pi\otimes_{\mathbb{Z}H}\pi_2Y$. As a $\mathbb{Z}A$-module, π_2Y is just $\mathbb{Z}A$, generated by the 2-sphere along which we take connected sum. As a $\mathbb{Z}H$-module, π_2Y is coker $(\mathbb{Z}H\xrightarrow{\cdot(1-x)}\mathbb{Z}H)$, a result of Andrews/Sumners [3]. Therefore, as a $\mathbb{Z}\pi$-module, $H_2(M)=\mathbb{Z}\pi\otimes_{\mathbb{Z}H}$ coker $(\mathbb{Z}H\xrightarrow{\cdot(1-x)}\mathbb{Z}H)=$ coker $(\mathbb{Z}\pi\xrightarrow{\cdot(1-x)}\mathbb{Z}\pi)$.

To compute φ, examine the lifts of $txt^{-1}x^{-2}$ which cut through S^3_1 (Figure 3) and compute $\varphi(1)=1+x^{-1}t^{-1}-t^{-1}-x^{-2}=1-t^{-1}+t^{-1}x^{-2}-x^{-2}=(1-t^{-1})\times(1-x^{-2})$. The proof of the following lemma is elementary.

LEMMA 3.1. *Let $g\in G$ be an element of infinite order in a group G. Then $\mathbb{Z}G\xrightarrow{(1-g)}\mathbb{Z}G$ is a monomorphism.* ■

Consequently, $\varphi:\mathbb{Z}\pi\to\mathbb{Z}\pi$ is a monomorphism, and $H_3(\hat{M})=0$. Writing $-x(1+x^{-1}t^{-1}-t^{-1}-x^{-2})=(1-x)(1+x^{-1}-t^{-1})$, we see that coker $\varphi\cong\mathbb{Z}\pi/(1-x)(1+x^{-1}-t^{-1})$.

To analyze ψ, replace Y for the moment by $S^1\times S^3$. The same surgery produces a knot with $\pi_1=G$, a one relator group, with the relator not a proper power. By Lyndon's theorem [18], the relation module $H_1(M)$ is freely generated by the lifts of $txt^{-1}x^{-2}$, so that $\psi_G:\mathbb{Z}G\to\mathbb{Z}G$ is an isomorphism. In our situation, $H_1(M)=\mathbb{Z}\pi\otimes_{\mathbb{Z}G}\mathbb{Z}G$, and $\psi=1\otimes\psi_G$ is again an isomorphism. Hence, ker $\psi=0$.

These calculations are identical for the exterior $\hat{X}'$ of $\hat{K}'$. We summarize our calculations:

PROPOSITION 3.2. *The knots $\hat{K}, \hat{K}'$ are quasi-aspherical, with π_2 given by the following exact sequence of $\mathbb{Z}\pi$-modules:*

$$0\longrightarrow\mathbb{Z}\pi\oplus\mathbb{Z}\pi\xrightarrow{\begin{pmatrix}1-x & 0\\ 0 & (1-x)(1+x^{-1}-t^{-1})\end{pmatrix}}\mathbb{Z}\pi\oplus\mathbb{Z}\pi\longrightarrow\pi_2\longrightarrow0.\quad ■$$

Remark. As above, replace Y by $S^1\times S^3$, giving a knot with $\pi_1=G$. In this case, $H_3(M)\cong\mathbb{Z}G$, generated by the lifts of the "fiber" S^3, and we compute $\varphi_G:H_3(M)\to\oplus_G H_2(S^1\times S^2)$ as $\varphi_G:\mathbb{Z}G\to\mathbb{Z}G$, $\varphi(1)=t^{-1}-x^{-1}-1$. Therefore, $\pi_2=\mathbb{Z}G/(t^{-1}-x^{-1}-1)$, a result in [17].

More generally, if we replace $txt^{-1}x^{-2}$ by a word $r(t,x)$ with exponent sum ±1 in x, we get an arbitrary 1-relator ribbon knot in S^4 with $\pi_1=(t,x\mid r)$ and $\pi_2=\mathbb{Z}\pi_1/\overline{(\partial r/\partial x)}$, where $\partial r/\partial x$ is the Fox derivative, and $\overline{(\sum n_g g)}=\sum n_g g^{-1}$.

Similarly, if $\pi=(t,x_1,\dots,x_n\mid r_1,\dots,r_n)$ is a Wirtinger presentation of a classical knot group, we can construct a ribbon 2-knot with this group by adding $\#_1^n S^1\times S^3$ to $S^1\times D^3$ and performing surgery on the curves r_i. By the asphericity of knots, ψ is an isomorphism, and we compute $\varphi=\overline{(\partial r_i/\partial x_j)}$, so that $\pi_2=(\mathbb{Z}\pi)^n/\overline{(\partial r_i/\partial x_j)}$. For a spun knot, this complements Andrew's and Lomonaco's computation $(\mathbb{Z}\pi)^n/(\partial r_i/\partial x_j)^t$ [2, 16]. Abelianizing, we find the classical fact that the Alexander matrix of a knot in S^3 is hermitian, up to trivial units.

§4. A cell complex for $\hat{X}$

We start with Figure 4, showing $S^1 \times D^3 \# Y$, the surgery curve, and intersections of the curve with two fibers of Y. We view $S^1 \times D^3 \# Y$ minus a neighborhood of $txt^{-1}x^{-2}$ as obtained by gluing $(S^1 \times D^3)^0$ and Y^0, minus neighborhoods

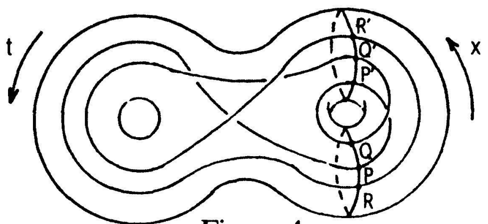


Figure 4

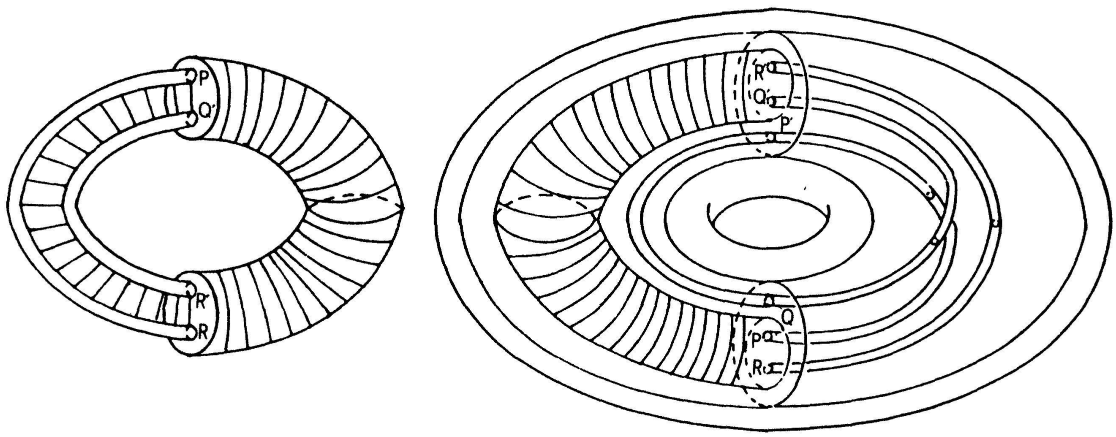
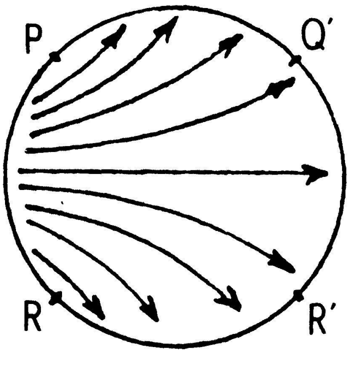


Figure 5

of arcs, along the four-times punctured "connector" S^3, Figure 5. In Y we have removed all the neighborhoods of arcs from a $B^3 \times S^1$, pictured as the inner solid torus. The region between the inner and outer tori is $X = (\Sigma_1 \# \Sigma_2)^0 \times_{\sigma_1 * \sigma_2} S^1$. Notice also that $(S^1 \times D^3)^0 -$(neighborhoods of arcs) has been deformed to S^3 with two 1-handles, plus a 2-dimensional "membrane" connecting PR to $Q'R'$.

Now glue in $S^2 \times D^2$ along the boundary of a neighborhood of $txt^{-1}x^{-2}$. $S^2 \times D^2$ deformation retracts onto $S^2 \times I$ via



This brings us to Figure 6, where the "membrane" has been stretched out to a 2-cell e^2 attached along $txt^{-1}x^{-2}$.

