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The Euler and Pontrjagin numbers of an n-manifold in $\mathbb{C}^n$

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Introduction

According to a theorem of H. Whitney, every smooth n-dimensional manifold M^n can be smoothly embedded in the Euclidean space $\mathbb{R}^{2n}$. Viewing $\mathbb{R}^{2n}$ as $\mathbb{C}^n$, one may ask for embeddings which have nice properties relative to the complex structure. The simplest properties relate to complex tangents. If M has no complex tangents, the embedding is said to be totally real. In general there are global obstructions to finding totally real embeddings. For example, if M is compact, orientable and totally real, then its Euler number and Pontrjagin classes must vanish, a result due to R. Wells [11].

In this paper we shall give an explicit formula for the Euler number of a compact real n-manifold M suitably immersed in a complex n-manifold. (The requirements on M hold generically if $n \le 5$.) We shall also give a formula for the Pontrjagin number of a compact, orientable M^4 generically immersed in $\mathbb{C}^4$. We must assume that M has only one-dimensional complex tangents which are non-degenerate in a certain sense, and occur along a smooth, compact, codimension two submanifold $N \subset M$. There is a smooth invariant function γ on N, $0 \le \gamma \le +\infty$. In section 5 we derive a relation among the Euler numbers $\chi(M)$, $e = \chi[\gamma < \frac{1}{2}]$, $h = (-1)^n \chi[\gamma > \frac{1}{2}]$, $\chi(M^\perp)$ (normal bundle), and the parabolic index p, which is described in section 1. As a special case we have the following.

THEOREM (0.1). *Let the compact, orientable n-manifold M be embedded in $\mathbb{C}^n$ as just described. Then its Euler number satisfies*

$$\chi(M) = e - h + p. \tag{0.1}$$

When $n = 2$ we have $p = 0$, since there are no parabolic points. In this case the

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theorem is due to E. Bishop [1], who reduced it to a theorem of Chern and Spanier [3]. Our method of proof is different, being based on the Poincare–Hopf formula for the Euler number. If M is totally real, (0.1) implies $\chi(M)=0$. For a direct proof see [11]. If the 2-sphere is embedded in $\mathbb{C}^2$ as above, then it must have at least 2 elliptic points by (0.1). Elliptic points are of interest since they contribute to the local hull of holomorphy of M. They have been studied most recently in [6], [7], and [10]. It is not known whether a 4-sphere generically embedded in $\mathbb{C}^4$ must have an elliptic point. If not then (0.1) gives $h=-2$, which puts some restriction on the topology of $N=N_h$.

In some cases the Pontrjagin numbers give more information about the complex-tangent structure. Let M be immersed in $\mathbb{C}^n$, and J denote the real operator corresponding to multiplication by $\sqrt{-1}$. Then $H_m = T_m(M) \cap JT_m(M)$ for $m \in N$, defines a complex line bundle H over N. N inherits a natural orientation from M (section 1).

THEOREM (0.2). *Suppose the compact, orientable 4-manifold M is generically immersed in $\mathbb{C}^4$. Then its Pontrjagin number satisfies*

$$p_1(M) = \chi(H). \tag{0.2}$$

This is proved in section 3, where we also show that $\chi(H)=0$ if M has no parabolic points. Thus if $p_1(M) \neq 0$, then M has a parabolic point, hence nearby elliptic and hyperbolic points. A generic embedding of $\mathbb{CP}_2$ in $\mathbb{C}^4$ therefore has a non-trivial hull of holomorphy.

In [8] H. F. Lai has given general formulas for the Euler and Pontrjagin classes of a wider class of submanifolds of $\mathbb{C}^n$. The relation of his work to the present paper is not clear. His formulas do not yield (0.1) or (0.2). In section 1 we describe the local properties of $M^n \subset \mathbb{C}^n$ near a complex tangent. In section 2 we study the real Grassmannian and give a transversality argument. Section 3 contains a general result about the intersection properties of Schubert varieties needed for Theorem (0.2). Section 4 is devoted to deriving suitable local equations for M near a complex tangent.

1. Complex tangents and the parabolic index

Let M^n be a smooth real n-manifold immersed in the complex n-manifold $\tilde{M}$. In a local holomorphic coordinate system $z = x + iy = (z^1, \ldots, z^n)$, M is given by

$$M: R = (r^1, \ldots, r^n) = 0, \qquad R = \bar{R}, \qquad dr^1 \wedge \cdots \wedge dr^n \neq 0,$$
$$\partial r^1 \wedge \cdots \wedge \partial r^n = B dz^1 \wedge \cdots \wedge dz^n. \tag{1.1}$$

Under a change of holomorphic coordinates $z \to z'$ and defining function $R \to R'$, the factor B changes by

$$B \to B' = \frac{\partial R'}{\partial R}\left(\frac{\partial z'}{\partial z}\right)^{-1} B. \tag{1.2}$$

Let F denote the normal bundle of M in $\tilde{M}$, and F^* its dual. Also, let K denote the canonical line bundle of $\tilde{M}$, and L and L^* the real line bundles $\Lambda^n F$ and $\Lambda^n F^*$, respectively. Then (1.2) says that the collection $\{B, B', \ldots\}$, which we denote simply by B, defines a section of the complex line bundle $K \otimes L^{*-1} = K \otimes L$ over M. In particular, the set

$$N = \{m \in M : B(m) = 0\} \tag{1.3}$$

is well defined and is precisely the set of points m at which M has a (non-trivial) complex tangent space H_m. If J_m denotes the real linear operator on the real tangent space $T_m\tilde{M}$ corresponding to multiplication by $\sqrt{-1}$, then $H_m = T_m \cap JT_m$, where $T = T(M)$ is the real tangent bundle of M.

We assume that $\dim_{\mathbb{C}} H_m = 1$, so that $H_m \otimes \mathbb{C} = H'_m \oplus H''_m$, $H''_m = \bar{H}'_m$, and H'_m is spanned by

$$X = \sum \xi^i \partial/\partial z^i, \qquad XR = \sum \xi^i (\partial R/\partial z^i)|_m = 0. \tag{1.4}$$

We further assume that the complex tangent H_m is non-degenerate, in that

$$\text{either } XB \neq 0 \quad \text{or} \quad \bar{X}B \neq 0. \tag{1.5}$$

Since X is determined up to $X \to cX$, c a non-zero complex constant, it follows that

$$\gamma(m) = \tfrac{1}{2}\,|XB/\bar{X}B| \in [0, \infty] \tag{1.6}$$

is a well defined biholomorphic invariant of M at m. It was first found by Bishop [1] in the following form.

We suppose that m is the origin of a coordinate system $(z_1, z^\alpha, 2 \le \alpha \le n-1, z_n)$ in which T_m is the (z_1, x^α)-space. M is given locally as the graph $R = 0$,

$$\begin{aligned} &r \equiv -z_n + F(z_1, x), \qquad F = q + x \cdot O(1) + O(2), \\ &r^\alpha \equiv -y^\alpha + f^\alpha(z_1, x), \qquad f^\alpha = \bar{f}^\alpha = O(2), \qquad 2 \le \alpha \le n-1, \\ &x = (x^\alpha), \qquad q = az_1^2 + bz_1\bar{z}_1 + c\bar{z}_1^2. \end{aligned} \tag{1.7}$$

Here and elsewhere $O(k)$ indicates a term which vanishes to order k at the origin. Then, up to a constant, $X = \partial/\partial z_1$, and

$$B = \frac{\partial(r, \bar{r}, r^\alpha)}{\partial(z_1, z_n, z^\beta)} = (1/2)^{n-2}\,\partial\bar{F}/\partial z_1 + O(2), \qquad XB = 2c, \qquad \bar{X}B = \bar{b},$$
$$\gamma(m) = |c/b|. \quad (1.8)$$

We also introduce

$$\Delta = |XB|^2 - |\bar{X}B|^2. \quad (1.9)$$

N is partitioned into

$$N_e = [\gamma < \tfrac{1}{2}], \qquad N_p = [\gamma = \tfrac{1}{2}], \qquad N_h = [\gamma > \tfrac{1}{2}], \quad (1.10)$$

the sets of elliptic, parabolic, and hyperbolic points. These sets correspond to $\Delta < 0$, $\Delta = 0$, $\Delta > 0$, respectively.

