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## Commutators of diffeomorphisms, III: a group which is not perfect

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The group of  $C^r$  diffeomorphisms of the real line with compact support is perfect if  $r \neq 2$  (cf. [1–3]). It is unknown whether this is the case if  $r = 2$  (cf. [4]).

In this note, we will give a very simple proof that the group  $G$  of compactly supported  $C^1$  diffeomorphisms of the real line whose first derivative has bounded variation is not perfect. For  $f \in G$ ,  $\log Df$  is a compactly supported function of bounded variation. Let  $D \log Df$  denote the derivative of  $\log Df$  in the sense of the theory of distributions. It is well known that  $D \log Df$  is a compactly supported Radon measure. In other words, if we think of  $D \log Df$  as a linear functional on the space of  $C^\infty$  functions on  $\mathbb{R}$ , then  $D \log Df$  has a unique linear continuous extension to the space of continuous functions on  $\mathbb{R}$ , where we provide this last space with the  $C^0$  topology.

A self homeomorphism  $f$  of  $\mathbb{R}$  induces automorphism  $f^*$  of the continuous functions on  $\mathbb{R}$ , defined by  $f^*u = u \circ f$ . The dual of  $f^*$  is an automorphism  $f_*$  of the space of compactly supported Radon measures on  $\mathbb{R}$ . Another way of describing  $f_*$  is to observe that if  $X$  is a Borel subset of  $\mathbb{R}$ , then  $(f_*\mu)(X) = \mu(f^{-1}X)$ , for any Radon measure  $\mu$ . If  $u$  is a compactly supported function of bounded variation, then

$$f_*^{-1}Du = D(u \circ f).$$

An easy way to see this is to use the fact that  $Du(I_{a,b}) = u(b-0) - u(a+0)$ , where  $I_{a,b}$  is the open interval  $(a, b)$ , and this uniquely specifies  $Du$  as a Radon measure. It follows that if  $f, g \in G$ , then

$$D \log D(f \circ g) = g_*^{-1}D \log Df + D \log Dg.$$

Any Radon measure  $\mu$  uniquely decomposes as a sum  $\mu = \mu_{\text{reg}} + \mu_{\text{sing}}$  where the regular part  $\mu_{\text{reg}}$  vanishes on all Borel sets of zero Lebesgue measure and the singular part  $\mu_{\text{sing}}$  has support in some Borel set  $X$  of zero Lebesgue measure, in the sense that  $\mu_{\text{sing}}(Y) = \mu_{\text{sing}}(Y \cap X)$  for all Borel sets  $Y$ . Note that if  $g \in G$ , then  $g$  is Lipschitz, so it preserves the decomposition of a Radon measure into its

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regular and singular parts, i.e.

$$(g_*\mu)_{\text{reg}} = g_*(\mu_{\text{reg}}), (g_*\mu)_{\text{sing}} = g_*(\mu_{\text{sing}}).$$

We let  $\int \mu$  denote the total mass of  $\mu$ , i.e.  $\mu(\mathbb{R})$ . For  $f \in G$ , we define

$$\pi(f) = \int (D \log Df)_{\text{reg}}.$$

Since  $\int g_*\mu = \int \mu$ , we have

$$\pi(fg) = \int (g_*^{-1} D \log Df + D \log Dg)_{\text{reg}} = \pi(f) + \pi(g).$$

In other words  $\pi : G \rightarrow \mathbb{R}$  is a homomorphism.

In fact, the homomorphism  $\pi$  is surjective. This is easy to prove: Consider  $a \in \mathbb{R}$  and construct a compactly supported real valued function  $u$  of a real variable such that  $u(0) = a$ ,  $u(1) = 0$ ,  $u$  is  $C^1$  outside of  $[0, 1]$ , and  $u|_{[0, 1]}$  is a monotone function whose derivative (as a Radon measure) is totally singular with respect to Lebesgue measure. Such a function may be constructed, for example, by letting  $u$  be the primitive of an appropriate totally singular measure in  $[0, 1]$  and extending  $u$  to be  $C^1$  outside of  $[0, 1]$ . We further require that  $\int_{-\infty}^{\infty} (e^{u(t)} - 1) dt = 0$ . This may be arranged by altering  $u$  (if necessary) outside the interval  $[0, 1]$ . Let  $f$  be the primitive of  $e^u$  which is the identity near  $-\infty$ . Then  $f \in G$  and

$$\pi(f) = \int_{-\infty}^0 Du + \int_1^{\infty} Du = a,$$

since  $(Du)_{\text{reg}}|_{[0, 1]} = 0$  and  $Du_{\text{reg}} = Du$ , elsewhere. We have proved:

**THEOREM.**  $\pi : G \rightarrow \mathbb{R}$  is a surjective homomorphism.

More generally, let  $R$  be a family of Borel subsets of  $\mathbb{R}$  which is  $G$ -invariant, closed under countable unions, and satisfies the condition that if  $X \in R$  and  $Y$  is a Borel subset of  $X$ , then  $Y \in R$ . Each compactly supported Radon measure  $\mu$  has a unique decomposition

$$\mu = \mu_{R\text{-reg}} + \mu_{R\text{-sing}}$$

where the  $R$ -regular part of  $\mu$  vanishes on all members of  $R$  and the  $R$ -singular part has support in a member of  $R$ .

For  $f \in G$ , let

$$\pi_R(f) = \int D(\log Df)_{R\text{-reg}}.$$

The same argument as before shows that  $\pi_R$  is a homomorphism.

We may obtain many different homomorphisms of  $G$  onto  $\mathbb{R}$  this way, for example, by taking  $R$  to be the set of subsets of Hausdorff dimension  $\leq \alpha$ , for  $0 \leq \alpha < 1$ , or of vanishing  $\alpha$ -dimensional Hausdorff measure, for  $0 \leq \alpha \leq 1$ , or of Hausdorff dimension  $< \alpha$ , for  $0 < \alpha \leq 1$ , or the family of subsets which for any  $\varepsilon > 0$  can be covered by a countable family of intervals whose lengths satisfy  $\sum_{i=1}^{\infty} -(\log l_i)^{-1} < \varepsilon$ , etc.

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