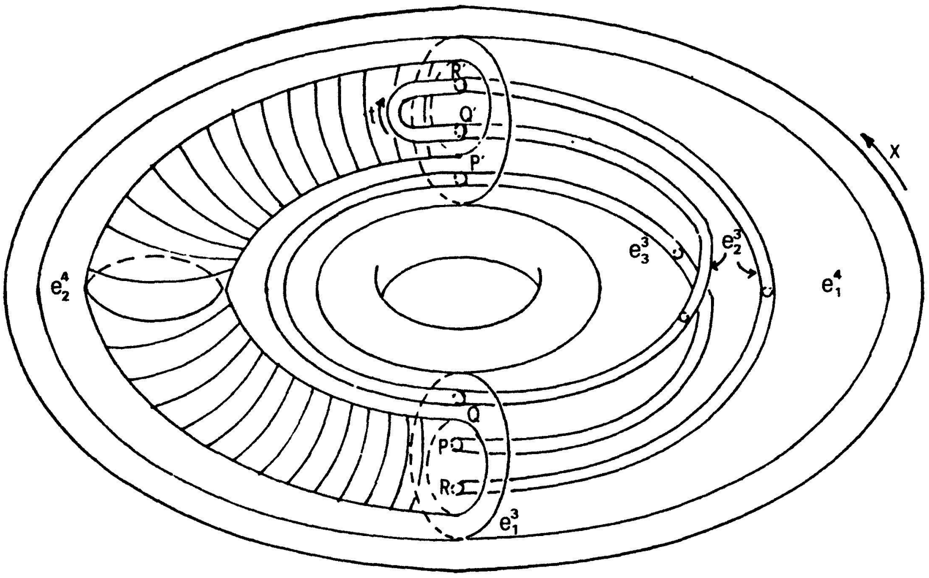


Figure 6

We have that

$$\partial_3 e_1^3 = a_1 + a_2 - P - Q - R \quad \text{(see Figure 7)}$$

$$\partial_3 e_2^3 = Q + xtx^{-1}R = Q + x^{-1}tR$$

$$\partial_3 e_3^3 = P - xQ.$$

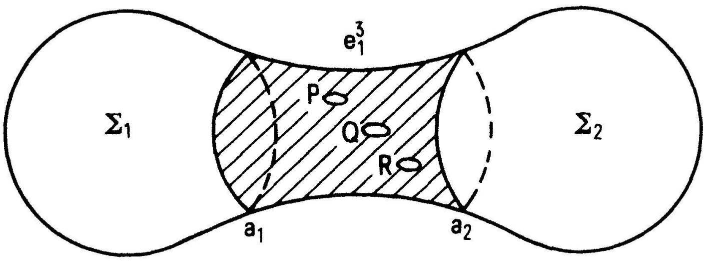


Figure 7

Finally, collapse e_1^4 and e_2^4, and cancel e_2^3, e_3^3 against Q, R, replacing e_1^3 by e^3. This, along with Lemma 2.1, gives $C_*(\tilde{X})$:

$$\mathbb{Z}\pi(e^3) \oplus (\mathbb{Z}\pi \otimes_{\mathbb{Z}H} C_3\tilde{X}) \xrightarrow{\partial_3} \mathbb{Z}\pi(P) \oplus \mathbb{Z}\pi(e^2) \oplus (\mathbb{Z}\pi \otimes_{\mathbb{Z}H} C_2\tilde{X}) \xrightarrow{\partial_2}$$

$$\mathbb{Z}\pi(t) \oplus (\mathbb{Z}\pi \otimes_{\mathbb{Z}H} C_1\tilde{X}) \xrightarrow{\partial_1}$$

where

$$\partial_i \big|_{\mathbb{Z}\pi\otimes_{\mathbb{Z}H}C_*(\tilde{X})} = 1 \otimes \partial_i^{\tilde{X}}$$

$$\partial_3(e^3) = a_1 + a_2 - (1 + x^{-1} - t^{-1})P$$

$$\partial_2(P) = 0.$$

From §3, we know e^2 does not contribute to $\ker \partial_2$. Since the Σ_i are aspherical, §2 gives $\ker \partial_2^{\tilde{X}} = \mathbb{Z}H(a_1) \oplus \mathbb{Z}H(a_2)$, $\pi_2 X \cong \mathbb{Z}H/(1-x) \oplus \mathbb{Z}H/(1-x)$. Finally, $\pi_2 \hat{X}$ is generated by a_1 and P, subject to the relations: $(1-x)a_1 = 0$, $(1-x)(1+x^{-1}-t^{-1})P = 0$. Therefore,

$$\pi_2 \hat{X} = \mathbb{Z}\pi(a_1)/(1-x) \oplus \mathbb{Z}\pi(P)/(1-x)(1+x^{-1}-t^{-1}),$$

in agreement with Proposition 3.2.

We now describe $k(\hat{X})$. Add a free summand $\bar{C}_3(\hat{X}) = \mathbb{Z}\pi(a_1) \oplus \mathbb{Z}\pi(P)$ to $C_3(\hat{X})$, and extend ∂_3 to $\bar{C}_3$ so as to kill $\pi_2 : \bar{\partial}_3(a_1) = a_1, \bar{\partial}_3(P) = P$. In order to restrict $k(\hat{X}) \in H^3(\pi; \pi_2)$ to $H^3(H; \pi_2)$, also define $\bar{C}_3(X) = \mathbb{Z}H(a_1) \oplus \mathbb{Z}H(a_2)$, $\bar{\partial}_3(a_i) = a_i$. The natural inclusion $C_*(\tilde{X}) \hookrightarrow \mathbb{Z}\pi \otimes_{\mathbb{Z}H} C_*(\tilde{X})$ extends to a chain map $C_*(\tilde{X}) \oplus \bar{C}_3(X) \to C_*(\hat{X}) \oplus \bar{C}_3(\hat{X})$ by defining

$$\bar{C}_3(X) \to \bar{C}_3(\hat{X}) \oplus \mathbb{Z}\pi(e^3)$$

$$a_1 \to a_1$$

$$a_2 \to -a_1 + (1 + x^{-1} - t^{-1})P + e^3.$$

We collect this information in Figure 8. The top row will allow us to further restrict $k(\hat{X})$ to $H^3(H_i; \pi_2)$.

Observe now that this discussion applies almost verbatim for K'. The only difference is $\partial_3 e^3 = -a_1 + a_2 - (1 + x^{-1} - t^{-1})P$, so that

$$\bar{C}_3(X') \to \bar{C}_3(\hat{X}') \oplus \mathbb{Z}\pi(e^3)$$

$$a_1 \mapsto a_1$$

$$a_2 \mapsto a_1 + (1 + x^{-1} - t^{-1})P + e^3.$$

§5. Computation of the k-invariant

We now identify $k(\hat{X})$ as an element of $H^3(\pi_1; \pi_2)$. We need the following: recall $\pi = G *_{\mathbb{Z}} H$

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}H_i \arrow[r, "{(1-x,*)}"] & \mathbb{Z}H_i(a_i) \oplus C_3\tilde{X}_i \arrow[r, "\partial_3"] \arrow[d, hook] & C_2\tilde{X}_i \arrow[r, "\partial_2"] \arrow[d, hook] & C_1\tilde{X}_i \arrow[r] \arrow[d, hook] & \mathbb{Z}A_i \arrow[r] \arrow[d, hook] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
& & \overbrace{\mathbb{Z}H(a_1) \oplus \mathbb{Z}H(a_2)}^{\bar{C}_3X} \oplus C_3\tilde{X} \arrow[r, "\partial_3"] \arrow[d, "{\begin{array}{c}\bar{C}_3X\\ \downarrow\\ \bar{C}_3\hat{X}\end{array}\begin{pmatrix} 1 & 0 \\ -1 & 1+x^{-1}-t^{-1} \end{pmatrix}}"'] & C_2\tilde{X} \arrow[r, "\partial_2"] \arrow[d, hook] & C_1\tilde{X} \arrow[r] \arrow[d, hook] & \mathbb{Z}H \arrow[r] \arrow[d, hook] & \mathbb{Z} \arrow[r] \arrow[d, hook] & 0 \\
& & \underbrace{\mathbb{Z}\pi(a_1) \oplus \mathbb{Z}\pi(P)}_{\bar{C}_3\hat{X}} \oplus \mathbb{Z}\pi(e^3) \oplus (\mathbb{Z}\pi \otimes_{\mathbb{Z}H} C_3\tilde{X}) \arrow[r, "\partial_3"] & \mathbb{Z}\pi(P) \oplus \mathbb{Z}\pi(e^2) \oplus (\mathbb{Z}\pi \otimes_{\mathbb{Z}H} C_2\tilde{X}) \arrow[r, "\partial_2"] & \mathbb{Z}\pi(t) \oplus (\mathbb{Z}\pi \otimes_{\mathbb{Z}H} C_1\tilde{X}) \arrow[r] & \mathbb{Z}\pi \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}
\ker \partial_2 = \mathbb{Z}H(a_1) \oplus \mathbb{Z}H(a_2) \arrow[r, hook] \arrow[d] & C_2\tilde{X} \\
\pi_2 X = \mathbb{Z}H/(1-x) \oplus \mathbb{Z}H/(1-x) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{Z}\pi(P) \oplus \mathbb{Z}\pi(e^2) \oplus (\mathbb{Z}\pi \otimes_{\mathbb{Z}H} C_2\tilde{X}) \\
\ker \partial_2 = \mathbb{Z}\pi(P) \otimes \mathbb{Z}\pi(a_1) \oplus \mathbb{Z}\pi(a_2) \arrow[u, phantom, "\cup"] \arrow[d] \\
\pi_2 = \mathbb{Z}\pi/(1-x)(1+x^{-1}-t^{-1}) \oplus \mathbb{Z}\pi/(1-x)
\end{tikzcd}
$$