Next we examine more closely a parabolic point m. Since the determinant (1.9) vanishes, (1.5) implies that the system

$$aXB + b\bar{X}B = 0, \qquad aX\bar{B} + b\bar{X}\bar{B} = 0,$$

has a non-trivial solution (a, b) unique up to $(a, b) \to (a', b') = (\mu a, \mu b)$. Complex conjugation of these equations shows that $\bar{b} = \lambda a$, $\bar{a} = \lambda b$, for some λ, $|\lambda| = 1$. The factor μ can be adjusted so that $b' = \bar{a}'$. Hence, there is an a, unique up to $a \to \rho a$, $\rho \in \mathbb{R}$, for which

$$YB = 0, \qquad Y = aX + \bar{a}\bar{X}. \quad (1.11)$$

It's easily to be seen that the changes $X \to cX$ and (1.2) can change Y by at most a real factor. Thus Y spans an intrinsic real line $l_m \subset H_m$, which we call the *parabolic line* at m.

Now we assume that

$$dB \wedge d\bar{B} \neq 0 \quad \text{on} \quad B = 0, \quad (1.12)$$

so that N is a real submanifold of M of codimension 2. The conormal bundle S^* of N in M is spanned by the coframes dB along N, and hence is the restriction of $(K \otimes L)^*$ to N. So the *normal bundle* S of N in M is the complex line bundle $K \otimes L$ restricted to N. The non-degeneracy condition (1.5) precludes H_m being contained in $T_m(N)$. It follows from (1.11) and (1.9) that m is a parabolic point precisely when $T_mN \cap H_m = l_m$ is one dimensional. We denote by φ the composite vector bundle mapping

$$\varphi : H \hookrightarrow T(M)|_N \to (T(M)|_N)/T(N) \equiv S. \tag{1.13}$$

Since dB is local coframe for S and X is a $(1, 0)$-frame for H_m, consideration of the sign of (1.9) shows that φ_m is orientation reversing if m is elliptic and orientation preserving if m is hyperbolic. It is singular of real rank one if m is parabolic.

We assume still further that $d\gamma \neq 0$ when $\gamma = \frac{1}{2}$, so that N_p is a smooth $(n-3)$-dimensional manifold. We may now proceed to define the parabolic index. For each $m \in N_p$ we have two lines in $T_m(N)$, the parabolic line l_m and the line k_m determined by the normal vector $\nabla\gamma$ (gradient relative to any convenient metric on N). This gives us two sections l, k of the projective bundle $P \to N_p$, which has as fiber over $m \in N_p$ all lines through the origin in $T_m(N)$.

We next describe an orientation on the open subset of P consisting of lines not tangent to N_p. Let x^α, $3 \leq \alpha \leq n-1$, be local coordinates on N_p. Let y be a local defining function for $N_p : y = 0$, with $\partial\gamma/\partial y > 0$. Then (x, y) are local coordinates on N, and any line $L \in P$, not tangent to N_p, is spanned by a unique vector

$$\frac{\partial}{\partial y} + \sum_{\alpha=3}^{n-1} w^\alpha \frac{\partial}{\partial x^\alpha} \in L. \tag{1.14}$$

Thus (x, w) are coordinates for L, and

$$\Omega_P = dx^3 \wedge \cdots \wedge dx^{n-1} \wedge dw^3 \wedge \cdots \wedge dw^{n-1} \tag{1.15}$$

defines a local volume form on the $(2n-6)$-dimensional manifold P. If $(\tilde{x}, \tilde{y})$ is another such coordinate system, then one easily sees that

$$\tilde{\Omega}_P = (\det \partial\tilde{x}/\partial x)^2 (\partial\tilde{y}/\partial y)^{-n+3} \Omega_P.$$

Since $\partial\tilde{y}/\partial y > 0$, we have a well defined orientation.

The *parabolic index* is defined as the intersection number of $l(N_p)$ and $k(N_p)$ relative to this orientation. This is possible since k_m is never tangent to N_p. More precisely, we take a slight perturbation of k, if necessary, so that $k(N_p)$ and $l(N_p)$ intersect transversely in P at a finite number of points. At a point $m \in N_p$ where $l_m = k_m$, we choose local coordinates as in the previous paragraph. Then $\partial l/\partial x^\alpha$, $\partial k/\partial x^\alpha$, $3 \leq \alpha \leq n-1$, give frames for $l(N_p)$ and $k(N_p)$ at m. The intersection index at m is given by the sign

$$\mathrm{ind}_{P,m}(l, k) = \mathrm{sgn}\, \Omega_P(\partial l/\partial x, \partial k/\partial x). \tag{1.16}$$

This is well defined since a change of orientation of the local coordinates x changes the orientations on *both* $l(N_p)$ and $k(N_p)$, so that (1.16) remains unchanged. The parabolic index is

$$p = \sum \{\mathrm{ind}_{P,m}(l, k) : m \in N_p, l_m = k_m\}. \tag{1.17}$$

2. The Grassmann manifold and transversality

In this section we consider an n-manifold M immersed in $\mathbb{C}^n$. Its Gauss map g associates to each $m \in M$ the real tangent plane $T_m(M)$. It is a smooth mapping of M into $Gr(n; n) \equiv Gr$, the n^2-dimensional Grassmann manifold of real n-planes through the origin of $\mathbb{C}^n \cong \mathbb{R}^{2n}$. We define

$$\begin{aligned} C_k &= \{V \in Gr : \dim_{\mathbb{C}} V \cap JV = k\}, \qquad 0 \le k \le n/2; \\ C &= C_1 \cup \cdots \cup C_{[n/2]}. \end{aligned} \tag{2.1}$$

C_0 is the dense open subset of totally real planes, and $Gr = C_0 \cup \cdots \cup C_{[n/2]}$ is a disjoint union. For each k, $0 < k \le n/2$, C_k fibers over the complex Grassmannian $Gc(k, n-k)$ of complex k-planes in $\mathbb{C}^n$. In fact, for V in C_k, $V \cap JV \cong \mathbb{C}^k$ and $V \cap (V \cap JV)^{\perp}$ is a totally real $(n-2k)$-plane in $\mathbb{C}^{n-k}$. So the fiber is an open subset of $Gr(n-2k; n-k)$, which has dimension $n(n-2k)$, while the base has real dimension $2k(n-k)$. Thus C_k is a real submanifold of codimension $2k^2$. If $n = 2m$ is even, $C_m = Gc(m; n)$; while if $n = 2m+1$, C_m is a bundle over $Gc(m; n)$ with fiber the real projective space $\mathbb{RP}(2m+1)$.

We repeat some of the constructions of section one in this "universal" setting. For $V \in Gr$ there are two natural vector spaces; V itself and $F_V = \mathbb{C}^n/V$. We also have the two real line bundles $L = \bigwedge^n F$, $L^* = \bigwedge^n F^*$, and the complex line bundle $K = \bigwedge^n (Gr \times \mathbb{C}^{n*}) \cong Gr \times \mathbb{C}$. We refer back to (1.1) where now the dr^i are n independent real linear forms on $\mathbb{C}^n$ annihilating a fixed V in Gr, and the dz^i are a basis of complex linear functions on $\mathbb{C}^n$. As before it follows that B is a section of $K \otimes L$, having as zero set precisely C. We restrict this bundle to C_1, where it is identified with the normal bundle S of C_1 in Gr. Its dual bundle has local frames dB restricted to C_1. Also, we have the complex line bundle $H \to C_1$, $H_V = V \cap JV$, $H \otimes \mathbb{C} = H' \oplus H''$.

In addition we have a real 4-plane bundle $E \to C_1$ defined by $E_V = \mathrm{Hom}_{\mathbb{R}}(H_V, S_V)$. Each element of E_V may be described by an equation

$$dB = XB\theta + \bar{X}B\bar{\theta},$$

where X is a frame vector for H' and θ is its dual. If we denote by E_0 the zero section of E, which corresponds to $XB = \bar{X}B = 0$, then we have a well defined function $\gamma : (E - E_0) \to [0, \infty]$ given by (1.6). Parallel to (1.10) we have the disjoint union $E = E_0 \cup E_e \cup E_p \cup E_h$, where E_e, E_p, E_h are the sets where $\gamma < \frac{1}{2}$, $\gamma = \frac{1}{2}$, $\gamma > \frac{1}{2}$, respectively.