Figure 8

(1) Cohomological dimension $G = 2$ by Lyndon's theorem (or, more simply, G is an HNN extension of $\mathbb{Z}$)

(2) $H_i = A_i \rtimes \mathbb{Z}$ has $\Sigma_i \times_{\sigma_i} S^1$ as a $K(H_i, 1)$, so Lemma 2.2 applies.

The Mayer–Vietoris sequences for the amalgamations $\pi = G *_{\mathbb{Z}} H = G *_{\mathbb{Z}} (H_1 *_{\mathbb{Z}} H_2)$, together with Lemma 2.2, yield, for $i = 3$ or 4,

$$H^i(\pi; \mathbb{Z}\pi) \cong H^i(H; \mathbb{Z}\pi) \cong H^i(H_1; \mathbb{Z}\pi) \oplus H^i(H_2; \mathbb{Z}\pi)$$
$$= \begin{cases} (\mathbb{Z} \otimes_{\mathbb{Z}H_1} \mathbb{Z}H) \oplus (\mathbb{Z} \otimes_{\mathbb{Z}H_2} \mathbb{Z}H); & i = 4 \\ 0; & i = 3. \end{cases}$$

Also, the long exact sequence for the coefficient sequence

$$0 \longrightarrow \mathbb{Z}\pi \oplus \mathbb{Z}\pi \xrightarrow{\begin{pmatrix} 1-x & 0 \\ 0 & (1-x)(1+x^{-1}-t^{-1}) \end{pmatrix}} \mathbb{Z}\pi \oplus \mathbb{Z}\pi \longrightarrow \pi_2 \longrightarrow 0$$

yields

$$\begin{array}{ccccccc}
0 \longrightarrow & H^3(\pi_1; \pi_2) & \xrightarrow{\delta} & H^4(\pi; (\mathbb{Z}\pi)^2) & \longrightarrow & H^4(\pi; (\mathbb{Z}\pi)^2) & \longrightarrow \\
& \cup & & & & & \\
& k & & \cong & & \cong & \\
& \cong & & & & & \\
0 \longrightarrow & H^3(H_1; \pi_2) \oplus H^3(H_2; \pi_2) & \xrightarrow{\delta} & H^4(H_1; (\mathbb{Z}\pi)^2) \oplus H^4(H_2; (\mathbb{Z}\pi)^2) & \longrightarrow & H^4(H_1; (\mathbb{Z}\pi)^2) \oplus H^4(H_2; (\mathbb{Z}\pi)^2) & \\
& \cup & & \cong & & \cong & \\
& (k_1, k_2) & & (\mathbb{Z} \otimes_{\mathbb{Z}H_1} \mathbb{Z}\pi)^2 \oplus (\mathbb{Z} \otimes_{\mathbb{Z}H_2} \mathbb{Z}\pi)^2 & \longrightarrow & (\mathbb{Z} \otimes_{\mathbb{Z}H_1} \mathbb{Z}\pi)^2 \oplus (\mathbb{Z} \otimes_{\mathbb{Z}H_2} \mathbb{Z}\pi)^2. &
\end{array}$$

While it would be difficult to determine $H^3(\pi_1; \pi_2)$ exactly, we need only identify $\delta(k_1)$, $\delta(k_2)$.

First consider H_1. From Figure 8, we see that $k(\hat{X})$ restricts to $k_1 \in H^3(H_1; \pi_2)$ given by

$$\begin{tikzcd}
\mathbb{Z}H_1 \arrow[r, hook] \arrow[rr, bend right, "\tilde{k}_1"'] \arrow[rrr, bend right=40, "k_1"'] & \mathbb{Z}H \arrow[r, hook] & \mathbb{Z}\pi \arrow[r, hook] & \mathbb{Z}\pi/(1-x).
\end{tikzcd}$$

From the diagram

$$\begin{tikzcd}
 & & \tilde{k}_1 \arrow[r] & k_1 & \\
0 \arrow[r] & C^3(H_1; \mathbb{Z}\pi) \arrow[r, "\cdot(1-x)"] \arrow[d, "d^3"] & C^3(H_1, \mathbb{Z}\pi) \arrow[r] \arrow[d, "d^3"] & C^3(H_1, \pi_2) \arrow[r] \arrow[d, "d^3"] & 0 \\
0 \arrow[r] & C^4(H_1, \mathbb{Z}\pi) \arrow[r, "\cdot(1-x)"] & C^4(H_1, \mathbb{Z}\pi) \arrow[r] & C^4(H_1, \pi_2) \arrow[r] & 0,
\end{tikzcd}$$

we have $d^3\tilde{k}_1(1)=\tilde{k}_1\partial_4(1)=\tilde{k}_1(1-x)=1-x$, so that $\delta(k_1)$ is the natural inclusion $\mathbb{Z}H_1 \hookrightarrow \mathbb{Z}\pi$. This means $[\delta(k_1)]=(1\otimes 1, 0, 0, 0)$.

Similarly, the restriction of k to H_2 is given by

$$\begin{tikzcd} \mathbb{Z}H_2 \arrow[r, hook] \arrow[rr, bend right, "\tilde{k}_2"'] \arrow[rrr, bend right=50, "k_2"'] & \mathbb{Z}H \arrow[r, "{(-1,\,1+x^{-1}-t^{-1})}"] & \mathbb{Z}\pi\oplus\mathbb{Z}\pi \arrow[r] & \pi_2. \end{tikzcd}$$

From the diagram

$$\begin{tikzcd}[column sep=small] & & \tilde{k}_2 \arrow[r] & k_2 & \\ 0 \arrow[r] & C^3(H_2,(\mathbb{Z}\pi)^2) \arrow[r, "{\begin{pmatrix} 1-x & 0 \\ 0 & (1-x)(1+x^{-1}-t^{-1}) \end{pmatrix}}"] \arrow[d, "d^3"] & C^3(H_2,(\mathbb{Z}\pi)^2) \arrow[r] \arrow[d, "d^3"] & C^3(H_2,\pi_2) \arrow[r] \arrow[d, "d^3"] & 0 \\ 0 \arrow[r] & C^4(H_2,(\mathbb{Z}\pi)^2) \arrow[r] & C^4(H_2,(\mathbb{Z}\pi)^2) \arrow[r] & C^4(H_2,\pi_2) \arrow[r] & 0, \end{tikzcd}$$

we have $d^3\tilde{k}_2(1)=\tilde{k}_2\partial_4(1)=\tilde{k}_2(1-x)=(-(1-x),\ (1-x)(1+x^{-1}-t^{-1}))$, so that $[\delta(k_2)]=(0,0,-1\otimes 1, 1\otimes 1)$. Thus, in $(\mathbb{Z}\otimes_{\mathbb{Z}H_1}\mathbb{Z}\pi)^2\oplus(\mathbb{Z}\otimes_{\mathbb{Z}H_2}\mathbb{Z}\pi)^2$,

$$k(\hat{X})=(1\otimes 1, 0, -1\otimes 1, 1\otimes 1).$$

The same arguments applied to $\hat{X}'$ yield $k(\hat{X}')=(1\otimes 1, 0, 1\otimes 1, 1\otimes 1)$. The similarity of these calculations to those in §2 should now be evident.

§6. Automorphisms of π

The purpose of this section is to put an element of Aut π into a standard form. We begin with two lemmas in combinatorial group theory.

LEMMA 6.1. *Given a free product with amalgamation $A *_C B$, let $a\in A$ be such that there is no $\bar{a}\in A$ with $\bar{a}a\bar{a}^{-1}\in C$. Then $waw^{-1}\in A$ implies $w\in A$.*

Proof. Recall that each $w\in A *_C B$ has a unique normal form $w=cd_1\cdots d_n$, where the d_i are chosen alternately from fixed coset representatives for $C\backslash A$ and $C\backslash B$. We say w as above has length n, [20]. Elements of A and B have length ≤ 1. Suppose $waw^{-1}\in A$. If $d_n\notin A$, $waw^{-1}=cd_1\cdots d_n a d_n^{-1}\cdots d_1^{-1}c^{-1}$ has length $2n+1>1$, a contradiction. Thus $d_n\in A$. Since $d_n a d_n^{-1}\notin C$, waw^{-1} has length $2n-1$. Hence, $n=1$, and $w=cd_n\in A$. ∎

The next lemma complements the well-known fact that torsion elements in a free product lie in a conjugate of one factor [20, p. 187].

LEMMA 6.2. *Given $w \in A * B$ with $1 \neq w^l \in A$ for some positive integer l, then $w \in A$.*

Proof. It suffices to prove: If $w^l \in cAc^{-1}$ with $c \in A$ or $c \in B$, then $w \in cAc^{-1}$. Write $w = c_1 \cdots c_n$, with c_i alternately in A or B. We induct on n. If $n = 1$, either $w \in A$, in which case there is nothing to prove, or $w \in B$. But this is impossible, since $c \in A$ implies $w^l \in A \cap B$, absurd, or $c \in B$ and $w^l \in cAc^{-1} \cap B$, also absurd.

If $n > 1$ and c_1, c_n are from different factors, then w^l has length $ln \geq 4$. But $w^l \in cAc^{-1}$ has length ≤ 3. Thus, c_1 and c_n are from the same factor, and $\bar{w} = c_1^{-1} w c_1 = c_2 \cdots c_{n-1}(c_n c_1)$ has length $\leq n-1$, with $\bar{w}^l \in (c_1^{-1}c)A(c_1^{-1}c)^{-1}$. Since c and c_1 are from the same factor, induction gives $\bar{w} \in (c_1^{-1}c)A(c_1^{-1}c)^{-1}$, hence $w \in cAc^{-1}$. ■

Recall $\pi = G *_{\mathbb{Z}} H = G *_{\mathbb{Z}} (H_1 *_{\mathbb{Z}} H_2)$, where $H_i = A_i \rtimes \mathbb{Z} = (A_i, x \mid xax^{-1} = \sigma_i(a))$, $A_i = \pi_1 \Sigma_i$, Σ_i a closed, orientable, aspherical 3-manifold, and $G = (t, x \mid txt^{-1} = x^2)$. Actually, all we need is that A_i is a 3-dimensional Poincaré duality group. Our result concerning Aut π is

PROPOSITION 6.3. *Let $\alpha \in \operatorname{Aut} \pi$. Up to conjugation, α has the form*

$$\alpha : \begin{cases} A_1 \to A_i \\ A_2 \to zA_j z^{-1} \\ \ \ x \to x^{\pm 1}, \quad \text{where } zx = xz. \end{cases}$$

Here $\{i, j\} = \{1, 2\}$ as sets. If $A_1 \cong A_2$, α can possibly interchange A_1 and A_2.

Proof. From the structure theorem for subgroups of amalgamated products [20, p. 243] $\alpha(A_1)$ is a free product of subgroups of conjugates of H_i or G, amalgamated along conjugates of subgroups of $\mathbb{Z}$. We claim that $\alpha(A_1)$ is contained in either H_i or G. Otherwise, $\alpha(A_1) = B_1 *_C B_2$, with $C = \{1\}$ or $\mathbb{Z}$, $B_1 \neq C \neq B_2$. The Mayer–Vietoris sequence for this decomposition, with $\mathbb{Z}_2$-coefficients, yields $0 \to H_3(B_1) \oplus H_3(B_2) \to \mathbb{Z}_2 \to 0$. This forces one of the B_i to have a $\mathbb{Z}_2$-fundamental class, hence have finite index in $\alpha(A_1)$, contradicting the non-triviality of the splitting. (Alternatively, such a decomposition of $\alpha(A_1)$ would give rise to an incompressible annulus or 2-sphere in Σ_1 [14], a contradiction.)

If $\alpha(A_1) \subset G$, then $\alpha(A_1) \subset [G, G]$, since $A_1 \subset [\pi, \pi]$ and $\pi/[\pi, \pi] \cong G/[G, G] \cong \mathbb{Z}(t)$. But $[G, G] \otimes \mathbb{Q} \cong \mathbb{Z}[\frac{1}{2}] \otimes \mathbb{Q} \cong \mathbb{Q}$, so the only non-trivial, finitely generated subgroup of $[G, G]$ is $\mathbb{Z}$, a contradiction.

Thus $\alpha(A_1) \subset H_i = A_i \rtimes \mathbb{Z}$. Now $\alpha(A_1) \not\subset \mathbb{Z}$, so $\alpha(A_1) \cap A_i \neq \{1\}$, say $a \in \alpha(A_1) \cap A_i$. If $y \in H_i$, then $yay^{-1} \in A_i$. Also, since x normalizes A_1, $\alpha(x) a \alpha(x)^{-1} \in H_i$. Applying Lemma 6.1, we see that $\alpha(x) \in H_i$, so $\alpha(H_1) \subset H_i$. Since A_1 and A_i are the commutator subgroups of H_1 and H_i respectively, $\alpha(A_1) \subset A_i$.

We claim that $\alpha(A_1) = A_i$. Otherwise, let $[A_i : \alpha(A_1)] = l > 1$. Since the A_i are Poincaré duality groups, $l < \infty$. Thus, there exists $a_i \in A_i - \alpha(A_1)$, $a_i^l \in \alpha(A_1)$. Then $a = \alpha^{-1}(a_i) \notin A_1$, but $a^l \in A_1$.

Let $\langle A_1 \rangle$ be the normal closure of A_1 in π. The map $\pi \to \pi/\langle A_1 \rangle = G *_{\mathbb{Z}} H_2$ takes a to $\bar{a}$. If $a \notin \langle A_1 \rangle$, then $\bar{a} \neq 1$. But $\bar{a}^l = 1$, contradicting the fact that $G *_{\mathbb{Z}} H_2$ is torsion free. So $a \in \langle A_1 \rangle$. Now $\langle A_1 \rangle$ is a free product of conjugates of A, since the only possible amalgamating subgroups are conjugates of subgroups of $\mathbb{Z}$, but $\mathbb{Z}$ maps monomorphically to $\pi/\langle A_1 \rangle$. Thus, $a \in \langle A_1 \rangle = A_1 * B$, $a \notin A_1$, $a^l \in A_1$. This contradicts Lemma 6.2 and therefore proves our claim: $\alpha(A_1) = A_i$.

Similarly, $\alpha(A_2) = zA_jz^{-1}$ for some z. We must have $\{i, j\} = \{1, 2\}$ as sets, since $\alpha(A_2) = zA_iz^{-1}$ gives $A_1 = \alpha^{-1}(A_i) = \alpha^{-1}(z)^{-1} A_2 \alpha^{-1}(z)$, an absurdity.

Notice also that Lemma 6.1 implies $N_\pi(A_i) = H_i$. Since $x \in N_\pi(A_1)$, $\alpha(x) \in N_\pi(A_i) = H_i$, so $\alpha(x) = x^k y_i$, $y_i \in A_i$. If $|k| > 1$, $\alpha|_{H_1} : H_1 \to H_i$ has index $|k|$, so $x \notin \alpha(H_1)$. But then $\alpha^{-1}(A_i) = A_1$, $x \in N_\pi(A_i)$ implies $\alpha^{-1}(x) \in N_\pi(A_1) = H_1$, a contradiction. Thus, $\alpha(x) = x^{\pm 1} y_i$. Similarly, $x \in N_\pi(A_2)$, so that $\alpha(x) = x^{\pm 1} y_i \in N_\pi(zA_jz^{-1})$, so $z^{-1} x^{\pm 1} y_i z \in N_\pi(A_j)$, hence $z^{-1} x^{\pm 1} y_i z = x^k y_j$, $y_j \in A_j$. Let $\mu_{z^{-1}}$ denote conjugation by z^{-1}. Then

$$\mu_{z^{-1}} \circ \alpha|_{H_2} : \begin{cases} A_2 \to A_2 \\ x \to x^k y_j, \end{cases}$$

so as above, $k = \pm 1$.