For M immersed in $\mathbb{C}^n$ with at most one dimensional complex tangents, it is clear that g^*H and g^*S are the corresponding bundles of section 1. For each $m \in N$ we have

$$dg_m : H_m \cong H_{g(m)} \to T_{g(m)}Gr \to T_{g(m)}Gr/T_{g(m)}C_1 \cong S_{g(m)},$$

which defines a map $dg : N \to E$. The degenerate points form the set $dg^{-1}(E_0)$. Clearly, $\gamma \circ dg$ is the invariant (1.6).

PROPOSITION (2.1). *Let $f : M^n \to \mathbb{C}^n$ be a smooth immersion.* a) *A generic small perturbation of f results in an immersion with the following properties: M has no complex k-dimensional tangents if $2k^2 > n$, while if $2k^2 \leq n$, the points with such tangents form a submanifold of codimension $2k^2$.* b) *Suppose in addition that the immersion f has only one dimensional complex tangents which occur along the set N. After a generic small perturbation, N is a compact smooth $(n-2)$-manifold, and the set N_0 of degenerate points is a smooth $(n-6)$-manifold.* c) *Assume further that N_0 is empty. Then after a generic small perturbation the parabolic set N_p forms a compact smooth $(n-3)$-manifold along which $d\gamma \neq 0$.*

Remark. As a consequence a generic M^n in $\mathbb{C}^n$ has the following characteristics:

i) $n = 2$ —isolated elliptic or hyperbolic points;

ii) $n = 3, 4, 5$ —at most one-dimensional non-degenerate complex tangents with N_p smooth;

iii) $n = 6$ —at most one-dimensional complex tangents and at most isolated degenerate points;

iv) $n = 8$ —at most isolated 2-dimensional complex tangents.

Case i) is due to Hunt and Wells [5].

The main ingredient in the proof is the parametric transversality theorem (see e.g. Hirsch [4], p. 79). Let f be any immersion of M with Gauss map $g = f_f$. Let Q denote the set of all real affine transformations $A(z) = A'(z) + a$, $A' \in Gl(2n, \mathbb{R})$, $a \in \mathbb{C}^n$. Then $G(A, z) \equiv g_{Af}(z) = A' \circ g_f(z)$ defines a smooth map

$$G : Q \times M \to Gr. \tag{2.2}$$

We claim that the mapping G is transverse to C_k for every $k \leq [n/2]$. This means that $TC_k + T\mathcal{I}mG = TGr$, at every point $G(A, z) \in C_k$, where $T\mathcal{I}mG \equiv DG(T(Q \times M))$. This will follow from $T\mathcal{I}mG = TGr$, which has nothing to do with complex tangents. We let $\mathbb{C}^n = V \oplus V^\perp$ with coordinates (x_1, x_2), and restrict to the submanifold of Q of maps of the form $(x_1, x_2) \to (x_1, x_2 - Bx_1)$, $B \in \mathrm{Hom}_{\mathbb{R}}(V, V^\perp)$. It is clear that DG maps the tangent space of this submanifold at $B = 0$ onto $T_V Gr$.

By the parametric transversality theorem, the set Q_k of A's for which $z \to G(A, z)$ is transverse to C_k is residual. It follows that $Q_1 \cap \cdots \cap Q_{[n/2]}$ is also residual and therefore dense. Thus there are affine (or even linear) mappings A, arbitrarily close to the identity, for which $A \circ f$ is transverse to every C_k. Part a) follows by transversality. Under the additional assumption of b) N is a compact smooth $(n-2)$-manifold.

For parts b) and c) we must use mappings quadratic in $(z, \bar{z})$: $A(z) = a + A'(z) + A''(z^2)$. We first replace f by a perturbation as in a), so that the Gaussian image $g(M)$ remains disjoint from C_k, $k > 1$, and intersects C_1 transversely. We then restrict to quadratic mappings A with A' unitary and A'' so small that this situation is preserved. We let Q denote this set of maps. For A'' small enough, perturbation by $A \in Q$ will result in a new manifold N_A close to the original N in the following sense. N_A will lie in a small tubular neighborhood $u = U\{D_m : m \in N\}$ and will intersect each normal 2-disc D_m in a unique point $\eta_A(m)$, giving a diffeomorphism $\eta_A : N \to N_A$. We consider the composite map.

$$G'(A, m) = dg_{Af} \circ \eta_A(m), \qquad G' : Q \times N \to E.$$

We claim that the map G' is transverse to E_0 under the assumptions of b) and transverse to E_p under those of c). Assume $G'(A, m) \in E_0$. After an affine unitary coordinate change we may assume that $m = 0$ and that M is given as in (1.7). We then restrict G' to the submanifold of Q consisting of mappings of the special form

$$A : \begin{cases} z_1 \to z_1, \; z^\alpha \to z^\alpha, & 2 \leq \alpha \leq n-1 \\ z_n \to z_n + Bz_1\bar{z}_1 + C\bar{z}_1^2, \end{cases}$$

which result in $(b, c) \to (b\text{-}B, c\text{-}C)$ in (1.7). Locally $E \cong C_1 \times \mathbb{R}^4$ and $E_0 \cong C_1 \times \{0\}$. The normal space to E_0 at m is $\{m\} \times \mathbb{R}^4 \cong \{m\} \times \mathbb{C}^2$ with coordinates (b, c). If we restrict G' further to (A, m) with A as described and $m = 0$, then it is clear that DG' at $(A, m) = (I, 0)$ maps onto the normal space. Thus G' is transverse to E_0. A similar argument shows that G' is transverse to $E_p \cong C_1 \times \text{Cone}$. Now b) and c) follow by the parametric transversality theorem since E_0 has codimension 4 and E_p codimension one in E.

3. The Pontrjagin number of a 4-manifold in $\mathbb{C}^4$

By a well-known theorem [9] the Pontrjagin class $p_k(T(M))$ equals $(-1)^k c_{2k}(T(M) \otimes \mathbb{C})$, where c denotes the Chern class. For $M^n \subset \mathbb{C}^n$ with Gauss map g, $p_k(T(M)) = g^* p_k(VGr)$, where $VGr \to Gr$ is the universal bundle. $Gr \subset Gc(n, n)$ and $(VGr) \otimes \mathbb{C}$ is the restriction of the complex universal bundle $VGc \to Gc$. Thus we must consider the Chern classes $c_k(VGc)$.

We begin by recalling some facts [2] about $Gc(n, r)$, the space of all complex n-planes $Z^n \subset \mathbb{C}^{n+r}$. For $0 \le k_1 \le \cdots \le k_n \le r$ and linear spaces $L_1 \subset \cdots \subset L_n$, $\dim L_j = k_j + j$, the Schubert variety is defined by

$$Z(k_1, \ldots, k_n) = \{Z \in Gc(n, r) \mid \dim Z \cap L_j \ge j\},$$

and has complex dimension $k_1 + \cdots + k_n$. The Chern class $c_k(VGc(n, r))$ is dual to $Z_k(n, r) \equiv Z(r-1, \ldots, r-1, r, \ldots, r)$, where $r-1$ appears k-times. It may also be defined as

$$Z_k(n, r) = \{Z \in Gc(n, r) \mid \dim Z \cap L \ge k\}, \qquad \dim L = r - 1 + k, \tag{3.1}$$

and decomposes into the disjoint union of

$$Z_k^0(n, r) = \{Z : \dim Z \cap L = k\},$$

and

$$Z_k^s(n, r) = \{Z : \dim Z \cap L > k\}.$$

$Z_k^0(n, r)$ is a complex manifold of complex codimension k, fibering over $Gc(k, r-1)$, and $Z_k^s(n, r) = Z(r-2, \ldots, r-2, r, \ldots, r)$ $(k+1 (r-2)\text{'s})$ has codimension $2(k+1)$. Thus a generic compact orientable real $2k$-manifold in $Gc(n, r)$, which we also denote by M, will be disjoint from $Z_k^s(n, r)$ and intersect $Z_k^0(n, r)$

transversely in finitely many points. $c_k(VGc(n, r))[M]$ is the sum of the intersection indices at these points.