LEMMA 6.4. *There is no* $z \in G = (t, x \mid txt^{-1} = x^2)$ *such that* $zxz^{-1} = x^{-1}$.

Proof. $G = [G, G] \rtimes \mathbb{Z}$ with $[G, G] \cong \mathbb{Z}[\frac{1}{2}]$ generated by $t^{-k} x t^k$, $k \geq 0$. Write $z = wt^n$, $w \in [G, G]$. If $n \geq 0$, $zxz^{-1} = wt^n x t^{-n} w^{-1} = w x^{2^n} w^{-1} = x^{2^n} \neq x^{-1}$. If $n < 0$, $zxz^{-1} = wt^n x t^{-n} w^{-1} = t^n x t^{-n} \neq x^{-1}$. ■

Lemma 6.4 implies, via projection to G, that $z^{-1} x^{\pm 1} y_i z = x^{\pm 1} y_j$, with corresponding exponents for x. We now have α in the form

$$\alpha : \begin{cases} A_1 \to A_i \\ A_2 \to zA_jz^{-1} \\ x \to x^{\pm 1} y_i, \quad z^{-1} x^{\pm 1} y_i z = x^{\pm 1} y_j. \end{cases}$$

LEMMA 6.5. *Assume that $\alpha \in \mathrm{Aut}\,\pi$ satisfies*

$$\alpha: \begin{cases} A_1 \to A_i \\ A_2 \to zA_jz^{-1} \\ x \to x^{\pm 1}y_i, \qquad z^{-1}x^{\pm 1}y_iz = x^{\pm 1}y_j. \end{cases}$$

Then α can be conjugated to the form

$$\alpha: \begin{cases} A_1 \to A_i \\ A_2 \to z_1A_jz_1^{-1} \\ x \to x^{\pm 1}, \quad \textit{with} \quad z_1^{-1}x^{\pm 1}z_1 = x^{\pm 1}y_j. \end{cases}$$

Assuming the lemma, we finish the proof. Let α be as in the conclusion of the lemma. Then conjugation by z_1^{-1} puts α in the form

$$\alpha: \begin{cases} A_2 \to A_j \\ A_1 \to z_1^{-1}A_iz_1 \\ x \to x^{\pm 1}y_j, \qquad z_1x^{\pm 1}y_jz_1^{-1} = x^{\pm 1}. \end{cases}$$

Reversing the roles of A_1 and A_2, we apply the lemma again, conjugating α to

$$\alpha: \begin{cases} A \to A_j \\ A_1 \to z_2A_iz_2^{-1} \\ x \to x^{\pm 1}, \qquad z_2^{-1}x^{\pm 1}z_2 = x^{\pm 1}. \end{cases}$$

Conjugation by z_2^{-1} now puts α in its final form, and proves the proposition. ■

Proof of Lemma 6.5. If $y_i = 1$, there is nothing to prove. If $y_i \neq 1$, there are two cases:

CASE 1. There exists $y \in A_i$ such that $y_i = \sigma_i^{\mp 1}(y^{-1})y$.

In this case, $yx^{\pm 1}y_iy^{-1} = x^{\pm 1}\sigma_i^{\mp 1}(y)y_iy^{-1} = x^{\pm 1}$. Hence

$$\mu_y \circ \alpha : \begin{cases} A_1 \to A_i \\ A_2 \to z_1 A_j z_1^{-1} \\ x \to x^{\pm 1}, \qquad z_1 = yz, \quad \text{and} \end{cases}$$

$$z_1^{-1}x^{\pm 1}z_1 = z^{-1}y^{-1}x^{\pm 1}yz = z^{-1}x^{\pm 1}\sigma^{\mp 1}(y^{-1})yz = z^{-1}x^{\pm 1}y_iz = x^{\pm 1}y_j.$$

CASE 2. There does not exist $y \in A_i$ such that $y_i = \sigma_i^{\pm 1}(y^{-1})y$.

Write $z^{-1} = x^k d_1 \cdots d_n$, $d_i \in \mathbb{Z}\backslash G, \mathbb{Z}\backslash H_1$, or $\mathbb{Z}\backslash H_2$ alternately. If $d_n \notin H_i$, then $z^{-1}x^{\pm 1}y_iz = x^k d_1 \cdots d_n x^{\pm 1}y_i d_n^{-1} \cdots d_1^{-1}x^{-k}$ has length $2n+1$. But length $x^{\pm 1}y_j \leq 1$. Thus, either $n = 0$, i.e. $z^{-1} = x^k$, in which case the lemma is trivial, or $d_n \in H_i$, $d_n = x^r y$, $y \in A_i$. Then

$$\begin{aligned} z^{-1}x^{\pm 1}y_iz &= x^k d_1 \cdots d_{n-1}x^r yx^{\pm 1}y_iy^{-1}x^{-r}d_{n-1}^{-1} \cdots d_1^{-1}x^{-k} \\ &= x^k d_1 \cdots d_{n-1}x^{\pm 1}\sigma_i^r(\sigma_i^{\mp 1}(y)y_iy^{-1})d_{n-1}^{-1} \cdots d_1^{-1}x^{-k}, \end{aligned}$$

which has length $2n-1$, since $\sigma_i^{\mp 1}(y)y_iy^{-1} \neq 1$. This forces $n = 1$, so that $z \in H_i$. But this forces $y_i = 1$, and now

$$\mu_{z^{-1}} \circ \alpha : \begin{cases} A_1 \to A_i \\ A_2 \to A_j \\ x \to x^{\pm 1} \quad \text{is in the required form.} \end{cases} \blacksquare$$

The proof of Lemma 6.5 indicates how to find complicated elements of Aut π. Suppose $\sigma_1(y_1) = y_1$, so that $xy_1 = y_1x$. For instance, if $\sigma_1 : \Sigma_1 \to \Sigma_1$ fixes a circle, let y_1 be the class of the circle in $\pi_1(\Sigma_1)$. We can define an element of Aut π by

$$\alpha : \begin{cases} A_1 \xrightarrow{=} A_1 \\ A_2 \xrightarrow{\mu_z} zA_2z^{-1}, \qquad z = t^{-1}y_1t \\ x \longrightarrow x \\ t \longrightarrow t. \end{cases}$$

There is also considerable freedom in the choice of $\alpha(t)$, so long as $\alpha(t)$ has exponent sum ± 1 in t. It is easy to verify that the map

$$\alpha:\begin{cases} H \xrightarrow{=} H \\ t \to t^{-1}xt^2 \end{cases}$$

is an automorphism, with $\alpha^{-1}(t)=t^{-1}x^{-1}t^2$. Composition of these and similar α give elements of Aut π with both z and $\alpha(t)$ rather complicated.

§7. Proof of the Theorem

We now prove Theorem 1.2. Assume there is a map $f:\hat{X}\to\hat{X}'$ inducing an isomorphism $\alpha:\pi_1\hat{X}\to\pi_1\hat{X}'$ and an α-map $\beta:\pi_2\hat{X}\to\pi_2\hat{X}'$. Then α and β preserve k-invariants. The guiding principle here is that if we precede f by a map $\hat{X}\to\hat{X}$, or follow f by a map $\hat{X}'\to\hat{X}'$, the composed maps on π_1 and π_2, being geometrically realizable, preserve k-invariants.

By Proposition 6.3, we may conjugate α by an element of π into the form

$$\begin{cases} A_1 \xrightarrow{f_1} A_i \\ A_2 \xrightarrow{f_2} A_j \xrightarrow{\mu_z} zA_jz^{-1} \\ x \longrightarrow x^{\pm 1}, \end{cases}$$

where $\{i,j\}=\{1,2\}$, the f_i are isomorphisms, and $zx=xz$. Conjugations are geometrically realizable. Thus, we may assume α has the above form.

Applying α to the relations $xa_i=\sigma_i(a_i)x$, and using $zx=xz$, we find

$$\begin{aligned} f_1 &= \sigma_i^{\mp 1}\circ f_1\circ\sigma_1 \\ f_2 &= \sigma_j^{\mp 1}\circ f_2\circ\sigma_2. \end{aligned} \tag{1}$$

Since $\Sigma_i=K(A_i,1)$, we may find homotopy equivalences

$$\begin{aligned} &F_1:\Sigma_1\to\Sigma_i, \qquad (F_1)_*=f_1 \\ &F_2:\Sigma_2\to\Sigma_j, \qquad (F_2)_*=f_2. \end{aligned}$$

Assume $i=1$, $j=2$. Then, *since the Σ_l do not admit orientation reversing homotopy equivalences*, we may assume there are small balls B_l^3 pointwise fixed by F_l[8], so that the F_i fit together to give a homotopy equivalence $F:\Sigma_1 \# \Sigma_2 \to \Sigma_1 \# \Sigma_2$, with $F_*|_{A_1}=f_1$, $F_*|_{A_2}=f_2$. If $i=2$, $j=1$, a similar argument yields a homotopy equivalence $F:\Sigma_1 \# \Sigma_2 \to \Sigma_2 \# \Sigma_1$. In either case, we see from (1) that $F_* = \bar{\sigma}^{\mp 1}\circ F_* \circ \sigma$, where $\sigma = \sigma_1 * \sigma_2$, $\bar{\sigma} = \sigma_i * \sigma_j$. This is precisely the compatibility condition we require to construct a fiber-preserving homotopy equivalence

$$\tilde{F}: \Sigma_1 \# \Sigma_2 \times_\sigma S^1 \to \Sigma_i \# \Sigma_j \times_{\bar{\sigma}} S^1,$$
$$\tilde{F}|_{\Sigma_1 \# \Sigma_2} = F$$
$$\tilde{F}(x) = x^{\pm 1}.$$

We may assume $\tilde{F}$ preserves a $D^3 \times S^1$, in which we take connected sum with $S^1 \times D^3$ and do surgery, so that $\tilde{F}$ extends to a homotopy equivalence $H:\hat{X} \to \hat{X}$. Finally, replacing the map $f:\hat{X} \to \hat{X}'$ by $f\circ H^{-1}$ enables us to assume that α has the form

$$\alpha:\begin{cases} A_1 \xrightarrow{=} A_1 \\ A_2 \xrightarrow{\mu_z} zA_2z^{-1} \\ x \longrightarrow x. \end{cases}$$

Now examine k-invariants. From $\alpha|_{H_1} = \mathrm{id}$ we find

$$\begin{tikzcd}
k' \in H^3(\pi_1;\pi_2) \arrow[r, "\alpha^*"] \arrow[d] & H^3(\pi_1;(\pi_2)_\alpha) \arrow[d] & H^3(\pi_1;\pi_2) \ni k \arrow[l, "\beta_*"'] \arrow[d] \\
k_1' \in H^3(H_1;\pi_2) \arrow[r, "="] & H^3(H_1;\pi_2) & H^3(H_1;\pi_2) \ni k_1 \arrow[l, "\beta_*"']
\end{tikzcd}$$

so that $\beta_*(k_1) = k_1'$. Similarly, $\mu_{z^{-1}}\circ\alpha|_{H_2} = \mathrm{id}$ yields

$$\begin{tikzcd}
k' \in H^3(\pi_1;\pi_2) \arrow[r, "(\mu_{z^{-1}})^*"] \arrow[d] & H^3(\pi_1;(\pi_2)_{\mu_{z^{-1}}}) \arrow[r, "\alpha^*"] & H^3(\pi_1;(\pi_2)_{\mu_{z^{-1}}\alpha}) \arrow[d] & H^3(\pi_1;(\pi_2)_\alpha) \arrow[l, "(\cdot z^{-1})_*"'] & H^3(\pi_1;\pi_2) \in \arrow[l, "\beta_*"'] \arrow[d] \\
k_2' \in H^3(H_2;\pi_2) \arrow[rr, "="] & & H^3(H_2;\pi_2) & & H^3(H_2;\pi_2) \ni k_2 \arrow[ll, "(z^{-1}.\beta)_*"']
\end{tikzcd}$$

so that $(z^{-1}\beta)_* k_2 = k_2'$.

The α-map β lifts to an α-map $\theta:(\mathbb{Z}\pi)^2 \to (\mathbb{Z}\pi)^2$ which restricts to $\bar{\theta}$:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}\pi \oplus \mathbb{Z}\pi \arrow[rr, "{\begin{pmatrix} 1-x & 0 \\ 0 & (1-x)(1+x^{-1}-t^{-1}) \end{pmatrix}}"] \arrow[d, "\bar{\theta}=\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}"] & & \mathbb{Z}\pi \oplus \mathbb{Z}\pi \arrow[r] \arrow[d, "\theta=\begin{pmatrix} a & b \\ c & d \end{pmatrix}"] & \pi_2 \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & \mathbb{Z}\pi \oplus \mathbb{Z}\pi \arrow[rr] & & \mathbb{Z}\pi \oplus \mathbb{Z}\pi \arrow[r] & \pi_2 \arrow[r] & 0,
\end{tikzcd}$$

$a, b, c, d, \bar{a}, \bar{b}, \bar{c}, \bar{d} \in \mathbb{Z}\pi$. From the commutativity of the diagram we find

$$(1-x)a = \bar{a}(1-x) \tag{2a}$$

$$(1-x)b = \bar{b}(1-x)(1+x^{-1}-t^{-1}) \tag{2b}$$

$$(1-x)(1+x^{-1}-\alpha(t^{-1}))c = \bar{c}(1-x) \tag{2c}$$

$$(1-x)(1+x^{-1}-\alpha(t^{-1}))d = \bar{d}(1-x)(1+x^{-1}-t^{-1}). \tag{2d}$$

Recall that we identify $H^3(\pi_1; \pi_2)$ with its image in $H^4(\pi; (\mathbb{Z}\pi)^2)$, and that the action of $\bar{\theta}$ here is given by right multiplication on the cosets $H_i\backslash\pi$. In particular,

$$\begin{aligned}
H^3(H_1; \pi_2) \hookrightarrow H^4(H_1; (\mathbb{Z}\pi)^2) &\cong \Big(\bigoplus_{H_1\backslash\pi} \mathbb{Z}\Big) \oplus \Big(\bigoplus_{H_1\backslash\pi} \mathbb{Z}\Big) \\
k_1, k_1' \longrightarrow\ & (H_1 \cdot 1, 0),
\end{aligned}$$

so $\beta_* k_1 = k_1'$ gives

$$H_1 \cdot \bar{a} = H_1 \cdot 1 \tag{3a}$$

$$H_1 \cdot \bar{b} = 0. \tag{3b}$$

Also,

$$\begin{aligned}
H^3(H_2; \pi_2) \hookrightarrow H^4(H_2; (\mathbb{Z}\pi)^2) &= \Big(\bigoplus_{H_2\backslash\pi} \mathbb{Z}\Big) \oplus \Big(\bigoplus_{H_2\backslash\pi} \mathbb{Z}\Big) \\
k_2 \longrightarrow\ & (-H_2 \cdot 1, H_2 \cdot 1) \\
k_2' \longrightarrow\ & (H_2 \cdot 1, H_2 \cdot 1),
\end{aligned}$$

so $(z^{-1} \cdot \beta)_* k_2 = k_2'$ gives

$$H_2 \cdot z^{-1}\bar{c} - H_2 \cdot z^{-1}\bar{a} = H_2 \cdot 1 \tag{4a}$$

$$H_2 \cdot z^{-1}\bar{d} - H_2 \cdot z^{-1}\bar{b} = H_2 \cdot 1. \tag{4b}$$

Equations (2), (3), (4) are the condition that α and β preserve k-invariants, and we see that the question of whether a map $\hat{X} \to \hat{X}'$ inducing α exists has been reduced to a question of whether these equations over $\mathbb{Z}\pi$ have solutions. *We claim that* (2), (3), (4) *have no solutions.*

To see this, consider the projection $\pi = G *_{\mathbb{Z}} H \to G \to D_3 = (t, x \mid txt^{-1} = x^2, t^2 = 1)$ onto the dihedral group of order 6, say $\lambda : \pi \to D_3$. This induces a ring homomorphism $\lambda : \mathbb{Z}\pi \to \mathbb{Z}D_3$. Notice that $\langle x \rangle = \mathbb{Z}_3 \subset D_3$, $\mathbb{Z}_3 \backslash D_3 = \mathbb{Z}_2$. Let $p_i : \mathbb{Z}\pi \to \mathbb{Z}[H_i \backslash \pi]$ be the (abelian group) homomorphism induced by $\pi \to H_i \backslash \pi$, and $p : \mathbb{Z}D_3 \to \mathbb{Z}[\mathbb{Z}_3 \backslash D_3]$ the (ring) homomorphism induced by $D_3 \to \mathbb{Z}_2$. Since $\lambda(H_i) = \langle x \rangle \subset D_3$, there are maps $\lambda_i : H_i \backslash \pi \to \mathbb{Z}_3 \backslash D_3$ which induce (abelian group) homomorphisms $\lambda_i : \mathbb{Z}[H_i \backslash \pi] \to \mathbb{Z}[\mathbb{Z}_3 \backslash D_3]$. These yield a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}\pi & \xrightarrow{\lambda} & \mathbb{Z}D_3 \\ {\scriptstyle p_i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \mathbb{Z}[H_i \backslash \pi] & \xrightarrow{\lambda_i} & \mathbb{Z}[\mathbb{Z}_3 \backslash D_3]. \end{array}$$