We fix k and $L_{r-2+k} \subset L_{r-1+k}$, subscripts denoting dimension, and consider the corresponding varieties $Z_{k-1}(n, r) \supset Z_k(n, r)$, $Z^s_{k-1}(n, r) \supset Z^s_k(n, r)$. A generic M^{2k} will miss $Z^s_{k-1}(n, r)$, since it has real codimension $4k$. Thus

$$M \cap Z_{k-1}(n, r) = M \cap Z^0_{k-1}(n, r) \equiv N$$

is a smooth compact oriented 2-manifold containing the finite set $Z_k(n, r) \cap M$. For $Z \in Z^0_{k-1}(n, r)$, we set

$$A_Z = Z \cap L_{r-2+k} \quad \text{and} \quad B_Z = Z \cap A_Z^{\perp},$$

so that $Z = A_Z \oplus B_Z$ is an orthogonal direct sum relative to the standard hermitian inner product on $\mathbb{C}^{n+r}$. We define a smooth map β by

$$\beta : Z^0_{k-1}(n, r) \to Gc(n-k+1, r+k-1), \qquad \beta(Z) = B_Z. \tag{3.2}$$

For $Z \in Z^0_{k-1}(n, r)$, $Z \in Z^0_k(n, r)$ if and only if $B_Z \in Z_1(n-k+1, r+k-1)$; i.e.

$$Z_k(n, r) \cap Z^0_{k-1}(n, r) = \beta^{-1}(Z_1(n-k+1, r+k-1)).$$

LEMMA (3.1)

$$c_k(VGc(n, r))[M] = c_1(VGc(n-k+1, r+k-1))[\beta N]. \tag{3.3}$$

Proof. This comes down to comparing two intersection indices. First, we have the equality of oriented vector spaces at $m \in M \cap Z^0_k(n, r)$

$$T_m Z^0_k(n, r) \oplus T_m M = c_k T_m Gc(n, r),$$

where $c_k = \pm 1$. If S is the normal bundle of $Z^0_{k-1}(n, r)$ in $Gc(n, r)$, then its restriction to N is that of N in M, so

$$T_m Gc = T_m Z^0_{k-1} \oplus S_m \quad \text{and} \quad T_m M = T_m N \oplus S_m.$$

It follows that

$$T_m Z^0_k(n, r) \oplus T_m N = c_k T_m Z^0_{k-1}(n, r). \tag{3.4}$$

The map β is not holomorphic since it involves the orthogonal complement $A_Z^\perp$. However, a slight local deformation of it is. For all $Z \in Z_{k-1}^0(n, r)$ sufficiently near m we may replace $A_Z^\perp$ by $A_m^\perp$, then $\alpha(z) = Z \cap A_m^\perp$ is holomorphic and approximates β near m. Clearly, $\alpha^{-1}(Z_1^0(n-k+1, r+k-1)) = Z_k^0(n, r)$. By (3.4)

$$T_{\alpha m}(\alpha Z_k^0(n, r)) \oplus T_{\alpha m}(\alpha N) = c_k T_{\alpha m}(\alpha Z_{k-1}^0(n, r)), \tag{3.5}$$

where the orientations agree with those from $Gc(n-k+1, r+k-1)$, since α is holomorphic. If $\tilde{S}$ is the normal bundle of $\alpha Z_{k-1}^0(n, r)$ in $Gc(n-k+1, r+k-1)$ it is also the normal bundle of $\alpha Z_k^0(n, r)$ in $Z_1^0(n-k+1, r+k-1)$. Adding $\tilde{S}_{\alpha m}$ to both sides of (3.5) gives

$$T_{\alpha m} Z_1^0(n-k+1, r+k-1) \oplus T_{\alpha m}(\alpha N) = c_k T_{\alpha m} Gc(n-k+1, r+k-1).$$

Now $\alpha m = \beta m$, and if we continuously deform α back to β, we see that $c_k = c_1$, where c_1 is the intersection index at βm entering into the right hand side of (3.3). Summing over all such m in $M \cap Z_k^0(n, r)$ gives (3.3).

Note. The same argument gives

$$c_k(VGc(n, r)[M^{2k}] = c_{k-l}(VGc(n-l, r+l))[M \cap Z_l^0(n, r)],$$

when the intersections are nice, which is generically so when $2l+2>k$.

We return to the study of $Gr(n; n)$. We set $\mathbb{C}^n = (W, J)$, $W \otimes \mathbb{C} \equiv W^c \equiv W' \oplus W''$, $\bar{W}'' = W' = \{w \in W^c : Jw = iw\}$. For a subspace $V \subset W$, $V^c \subset W^c$, $V^c \supset V' \oplus V''$, where $V' = V^c \cap W'$, $V'' = V^c \cap W''$. The map $V \to V^c$ embeds $Gr(n; n)$ in $Gc(n, n)$ as a totally real submanifold.

LEMMA (3.2). a) $V^c = V' \oplus V''$ if and only if $JV = V$.
b) $V' = H'$ if $H = V \cap JV$.
c) $C = Gr(n; n) \cap Z_1(n, n)$.

Proof. a) If $JV = V$, then $JV^c = V^c$, and any $w \in V^c$ is the sum $\frac{1}{2}(w - iJw) + \frac{1}{2}(w + iJw) \in V' \oplus V''$. If $V^c = V' \oplus V''$, then $JV^c = JV' + JV'' = V' \oplus V'' = V^c$; so $JV = V$. b) $V' \supset H'$ is clear since $V^c \supset H^c$. If $w \in V'$, then $Jw = iw$. So if $w = u + iv$, $u, v \in V$, then $u = Jv$, $v = -Ju$. Hence, $u, v \in H$ and b) holds. If we apply a) to H, then it follows that $V \in C$ if and only if $\dim_{\mathbb{C}} V^c \cap W' \geq 1$. So c) follows by taking $n = r$, $k = 1$, and $L_n = W'$ in (3.1).

We now turn to the proof of Theorem (0.2) of the introduction. Since M^4 is generically immersed in $\mathbb{C}^4$ it has the properties of Remark (ii) following Proposition (2.1). We have

$$p_1(M) = p_1(T(M))[M] = p_1(VGr(4;4))[gM]$$
$$= -c_2(VGc(4,4))[gM] = -c_1(VGc(3,5))[\beta gN],$$

by (3.3) with $n=r=4$, $k=2$. Now over the surface N (or rather βgN) $VGc(4,4) = A \oplus B = H' \oplus VGc(3,5)$ by Lemma 3.2). Thus, the total Chern class $c = 1 + c_1$ satisfies [9] $c(VGc(4,4)) = c(H')c(VGc(3,5))$ or $c_1(VGc(4,4)) = c_1(H') + c_1(VGc(3,5))$. Over $Gr(4;4)$ $VGc(4,4) = VGr(4;4) \otimes \mathbb{C}$, hence its first Chern class is a 2-torsion element. When pulled back to the compact orientable surface N it vanishes; thus $c_1(VGc(3,5)) = -c_1(H') = -c(H)$. Hence, $p_1(M) = \chi(H)$, since $\chi(H) = c_1(H)[N]$.

The bundle mapping φ (1.13) can be used to get a formula for $\chi(H)$. Since $\tilde{M} = \mathbb{C}^n$, the canonical bundle K is trivial. Since M is orientable, so is its normal bundle, hence the line bundle L is trivial. It follows that $S = K \otimes L$ is trivial. φ is a bundle isomorphism over N_h and an anti-isomorphism over N_e. Therefore H is trivial over any connected component of N which does not meet N_p. If we let N^0 be the union of the components of N which meet N_p, and H^0 the restriction of H to N^0, then $\chi(H) = \chi(H^0)$. We choose a section v of H^0 which does not vanish on $N_e^0 \cup N_p$ and has only isolated non-degenerate zeros in N_h^0. v gives a trivialization of H over N_p; hence, the parabolic line l gives a map $\tilde{l}: N_p \to \mathbb{RP}_1$, where N_p and $\mathbb{RP}_1$ have naturally induced orientations. We define the *H-parabolic index* p_H to be the degree of this mapping $\tilde{l}$. If w is a piecewise smooth section of H over N_p which spans l at each point, then $w = \mu v$, $\mu \neq 0$ and piecewise smooth. We have

$$p_H = \mathrm{Re}\,\frac{1}{\pi i}\int_{N_p} \frac{d\mu}{\mu}. \tag{3.6}$$

Note that w is determined up to $w \to \rho w$, with $\rho \neq 0$, real and piecewise smooth. It follows that (3.6) is not affected by this change. Also, v may be changed by $v \to \xi v$, $\xi \neq 0$ and smooth on $N_p \cup N_e^0$. Applying Stokes's theorem to $d\xi/\xi$ on N_e^0 shows that the integral in (3.6) remains unchanged. Thus p_H is well defined.

LEMMA (3.3). $\chi(H) = -p_H$.

Proof. This follows by comparing the index sums for v and $\varphi(v)$. We assume that v has been chosen so that l_m contains v_m at only a finite number of points m in N_p and that l crosses v transversely at such points. In otherwords $1 \in \mathbb{RP}_1$ is a regular value of $\tilde{l}$. At such a point m we choose local coordinates (x, y) on N so that $m = (0,0)$, N_p is given by $y = 0$, N_h by $y > 0$, and N_e by $y < 0$. We let $\zeta = \xi + i\eta$

be a local fiber coordinate on H^0 relative to v and choose a local frame v' and related coordinate $\zeta' = \xi' + i\eta'$ for S near m. We may assume that $\varphi(v)$ is a positive multiple of iv' at m. Then

$$\varphi: \begin{array}{ll} \xi' = a\xi + b\eta, & a(0) = b(0) = c(0) = 0, \\ \eta' = c\xi + d\eta, & d(0) > 0. \end{array}$$

$\Delta = ad - bc$, $\Delta(x, 0) = 0$, $\Delta_y(x, 0) > 0$. We let $(\xi, \eta) = (1, \lambda(x))$, $\lambda(0) = 0$, span l along N_p; then the sign of $\lambda_x(0)$ gives the intersection index of l with respect to v. Since $\varphi(l) = 0$, we have $c + d\lambda = 0$, so $\lambda_x(0) = -c_x(0)/d(0)$. Also, $a(x, 0) = (bc/d)(x, 0)$, so $a_x(0) = 0$, and $\Delta_y(0, 0) = a_y(0)d(0)$, so $a_y(0) > 0$. Finally, $\varphi(v) = \varphi(1, 0) = (a, c)$ has index at $m = (0, 0)$ given by the sign of

$$\frac{\partial(a, c)}{\partial(x, y)}(0, 0) = -(a_y c_x)(0, 0).$$

Thus the index of $\varphi(v)$ at m is the same as the H-parabolic index at m. Since $\varphi(v)$ has the same index as v at any zero of v(in N_h), we have $\chi(S) = \chi(H) + p_H$. But $\chi(S) = 0$, since S is trivial, and the lemma follows.

Theorem (0.2) and Lemma (3.3) give

COROLLARY (3.4). *If M^4 is compact, orientable and generically immersed in $\mathbb{C}^4$, then $p_1(M) = -p_H$. If $p_1(M) \neq 0$, then M must have elliptic, parabolic, and hyperbolic points.*

4. Local equations for M

To facilitate the study of M near a complex tangent, we shall simplify the presentation (1.7) by means of a local holomorphic coordinate change. In this section we prove the following.

PROPOSITION (4.1). *Suppose M has a non-degenerate one-dimensional complex tangent at a point m. Then holomorphic coordinates $z = (z_1, z^\alpha, 2 \leq \alpha \leq n-1, z_n)$ can be chosen so that $m = 0$ and M is given locally by*

$$M: \begin{array}{ll} z_n = F(z_1, x), & x = (x^2, \ldots, x^{n-1}), \\ y^\alpha = f^\alpha(z_1, x), & f^\alpha = \bar{f}^\alpha, 2 \leq \alpha \leq n-1. \end{array} \qquad (4.1)$$

If m is an elliptic or hyperbolic point, then

$$F=q+H,\ q=az_1^2+bz_1\bar{z}_1+a\bar{z}_1^2,\ H=O(3),$$
$$f^\alpha=b^\alpha z_1\bar{z}_1+h^\alpha,\qquad h^\alpha=O(3),\tag{4.2}$$

where $a\geq 0$, *and* b, b^α *are either* 0 *or* 1. *If m is a parabolic point, then*

$$F=Q+H,\qquad f^\alpha=O(4)$$
$$Q=\tfrac{1}{2}(z_1+\bar{z}_1)^2+i(z_1-\bar{z}_1)c(x),\qquad c(x)=c_\beta x^\beta,\tag{4.3}$$
$$H=(-i\eta(z_1+\bar{z}_1)+\eta_\beta x^\beta)z_1\bar{z}_1+O(4),$$

where β *is summed from* 2 *to* $n-1$, c_β, η_β *are real, and* η *is either* 0 *or* 1. *If the transversality condition* $dB\wedge d\bar{B}\neq 0$ *holds, then* $c(x)\not\equiv 0$. *In this case the parabolic line at m, which is the* y_1*-axis, is tangent to* N_p *if and only if* $\eta=0$.

We remark that (4.2) is already known [1], [10].

We begin with M in the form (1.7). If $b\neq 0$, we replace z_n by bz_n to make $b=1$. By a rotation $z_1\to\mu z_1$, $\mu\bar{\mu}=1$, we can make $c\geq 0$. Then by a change of the form

$$z_n\to z_n+(c-a)z_1^2+e_\alpha z_1z^\alpha+f_{\beta\alpha}z^\alpha z^\beta,$$
$$z^\alpha\to z^\alpha+2i(a^\alpha z_1^2+d_\beta^\alpha z_1z^\beta+f_{\beta\gamma}^\alpha z^\beta z^\gamma),\tag{4.4}$$

we can achieve (4.2) but with

$$H=c(x)z_1+\bar{c}(x)\bar{z}_1+O(3),\qquad c(x)=c_\beta x^\beta.\tag{4.5}$$

The b^α are either 0 or can be made 1 by $z^\alpha\to b^\alpha z^\alpha$.

We make the further change

$$z_1\to z_1+A(z),\qquad A(z)=A_\alpha z^\alpha,\tag{4.6}$$

under which $c(x)$ in (4.5) changes by

$$c(x)\to c(x)+2aA(x)+b\bar{A}(x),$$
$$\bar{c}(y)\to\bar{c}(x)+bA(x)+2a\bar{A}(x).$$

If $\gamma=|a/b|\neq\frac{1}{2}$, then the determinant $b^2-4a^2\neq 0$, and $A(z)$ can be chosen uniquely to make $c(x) \to 0$. If $\gamma = \frac{1}{2}$, we take $b = 1$ and $a = \frac{1}{2}$. Then (4.6) results in

$$c(x) \to c(x) + 2\,\mathrm{Re}\,A(x),$$

which may be used to make the $c(x)$ in (4.5) purely imaginary. The $x^\alpha x^\beta$ terms introduced by (4.6) can then be removed by a transformation of the form (4.4). This gives (4.2) and the form (4.3) for the quadratic term Q.

We must investigate the third order terms in the parabolic case. Since $b = 1$, the change $z^\alpha \to z^\alpha - b^\alpha z_n$, followed by one of the type (4.4), makes $f^\alpha \equiv h^\alpha = O(3)$ in (4.1). We put

$$\begin{aligned} h^\alpha &= h_0^\alpha + h_\beta^\alpha x^\beta + h_{\beta\gamma}^\alpha x^\beta x^\gamma + c_{\beta\gamma\rho}^\alpha x^\beta x^\gamma x^\rho + O(4),\\ h_0^\alpha &= c^\alpha z_1^3 + e^\alpha z_1^2 \bar{z}_1 + \bar{e}^\alpha z_1 \bar{z}_1^2 + \bar{c}^\alpha \bar{z}_1^3,\\ h_\beta^\alpha &= c_\beta^\alpha z_1^2 + e_\beta^\alpha z_1 \bar{z}_1 + \bar{c}_\beta^\alpha \bar{z}_1^2,\ e_\beta^\alpha \text{ real},\\ h_{\beta\gamma}^\alpha &= c_{\beta\gamma}^\alpha z_1 + \bar{c}_{\beta\gamma}^\alpha \bar{z}_1,\ c_{\beta\gamma\rho}^\alpha \text{ real}. \end{aligned}$$