Since $zx = xz$, $\lambda(z) \in \langle x \rangle \subset D_3$. Therefore, projecting (3a), (4a) to $\mathbb{Z}[\mathbb{Z}_3 \backslash D_3]$, we find

$$p\lambda(\bar{c}) = 2(\mathbb{Z}_3 \cdot 1). \tag{5}$$

Also, since t generates $H_1(\pi) \cong \mathbb{Z}$, $\alpha(t)$ must also, and thus $\lambda\alpha(t) = t$, tx, or tx^{-1}. Write $\lambda(c) = \sum n_g g$, $\lambda(\bar{c}) = \sum \bar{n}_g g$. Equations (2c) followed by λ, and (5), yield

$$(1 - x)(1 + x^{-1} - tx^j)(\textstyle\sum n_g g) = (\textstyle\sum \bar{n}_g g)(1 - x), \qquad j = 0, 1, -1 \tag{6a}$$

$$\bar{n}_1 + \bar{n}_x + \bar{n}_{x^{-1}} = 2 \tag{6b}$$

$$\bar{n}_t + \bar{n}_{tx} + \bar{n}_{tx^{-1}} = 0. \tag{6c}$$

For the reader's convenience, we write out (6a) when $j = 0$. Keeping in mind that $y \cdot (\sum n_g g) = \sum n_{y^{-1}g} g$ and $(\sum n_g g) \cdot y = \sum n_{gy^{-1}} g$, and writing

$$(1 - x)(1 + x^{-1} - t^{-1}) = x^{-1} - x - t + tx^{-1},$$

we find

$$n_x - n_{x^{-1}} - n_t + n_{tx^{-1}} = \bar{n}_1 - \bar{n}_{x^{-1}}$$
$$n_{x^{-1}} - n_1 - n_{tx} + n_t = \bar{n}_x - \bar{n}_1$$
$$n_1 - n_x - n_{tx^{-1}} + n_{tx} = \bar{n}_{x^{-1}} - \bar{n}_x$$
$$n_{tx^{-1}} - n_{tx} - n_1 + n_x = \bar{n}_t - \bar{n}_{tx^{-1}}$$
$$n_t - n_{tx^{-1}} - n_x + n_{x^{-1}} = \bar{n}_{tx} - \bar{n}_t$$
$$n_{tx} - n_t - n_{x^{-1}} + n_1 = \bar{n}_{tx^{-1}} - \bar{n}_{tx}.$$

Hence,

$$\bar{n}_1 - \bar{n}_{x^{-1}} = \bar{n}_t - \bar{n}_{tx}$$
$$\bar{n}_x - \bar{n}_1 = -\bar{n}_{tx^{-1}} + \bar{n}_{tx}.$$

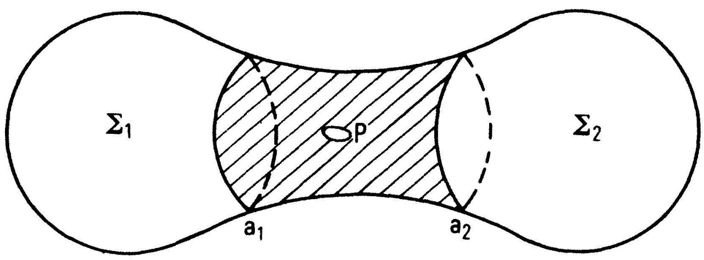
Combining these with (6b), (6c) leads to $3\bar{n}_1 - 2 = -3\bar{n}_{tx}$, a contradiction.

The cases $j = \pm 1$ are similar. For $j = 1$, $(1-x)(1+x^{-1}-tx) = x^{-1} - x + t - tx$, (6a) reduces to $\bar{n}_1 - \bar{n}_{x^{-1}} = \bar{n}_{tx} - \bar{n}_{tx^{-1}}$, $\bar{n}_x - \bar{n}_1 = \bar{n}_{tx^{-1}} - \bar{n}_t$, which combine with (6b), (6c) to give $3\bar{n}_1 - 2 = 3\bar{n}_{tx} + 3\bar{n}_t$. When $j = -1$, (6a) becomes $\bar{n}_1 - \bar{n}_{x^{-1}} = \bar{n}_{tx^{-1}} - \bar{n}_t$, $\bar{n}_x - \bar{n}_1 = \bar{n}_t - \bar{n}_{tx}$, which, together with (6b), (6c), give $3\bar{n}_1 - 2 = -3\bar{n}_t$. Thus, equations (2), (3), (4), when projected to $\mathbb{Z}D_3$, have no solutions, hence have no solutions in $\mathbb{Z}\pi$. This completes the proof of Theorem 1.2. ■

There are several aspects of the proof which deserve further comment. First why pick such complicated examples? Why not just consider the knots $K = K_1 \# K_2$ and $K' = -K_1 \# K_2$, with exterior $(\pm\Sigma_2 \# \Sigma_2)^\circ \times_\sigma S^1$? Notice that this is the same as performing surgery on the curve tx in $S^1 \times D^3 \# ((\pm\Sigma_1 \# \Sigma_2) \times_\sigma S^1)$. Why use the more complicated $txt^{-1}x^{-2}$?

Consider $(\pm\Sigma_1 \# \Sigma_2)^0 \times_\sigma S^1$. This deforms to $(\Sigma_1^{(2)} \vee \Sigma_2^{(2)}) \times_\sigma S^1$, where $\Sigma_i^{(2)}$ is the 2-skeleton of Σ_i. The orientation information is lost, and the complements are homotopy equivalent. Indeed, they push back to identical 3-complexes. (Notice that they cannot be homotopy equivalent (rel ∂) since the fibers are not homotopy equivalent (rel ∂). Similar examples, using lens spaces, are in [22].)

This may be seen algebraically as follows. We have the following picture for $(\pm\Sigma_1 \# \Sigma_2)^0$, where $\pm\Sigma_1 \# \Sigma_2$ gives $P = \pm a_1 + a_2$. Deforming to 2-skeleta gives a



homotopy equivalence $(\Sigma_1\#\Sigma_2)^0\to(-\Sigma_1\#\Sigma_2)^0$, inducing $\alpha=\mathrm{id}$ on π_1, which takes $a_i\to a_i$. Hence, $P\to 2a_1+P$, so we can write it, in the basis $\{a_1, P\}$, as $\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$.

For the knot exteriors, π_2 and $\beta:\pi_2\to\pi_2$ are given by

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}\pi\oplus\mathbb{Z}\pi & \xrightarrow{\begin{pmatrix}1-x & 0\\ 0 & 1-x\end{pmatrix}} & \mathbb{Z}\pi\oplus\mathbb{Z}\pi & \longrightarrow & \pi_2 & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\bar\theta} & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\beta} & & \\
0 & \longrightarrow & \mathbb{Z}\pi\oplus\mathbb{Z}\pi & \longrightarrow & \mathbb{Z}\pi\oplus\mathbb{Z}\pi & \longrightarrow & \pi_2 & \longrightarrow & 0.
\end{array}$$

Writing $\theta=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $\bar\theta=\begin{pmatrix}\bar a & \bar b\\ \bar c & \bar d\end{pmatrix}$ as before, the conditions imposed on θ, $\bar\theta$ are now

$$(1-x)a=\bar a(1-x)$$
$$(1-x)b=\bar b(1-x)$$
$$(1-x)c=\bar c(1-x)$$
$$(1-x)d=\bar d(1-x).$$

In particular, we can choose $\theta=\bar\theta=\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$, and now it is easy to see the k-invariants correspond, i.e. (3) and (4) are satisfied.

In our examples, however, with the more complicated module structure on π_2, the map $\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$ is not available—the generator P coming from the puncture cannot be combined so freely with the generator a coming from the separating 2-sphere in $\Sigma_1\#\Sigma_2$. Notice that, of conditions (2), (3), (4), we only used (2c), (3a), (4a). We can still pick $a=\bar a=1$, $b=\bar b=0$, $d=\bar d=1$, satisfying (2a), (2b), (2d), (3), (4b). But we cannot simultaneously satisfy (2c) and (4a). We cannot choose $\bar c\in\mathbb{Z}$. It must involve t, and thus (4a) will not hold. It seems clear that in trying to satisfy these equations one should only look inside $\mathbb{Z}G \subset \mathbb{Z}\pi$. Possibly one could show directly that there are no solutions here. It seems miraculous, however, that even in $\mathbb{Z}D_3$ there are no solutions. We regard this as clear evidence of a guiding moral force behind our examples.