The transformation

$$z^\alpha \to z^\alpha + 2i\{c^\alpha z_1^3 + c_\beta^\alpha z_1^2 z^\beta + c_{\beta\gamma}^\alpha z_1 z^\beta z^\gamma + \tfrac{1}{2} c_{\beta\gamma\rho}^\alpha z^\beta z^\gamma z^\rho\} \tag{4.7}$$

reduces h^α to the form

$$h^\alpha = (c^\alpha z_1 + \bar{c}^\alpha \bar{z}_1 + c_\beta^\alpha x^\beta) z_1 \bar{z}_1 + O(4). \tag{4.8}$$

The substitution

$$z^\alpha \to z^\alpha + 2i\{c^\alpha z_1 + \tfrac{1}{2} c_\beta^\alpha z^\beta\} z_n, \tag{4.9}$$

followed by another one of type (4.7) (to remove any newly introduced third order terms already removed by (4.7)) results in

$$\begin{aligned} z_n &= Q + H, \qquad & H &= O(3)\\ y^\alpha &= h^\alpha, & h^\alpha &= O(4). \end{aligned} \tag{4.10}$$

Next we consider the third order terms in h,

$$\begin{aligned} H &= H_0 + H_\alpha x^\alpha + H_{\alpha\beta} x^\alpha x^\beta + K_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma + O(4),\\ H_0 &= K_0 z_1^3 + K_1 z_1^2 \bar{z}_1 + K_2 z_1 \bar{z}_1^2 + K_3 \bar{z}_1^3,\\ H_\alpha &= K_{\alpha 0} z_1^2 + K_{\alpha 1} z_1 \bar{z}_1 + K_{\alpha 2} \bar{z}_1^2,\\ H_{\alpha\beta} &= K_{\alpha\beta 0} z_1 + K_{\alpha\beta 1} \bar{z}_1. \end{aligned} \tag{4.11}$$

We shall simplify this by means of a transformation of the form

$$\begin{aligned} &z_1 \to z_1 + A(z_1, z^\alpha, z_n), \qquad A = A_2 + A_0 z_n, \\ &z_n \to z_n + B(z_1, z^\alpha, z_n), \qquad B = B_3 + B_1 z_n, \\ &A_2 = A_{20} z_1^2 + A_{2\alpha} z_1 z^\alpha + A_{2\alpha\beta} z^\alpha z^\beta, \qquad A_0 = \text{const.}, \\ &B_3 = B_{30} z_1^3 + B_{3\alpha} z_1^2 z^\alpha + B_{3\alpha\beta} z_1 z^\alpha z^\beta + B_{3\alpha\beta\gamma} z^\alpha z^\beta z^\gamma, \\ &B_1 = B_{10} z_1 + B_{1\alpha} z^\alpha. \end{aligned} \tag{4.12}$$

This will not alter any of the previous normalizations. Note that

$$Q(z_1 + A, x) = Q(z_1, x) + (z_1' + \bar z_1)(A + \bar A) + Q(A, x).$$

Therefore, when we substitute (4.12) into (4.10), we get

$$H \to H + (z_1 + \bar z_1)(A + \bar A) - B + i(A - \bar A)c(x) + \tfrac{1}{2}(A + \bar A)^2, \tag{4.13}$$

in which we must make the substitution (4.10). We shall simplify the terms of H in order of increasing degree in x^α. This allows us to ignore the term $i(A - \bar A)c(x)$, and hence $Q(A, x)$, since $(A + \bar A)^2$ is of fourth order.

In simplifying H_0 we ignore terms in x^α and $z^\alpha = x^\alpha + O(4)$, so that

$$H_0 \to H_0 + (z_1 + \bar z_1)(A_2 + \bar A_2 + (A_0 + \bar A_0)Q) - B_3 - B_1 Q,$$

with $A_2 \equiv A_{20} z_1^2$, $B_3 = B_{30} z_1^3$, $B_1 \equiv B_{10} z_1$, $Q \equiv \frac{1}{2}(z_1 + \bar z_1)^2$. Comparison of coefficients shows that

$$\begin{aligned} &K_0 \to K_0 + A_{20} + \tfrac{1}{2}(A_0 + \bar A_0) - B_{30} - \tfrac{1}{2}B_{10}, \\ &K_1 \to K_1 + A_{20} + \tfrac{3}{2}(A_0 + \bar A_0) - B_{10}, \\ &K_2 \to K_2 + \bar A_{20} + \tfrac{3}{2}(A_0 + \bar A_0) - \tfrac{1}{2}B_{10}, \\ &K_3 \to K_3 + \bar A_{20} + \tfrac{1}{2}(A_0 + \bar A_0). \end{aligned}$$

By proper choice of A_{20} and B_{30} we can realize $K_0 = K_3 = 0$, after which $A_{20} = -\text{Re}\, A_0$, $B_{30} = -\frac{1}{2}B_{10}$. Then $K_1 - K_2 \to K_1 - K_2 - \frac{1}{2}B_{10}$, so that we can make $K_1 = K_2$, and restrict to $B_{10} = 0$. This leaves the change $K_1 \to K_1 + 2\,\text{Re}\, A_0$, by which we make $K_1 = -i\eta$, purely imaginary.

To simplify $H_\alpha x^\alpha$ in (4.11), we set $A_{20} = A_0 = B_{30} = B_{10} = 0$ in (4.12) and work mod $x^\alpha x^\beta$, $z^\alpha z^\beta$. With $A_0 = A_{20} = 0$, $i(A - \bar A)c(x) \equiv 0$, mod $x^\alpha x^\beta$, so

$$H_\alpha x^\alpha \to H_\alpha x^\alpha + (z_1 + \bar z_1)(A + \bar A) - B_3 - B_1 Q,$$

with $A \equiv A_{2\alpha} z_1 z^\alpha$, $B_3 \equiv B_{3\alpha} z_1^2 z^\alpha$, $B_1 \equiv B_{1\alpha} z_1^\alpha$, and $Q \equiv \frac{1}{2}(z_1 + \bar{z}_1)^2$. Comparison of coefficients gives

$$K_{\alpha 0} \to K_{\alpha 0} + A_{2\alpha} - B_{3\alpha} - \tfrac{1}{2} B_{1\alpha},$$
$$K_{\alpha 1} \to K_{\alpha 1} + A_{2\alpha} + \bar{A}_{2\alpha} - B_{1\alpha},$$
$$K_{\alpha 2} \to K_{\alpha 2} + \bar{A}_{2\alpha} - \tfrac{1}{2} B_{1\alpha}.$$

So we normalize to $K_{\alpha 0} = K_{\alpha 2} = 0$ and restrict to $A_{2\alpha} = B_{3\alpha} + \frac{1}{2} B_{1\alpha} = \frac{1}{2} \bar{B}_{1\alpha}$. It follows that $K_{\alpha 1} \to K_{\alpha 1} + \frac{1}{2}(\bar{B}_{1\alpha} - B_{1\alpha})$, so that we can make $K_{\alpha 1} = \eta_\alpha = \bar{\eta}_\alpha$, real.

Now we further restrict to $A_{2\alpha} = \beta_{3\alpha} = B_{1\alpha} = 0$ in (4.12) and work mod $x^\alpha x^\beta x^\gamma$, $z^\alpha z^\beta z^\gamma$. Again $i(A - \bar{A})c(x)$ can be ignored in (4.13). We have

$$H_{\alpha\beta} x^\alpha x^\beta \to H_{\alpha\beta} x^\alpha x^\beta + (z_1 + \bar{z}_1)(A + \bar{A}) - B_3,$$

where $A \equiv A_{2\alpha\beta} z^\alpha x^\beta$, $B_3 \equiv B_{3\alpha\beta} z_1 z^\alpha z^\beta$. This results in the change

$$K_{\alpha\beta 0} \to K_{\alpha\beta 0} + A_{2\alpha\beta} - B_{3\alpha\beta},$$
$$K_{\alpha\beta 1} \to K_{\alpha\beta 1} + \bar{A}_{2\alpha\beta}.$$

It's clear that we can make $K_{\alpha\beta 0} = K_{\alpha\beta 1} = 0$. Finally, we remove the term $K_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma$ by a transformation

$$z_n \to z_n + B_{3\alpha\beta\gamma} z^\alpha z^\beta z^\gamma.$$

This achieves the form (4.3). If $\eta \neq 0$, a dilation $(z_1, z^\alpha, z_n) \to (\lambda z_1, \lambda z^\alpha, \lambda^2 z_n)$ with λ real results in $\eta \to \lambda \eta$, so we can make $\eta = 1$.