We earlier mentioned the difference between homotopy equivalences and those rel boundary. Consider, for example, the square knot versus the granny knot. Spinning these two knots yields the same knot [12]. If we 7-twist-spin them, we obtain fibered knots, with fibers $(\pm\Sigma(2,3,7)\#\Sigma(2,3,7))^0$. The exteriors are homotopy equivalent, but not rel boundary. Using these in our construction essentially fixes the boundary, so the resulting knots are no longer homotopy equivalent.

Finally, we point out two ways in which these examples can be generalized. First, notice that $S^1 \times D^3 \# Y$ is the connected sum of the trivial knot exterior with Y. If we replace $S^1 \times D^3$ by the exterior of some more interesting knot $\bar{K}$, and again perform surgery on $txt^{-1}x^{-2}$, we obtain a knot in S^4 with group $\pi_1\bar{K} *_{\mathbb{Z}(t)} G *_{\mathbb{Z}(x)} H$, and one still expects Theorem 1.2 to be true. Secondly, there is nothing sacred about $txt^{-1}x^{-2}$, other than the exponent sum of x is ± 1. For instance, we could replace this by $tx^nt^{-1}x^{-n-1}$, replacing G by the Baumslag–Solitar group $(t, x \mid tx^nt^{-1} = x^{n+1})$, and one still expects Theorem 1.2 to hold, with computations in $\mathbb{Z}D_{2n+1}$ instead of $\mathbb{Z}D_3$.

§8. Generalizations

This section generalizes Theorem 1.2 from two knots to arbitrarily many:

THEOREM 8.1. *There are arbitrarily many knots in S^4 with isomorphic π_1 and π_2 but with distinct k-invariants.*

Proof (sketch). We start with n fibered knots K_i, with fiber $\mathring{\Sigma}_i$. As usual, we assume Σ_i are aspherical and admit no orientation reversing homotopy equivalences. To avoid certain technical difficulties involving interchanging factors, we further assume that all $\pi_1\Sigma_i$ are distinct.

Given an $(n-1)$-tuple $I = (\varepsilon_1, \ldots, \varepsilon_{n-1})$, $\varepsilon_i = \pm 1$, form $K_I = \varepsilon_1 K_1 \# \cdots \# \varepsilon_{n-1} K_{n-1} \# K_n$. Surgery on K_I yields Y_I, fibered over S^1 with fiber $= \#_{i=1}^n \Sigma_i$. Construct $\hat{X}_I$ by performing surgery on $txt^{-1}x^{-2}$ in $S^1 \times D^3 \# Y_I$. As before, $\hat{X}_I$ is the exterior of a knot $\hat{K}_I$ in S^4. We claim these 2^{n-1} knots have distinct k-invariants.

Write $\pi = \pi_1\hat{X}_I = G *_{\mathbb{Z}} H_1 *_{\mathbb{Z}} H_2 * \cdots *_{\mathbb{Z}} H_n$

$$H_i = A_i \rtimes \mathbb{Z}, \qquad A_i = \pi_1\Sigma_i.$$

As before, we compute $\pi_2 = \pi_2\hat{X}_I = (\mathbb{Z}\pi/1-x)^{n-1} \oplus \mathbb{Z}\pi/(1-x)(1+x^{-1}-t^{-1})$. Further, $\bar{C}_3(X_I) = \oplus_{i=1}^{n} \mathbb{Z}H(a_i)$, $\bar{C}_3(\hat{X}_I) = \oplus_{i=1}^{n-1} \mathbb{Z}\pi(a_i) \oplus \mathbb{Z}\pi(P)$, and $\bar{C}_3(X_I) \to \bar{C}_3(\hat{X}_I)$ is given by

$$\bar{C}_3(X_I) \xrightarrow{\left(\begin{array}{ccc|c} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ \hline -\varepsilon_1 & \cdots & -\varepsilon_{n-1} & 1+x^{-1}-t^{-1} \end{array}\right)} \bar{C}_3(\hat{X}_I). \text{ See §4.}$$

The exact sequence

$$0 \longrightarrow (\mathbb{Z}\pi)^n \xrightarrow{\left(\begin{array}{ccc|c} 1-x & & & 0 \\ & \ddots & & \vdots \\ & & 1-x & 0 \\ \hline 0 & \cdots & 0 & (1-x)(1+x^{-1}-t^{-1}) \end{array}\right)} (\mathbb{Z}\pi)^n \longrightarrow \pi_2 \longrightarrow 0$$

yields

$$H^3(\pi_1; \pi_2) \cong \bigoplus_{i=1}^{n} H^3(H_i; \pi_2) \to \bigoplus_{i=1}^{n} H^4(H_i; (\mathbb{Z}\pi)^n) \cong \bigoplus_{i=1}^{n} \Big(\bigoplus_{H_i \backslash \pi} \mathbb{Z} \Big)^n$$

$$k_I \to ((k_I)_i) = ((H_1 \cdot 1, 0, \ldots, 0), (0, H_2 \cdot 1, 0, \ldots, 0),$$
$$\ldots, (0, \ldots, 0, H_{n-1} \cdot 1, 0), (-\varepsilon_1 H_1 \cdot 1, \ldots, -\varepsilon_{n-1} H_{n-1} \cdot 1, H_n \cdot 1)).$$

Now suppose we have distinct $(n-1)$-tuples I, J. We can assume $\varepsilon_1^I = 1 = -\varepsilon_1^J$. Given an isomorphism α and an α-map β, we can arrange, as in §6, that $\alpha|_{H_1} = \mathrm{id}$, $\alpha(A_i) = z_i A_i z_i^{-1}$, $z_i x = x z_i$, $i = 2, \ldots, n$. Lifting β to maps $\theta, \bar{\theta}: (\mathbb{Z}\pi)^n \to (\mathbb{Z}\pi)^n$, $\theta = (a_{ij})$, $\bar{\theta} = (\bar{a}_{ij})$, we find

$$(1-x)a_{ij} = \bar{a}_{ij}(1-x) \qquad 1 \le i, j \le n-1$$

$$\left.\begin{array}{l} (1-x)a_{in} = \bar{a}_{in}(1-x)(1+x^{-1}-t^{-1}) \\ (1-x)(1+x^{-1}-\alpha(t^{-1}))a_{ni} = \bar{a}_{ni}(1-x) \end{array}\right\} \quad 1 \le i \le n-1$$

$$(1-x)(1+x^{-1}-\alpha(t^{-1}))a_{nn} = \bar{a}_{nn}(1-x)(1+x^{-1}-t^{-1}).$$

Assuming $\beta_*(k_I) = \alpha^*(k_J)$, we find, as in §7, that $(z_i^{-1}\beta)_*(k_I)_i = (k_J)_i$, $2 \le i \le n$, and $\beta_*(k_I)_1 = (k_J)_1$. This yields equations over $\mathbb{Z}\pi$. As before, we apply the projection $\lambda: \pi \to D_3$ to give equations over $\mathbb{Z}D_3$. Since $xz_i = z_i x$, we can ignore the z_i. The relevant equations which must be satisfied if the k-invariants are to be preserved are:

$$\mathbb{Z}_3\lambda(\bar{a}_{11})=\mathbb{Z}_3 \quad \text{and} \quad -\mathbb{Z}_3\lambda(\bar{a}_{11})+\mathbb{Z}_3\lambda(\bar{a}_{n1})=\mathbb{Z}_3,$$

or $\mathbb{Z}_3\lambda(\bar{a}_{n1})=2\mathbb{Z}_3$, the analogue of (5) of §7. But this equation, together with $(1-x)(1+x^{-1}-\alpha(t))\lambda(a_{n1})=\lambda(\bar{a}_{n1})(1-x)$, are (6) of §7, and lead to the same contradiction. This completes the proof. ■

We mention here two obvious questions raised by these results. First, are there infinitely many knots in S^4 distinguished by their k-invariants? Secondly, does the algebraic 3-type of a knot exterior determine its homotopy type, or are still higher invariants necessary (Question 1 of [17])?

(Added in proof.) Theorem 8.1 has been proven by the second author in his 1984 Columbia Univ. Ph.D. Thesis – using 2-twist spun 2-bridge knots $K_{p,q}$, with fiber $\mathring{L}(p, q)$. The surgery construction in §1 yields knots $\hat{K}_{p,q}$ with isomorphic π_1 and π_2. If $L(p, q)\not\simeq L(p, q')$, then $\hat{K}_{p,q}$ and $\hat{K}_{p,q'}$ have distinct k-invariants.

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