At a parabolic point (1.8) and (4.3) give

$$B = (i/2)^{n-2}(z_1 + \bar{z}_1 + ic(x) + i\eta(\bar{z}_1^2 + 2 z_1 \bar{z}_1) + \eta_\beta x^\beta \bar{z}_1) + O(3). \tag{4.14}$$

It follows that $dB \wedge d\bar{B} = 4^{1-n} ic(dx) \wedge dx_1 + O(1)$, so that $c(x) \neq 0$ if the transversality condition holds. We make a linear change in the coordinates $(x^2, \ldots, x^{n-1})$ so that $c(x) = x^2$, then N has the local equations

$$\begin{aligned} x_1 &= O(3), \\ x^2 &= -\eta y_1^2 + \sum_{\beta=3}^{n-1} \eta_\beta x^\beta y_1 + O(3). \end{aligned} \tag{4.15}$$

The conditions $Xr = X\bar{r} = Xr^{\alpha} = 0$, which determine X give

$$X = \frac{\partial}{\partial z_1} + (Q+H)_{z_1}\frac{\partial}{\partial z_n} + O(3). \tag{4.16}$$

Also,

$$\begin{aligned} XB &= B_{z_1} + O(3) = 1 + \eta y_1 + O(2), \\ \bar{X}B &= B_{\bar{z}_1} + O(3) = 1 + \sum_{\beta=3}^{n-1} \eta_\beta x^\beta + O(2). \end{aligned} \tag{4.17}$$

The condition (1.11) gives $a(0) + \bar{a}(0) = 0$, so we may take $a = a' + i$, a' real, $a'(0) = 0$. Then

$$YB = a'(2 + \eta y_1 + \eta_\beta x^\beta) + i(\eta y_1 - \eta_\beta x^\beta) + O(2),$$

so that $a' = O(2)$. Thus, in coordinates $(y_1, x^2, \ldots, x^{n-1})$

$$Y = \partial/\partial y_1 + O(2). \tag{4.18}$$

From (4.17) and (1.9)

$$\Delta = 2\eta y_1 - 2\sum_{\beta=3}^{n-1} \eta_\beta x^\beta + O(2); \tag{4.19}$$

so that $Y[\Delta] = 2\eta + O(1)$. It follows that Y is tangent to $N_p : \Delta = 0$ if and only if $\eta = 0$. If $\eta = 1$, then $Y[\Delta] > 0$ implies that Y points toward N_h.

5. A formula for the Euler number

To derive our formula we shall make use of the Poincare–Hopf theorem characterizing the Euler number $\chi(M)$ as the sum of the indices of the zeros of a vector field tangent to M. This does not require M to be orientable and is applicable to compact manifolds with boundary, provided the vector field points outward along the boundary. For M^n immersed in the complex n-manifold $\tilde{M}$ with normal bundle F, $\chi(F)$ denotes the sum of the indices of the zeros of a suitable section of F. The index at an isolated zero $m \in M$ is well defined since $T_m\tilde{M} = T_mM \oplus F_m$ as oriented vector spaces locally. A reversal of the local orientation of M near m results in a reversal of that of F as well as of TM.

In this section we prove the following, which does not require M to be orientable.

THEOREM (5.1). *Suppose that the compact n-dimensional manifold M is immersed in the complex n-dimensional manifold $\tilde{M}$ with at most nondegenerate, one-dimensional complex tangents as in section 1 and Proposition (2.1c). Then*

$$\chi(M)=\varepsilon_n\chi(F)+e-h+p, \qquad \varepsilon_n=(-1)^{(n-1)n/2}, \tag{5.1}$$

where $e=\chi(N_e)$, $h=(-1)^n\chi(N_h)$, and p is the parabolic index.

If M is also orientable and embedded in $\mathbb{C}^n$, then a theorem of Whitney (see [4] or [9]) asserts that $\chi(F)=0$. Theorem (0.1) follows immediately from this. As mentioned after Proposition (2.1) the assumptions of Theorem (5.1) are generic if $n\leq 5$. The remainder of this section is devoted to the proof of Theorem (5.1).

We choose some convenient hermitian metric on $\tilde{M}$ and denote by $\pi_m: T_m\tilde{M}\to F_m$, the orthogonal projection onto F_m along $T_m\equiv T_mM$. Then $\pi_m\circ J_m$ gives a linear mapping from T_m to F_m, which will be a linear isomorphism if m is a totally real point of M. If v is a vector field tangent to M, then πJv is a section of F. The idea of the proof is to relate the index sum of πJv to that of v for a suitable choice of v.

About any particular m in M we choose holomorphic coordinates $z=x+iy$ for $\tilde{M}$ centered at m. The orientation of $\tilde{M}$ is given by the local form

$$\tilde{\Omega}=\prod_{\alpha=1}^{n}\left(\frac{i}{2}dz^\alpha\wedge d\bar{z}^\alpha\right)=\varepsilon_n\,dx^1\wedge\cdots\wedge dx^n\wedge dy^1\wedge\cdots\wedge dy^n, \tag{5.2}$$

and the operator J is identified with $(x, y)\to(-y, x)$. Suppose m is a totally real point of M. Then the coordinates may be chosen so that T_m is the x-space and F_m is the y-space, which by (5.2) have the orientations

$$\Omega_T=dx^1\wedge\cdots\wedge dx^n, \qquad \Omega_F=\varepsilon_n\,dy^1\wedge\cdots\wedge dy^n.$$

Since π is smoothly deformable to $(x, y)\to(0, y)$ and $\pi\circ J$ to $(x, 0)\to(0, x)$, we have

$$(\pi\circ J)^*\Omega_F=c\Omega_T, \qquad \operatorname{sgn} c=\varepsilon_n.$$

It follows that the effect of πJ on the index of a vector field v with isolated zero at

m is

$$\operatorname{ind}_{F,m}(\pi Jv) = \varepsilon_n \operatorname{ind}_{M,m}(v),$$

so that

$$\sum_{m \notin N} \operatorname{ind}_{F,m}(\pi Jv) = \varepsilon_n \chi(M). \tag{5.3}$$

This proves (5.1) if M is totally real.

In the general case we start with a smooth vector field v_0 tangent to N with the following properties. It is to have only finitely many zeros m_j, $1 \leq j \leq l$, which are non-degenerate and lie in $N_e \cup N_h$, and is to be transverse to N_p and point toward N_h along N_p. Furthermore, the line field k along N_p spanned by v_0 is to satisfy $k_m = l_m$ for only finitely many $m \in N_p$, and at such m this intersection is transverse in the space P (see (1.15)). We find disjoint neighborhoods U_j of m_j in $N - N_p$ and smooth sections v_j of H, compactly supported in U_j, with $v_j(m_j) \neq 0$. Then we smoothly extend $v_0 + \sum v_j$ to a vector field v on M having a finite number of non-degenerate zeros. By construction v does not vanish on N; however πJv will have a zero at each m_j and at each m in N_p where $v(m) \in l_m \subset H_m$, as well as at each zero of v. There is much freedom in the choice of such a v, which we shall specify more precisely later.

Let m_j be one of the zeros of v_0, and choose coordinates as in (4.1), (4.2), so that (z_1, x^α) are coordinates on M. We may assume that the hermitian metric on $\tilde{M}$ has been chosen so that F_m coincides with the (y^α, z_n)-space for all m near m_j. The local orientations are given by

$$\Omega_T = \frac{i}{2} dz_1 \wedge d\bar{z}_1 \wedge dx^2 \wedge \cdots \wedge dx^{n-1}, \tag{5.4}$$

$$\Omega_F = \varepsilon_{n-2}\, dy^2 \wedge \cdots \wedge dy^{n-1} \wedge \frac{i}{2} dz_n \wedge d\bar{z}_n. \tag{5.5}$$

We set $G(z_1, x^\alpha) = (z_1, x^\alpha + if^\alpha, F)$, so that $G_{x_1}, G_{y_1}, G_{x^\alpha}$ span $T(M)$. In the local coordinates (z_1, x^α) on M we have

$$v = v_1 \partial/\partial z_1 + \bar{v}_1 \partial/\partial \bar{z}_1 + v^\alpha \partial/\partial x^\alpha, \tag{5.6}$$

so that as a vector in $\mathbb{C}^n$

$$v \equiv v[G] = (v_1, v^\alpha + iv[f^\alpha], v[F]),$$

where $v[\cdot]$ denotes directional derivative. It follows that

$$Jv \equiv iv[G] = (iv_1, -v[f^\alpha] + iv^\alpha, iv[F]),$$

so that

$$\begin{aligned}\pi Jv &\equiv iv[G] - c'G_{x_1} - c''G_{y_1} - c^\alpha G_{x^\alpha} \\ &= iv[G] - cG_{z_1} - \bar{c}G_{\bar{z}_1} - c^\alpha G_{x^\alpha} = (0, 0 + i*, *).\end{aligned}$$

Here $G_{z_1} = (1, if^\alpha_{z_1}, F_{z_1})$, $G_{\bar{z}_1} = (0, if^\alpha_{\bar{z}_1}, F_{\bar{z}_1})$, and $G_{x^\alpha} = (0, \delta^\beta_\alpha + if^\beta_{x^\alpha}, F_{x^\alpha})$, so that $c = iv_1$ and $c^\alpha = -v[f^\alpha]$. Hence, as a map from (z_1, x^α)-space to (y^α, z_n)-space, πJv has the form

$$\begin{aligned} y^\alpha &= v^\alpha - iv_1 f^\alpha_{z_1} + i\bar{v}_1 f^\alpha_{\bar{z}_1} + v[f^\beta] f^\alpha_{x^\beta}, \\ z_n &= iv[F] - iv_1 F_{z_1} + i\bar{v}_1 F_{\bar{z}_1} + v[f^\beta] F_{x^\beta}. \end{aligned} \tag{5.7}$$

If we substitute (5.7) into (5.5), we get (5.4) multiplied by the Jacobian factor

$$\varepsilon_{n-2} \frac{\partial(y^\alpha, z_n, \bar{z}_n)}{\partial(z_1, \bar{z}_1, x^\beta)}, \tag{5.8}$$

the sign of which gives the index of πJv at m_j. If we take into account (4.2), (5.7) becomes

$$\begin{aligned} y^\alpha &= v^\alpha - iv_1 b^\alpha \bar{z}_1 + i\bar{v}_1 b^\alpha z_1 + O(2), \\ z_n &= 2i\bar{v}_1 q_{\bar{z}_1} + O(2). \end{aligned}$$

We may assume that the H-component v_j added to v_0 is such that $v_1 \equiv 1$ near 0. Also, we assume that the extension of v from N to M is made so that the coefficients of v are locally independent of z_1. Then at the origin (5.8) has the value

$$4(b^2 - 4a^2) \det (\partial v_0^\alpha / \partial x^\beta)(0). \tag{5.9}$$

The sign of the determinant is the index of v_0 at m_j, and $b^2 - 4a^2$ is positive if m_j is elliptic and negative if m_j is hyperbolic. Hence,

$$\varepsilon_{n-2} \operatorname{ind}_{F,m_j}(\pi Jv) = \delta \operatorname{ind}_{N,m_j}(v_0), \tag{5.10}$$

where $\delta=+1$ if m_j is elliptic or $\delta=-1$ if m_j is hyperbolic. If we sum (5.10) over the m_j in N_e, the right hand side is $\chi(N_e)$. To get $\chi(N_h)$ we must use $-v_0$ which multiplies the determinant in (5.9) by $(-1)^{n-2}$. Thus we get

$$\varepsilon_{n-2}\sum_j \operatorname{ind}_{F,m_j}(\pi Jv)=\chi(N_e)-(-1)^n\chi(N_h), \tag{5.11}$$

which accounts for the term $e-h$ in (5.1).

Finally, we consider a zero of πJv at a point m in N_p which arises when $v(m)$, which spans the line k_m, lies in l_m. We first elaborate further on the construction of v along N_p. It is initially defined so that $k(N_p)$ intersects $l(N_p)$ transversely at m. Then it will be extended to N. We take coordinates as in (4.1), (4.3) with $c(x)=x^2$, so that N is given by (4.15). $(x^3,\dots,x^{n-1})$ gives coordinates on N_p, and $(y_1,x^3,\dots,x^{n-1})$ coordinates on N. In (5.6) we take $v_1=v^1+i$, $v^1=\bar{v}^1$, so that

$$v=\partial/\partial y_1+\sum_{j=1}^{n-1} v^j\partial/\partial x^i, \qquad v^j(0)=0. \tag{5.12}$$

The condition that v be tangent to N gives, via (4.15) and (5.12),

$$v^1=O(2), \qquad v^2=-2\eta y_1+\sum_{\beta=3}^{n-1}\eta_\beta x^\beta+O(2). \tag{5.13}$$

Thus we start with

$$v^\alpha=v^\alpha(x^3,\dots,x^{n-1}), \qquad v^\alpha(0)=0, \det\frac{\partial v^\alpha}{\partial x^\beta}(0)\neq 0, \qquad 3\leq\alpha,\beta\leq n-1, \tag{5.14}$$

and determine v^1 and v^2 by (5.13). We then extend this vector v locally from N_p to N by keeping (5.14) independent of y_1, and from N to M by keeping (5.14) independent of x_1 and x^2. Again we assume that F_m is the (y^α, z_n)-space for m near 0. Note that we may take $\eta=1$, since $l_0=k_0$ is transverse to N_p.

The parabolic index as defined in section 1 is computed relative to a coordinate system (x_*^α, y_*) with $y_*=0$ on N_p. Therefore we set (4.19)

$$y_*=\tfrac{1}{2}\Delta=y_1-\sum_{\beta=3}^{n-1}\eta_\beta x^\beta+O(2), \qquad x_*^\alpha=x^\alpha.$$

The chain rule in (1.14) gives

$$w_*^\alpha = w^\alpha(\partial y_*/\partial y_1 + w^\beta\, \partial y_*/\partial x^\beta)^{-1} = w^\alpha(1-\eta_\beta w^\beta + O(1))^{-1}.$$

Since $w^\alpha = O(1)$ for both Y (4.18) and v (5.12), and $\partial/\partial x_*^\beta = \partial/\partial x^\beta$ for functions defined along N_p, we have $\partial w_*^\alpha/\partial x_*^\beta(0) = \partial w^\alpha/\partial x^\beta(0)$. Thus the parabolic intersection index at $m=0$ is given (see (1.15)) by the sign of

$$\Omega_P\left(\frac{\partial l}{\partial x}, \frac{\partial k}{\partial x}\right)(0) = \det\begin{bmatrix} \delta_{\alpha\beta} & 0 \\ \delta_{\alpha\beta} & \partial v^\alpha/\partial x^\beta(0)\end{bmatrix}.$$

Hence,

$$\mathrm{ind}_{P,m}(l, k) = \mathrm{sgn}\det(\partial v^\alpha/\partial x^\beta(0))_{3\le\alpha,\beta\le n-1}. \tag{5.15}$$

For the index of πJv at m we again compute the determinant (5.8). We substitute (4.3) into (5.7) and ignore second order terms. By (5.12) and (5.13) we get

$$y^2 \equiv v^2 \equiv -2y_1 + \sum_{\beta=3}^{n-1} \eta_\beta x^\beta, \qquad y^\alpha \equiv v^\alpha(x^3, \dots, x^{n-1}), \qquad 3\le\alpha\le n-1,$$
$$z_n \equiv 2i\bar{v}_1 Q_{z_1} \equiv 4x_1 + 2ix^2.$$

Thus,

$$\frac{\partial(y^2, y^\alpha, z_n, \bar{z}_n)}{\partial(z_1, \bar{z}_1, x^2, x^\beta)}(0) = 16\det[\partial v^\alpha/\partial x^\beta(0)]_{3\le\alpha,\beta\le n-1}.$$

Comparison with (5.15) gives

$$\mathrm{ind}_{F,m}(\pi Jv) = \varepsilon_{n-2}\,\mathrm{ind}_{P,m}(l, k),$$

so that

$$\sum_{N_p} \mathrm{ind}_{F,m}(\pi Jv) = \varepsilon_{n-2}\, p. \tag{5.16}$$

Combining (5.3), (5.11), and (5.16) gives (5.1), since $\varepsilon_n\varepsilon_{n-2} = -1$.

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