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Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **60 (1985)**

PDF erstellt am: **16.08.2024**

Persistenter Link: https://doi.org/10.5169/seals-46313

Comment. Math. Helvetici 60 (1985) 280–283 0010–2571/85/020280–04$01.50 + 0.20/0

A non-immersion theorem for hyperbolic manifolds

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Let M^n be a complete hyperbolic n-manifold, so that $M^n = H^n/\Gamma$ where Γ is a discrete group of isometries of hyperbolic space H^n. The manifold M^n is said to be *elementary* if the limit set of Γ in the sphere at infinity has at most two points. It is not difficult to see that in the case of non-elementary M^n the group Γ has infinite limit set. The purpose of this paper is to establish the following

THEOREM. *A non-elementary complete hyperbolic manifold M^n cannot be isometrically immersed in $\mathbb{R}^{2n-1}$.*

To put this result in the proper perspective let us recall that a well-known result of Hilbert [4] states that H^2 cannot be isometrically immersed in $\mathbb{R}^3$. It has been conjectured for a long time that hyperbolic n-space H^n cannot be isometrically immersed in $\mathbb{R}^{2n-1}$, $n \geq 2$. In this regard our theorem represents a partial verification of this conjecture.

The local analysis in Hilbert's theorem carries over to the higher dimensional case but it is not clear how to use this information at the global level. One source of difficulty lies in the fact that the sine-Gordon system, a highly non-linear system of partial differential equations associated to the problem ([1], [5]) is not yet fully understood. In our proof we abandon this system and concentrate on the metric dissimilarities between H^n and $\mathbb{R}^n$. However, the crucial relation (2) below – that leads in the case of surfaces to Tchebychef nets and, ultimately, to a contradiction – is not used in its full force. A better understanding of its significance might lead to sharper results. On the other hand, it seems that the existence of Tchebychef nets in higher dimensions is a less compelling evidence of a contradiction. In two dimensions they yield coverings of H^2 by disjoint "squares" (quadrilaterals all of whose sides are asymptotic curves of equal lengths). Thinking about the effect that the isoperimetric inequality will have on the shape of the squares we eventually become convinced that they will have unbounded distortion and that singularities will arise. In the case $n = 3$, say, the "cubes" which are far

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away will also have some degenerating angles but, at least in principle, their labels can change in a complicated manner and it is not geometrically clear how this will force appearance of singularities.

In the first version of this paper our theorem was proved in the case $n=3$ only, where we used existence of incompressible surfaces to guarantee that the group Γ was "big" enough. We are indebted to Dennis Sullivan for pointing out to us that, by an argument of Klein, non-elementary groups contain non-trivial free subgroups, regardless of dimension. This enabled us to extend our proof to all dimensions.

Proof of the theorem. Let us start by establishing the fact that if $M^n = H^n/\Gamma$ is non-elementary then Γ contains non-trivial free subgroups. A hyperbolic (parabolic) isometry $g \in \Gamma$ has a fundamental domain in the Poincaré model which is the complement of two disjoint balls about the fixed points (two externally tangent balls about the fixed point). For a power g^n the balls shrink to the fixed point (s). Since the group is non-elementary we can find two isometries with disjoint sets of fixed points. Taking powers we can therefore assume that Γ contains isometries g_1, g_2 having fundamental regions X_1, X_2 satisfying $X_1 \cup X_2 = H^n$, $X_1 \cap X_2 \neq \emptyset$. It follows that the image of a point $p \in X_1 \cap X_2$ by a non-trivial word in g_1 and g_2 lies outside $X_1 \cap X_2$. In particular, there are no relations and the group generated by g_1 and g_2 is free.

Suppose now that $j: M^n \to \mathbb{R}^{2n-1}$ is an isometric immersion. According to the previous paragraph we can pass to a covering and suppose that $\pi_1(M^n)$ is free on two generators. Let $p: H^n \to M^n$ be the canonical projection. As a consequence of Cartan's theorem on exteriorly orthogonal forms it it is possible to diagonalize simultaneously all second fundamental forms of the immersion $p \circ j$ by orthonormal "principal" vectors $e_1, \ldots, e_n$ ([3]). Moreover, they can be extended to a global frame on H^n. Furthermore, the functions

$$a_i = \left(1 + \sum_{\lambda=n+1}^{2n-1} b_{ii}^{\lambda} b_{ii}^{\lambda}\right)^{-1/2} \quad \text{(see [3], eq. 10),} \tag{1}$$

where the b_{ii}^{λ} are the components of the $(n-1)$ second fundamental forms associated to an orthonormal basis of the normal space of the immersion, satisfy

$$\sum_{i=1}^{n} a_i^2 = 1, \qquad [a_i e_i, a_j e_j] = 0 \tag{2}$$

(see bottom of page 161 and top of page 166 of [3]).

Following the flow lines of the (complete) vector fields $a_i e_i$ we get a parametrization $J:\mathbb{R}^n \to H^n$ whose coordinate vectors are the vectors $a_i e_i$. The pull back of the hyperbolic metric by J is $g=\sum_{i=1}^n a_i^2\, dx_i^2$. We then have the following situation

$$(\mathbb{R}^n, g) \xrightarrow{J} H^n \xrightarrow{p} M^n \xrightarrow{i} \mathbb{R}^{2n-1}.$$

Let L be a compact neighborhood of two loops generating $\pi_1(M^n)$ and let C be a connected component of $J^{-1}(p^{-1}(L))$. For a fixed $q \in C$ above the base point, let C^s be the set of points in C that can be joined to q by a curve having g-length not greater than s and lying entirely in C. Let also $B(r)$ be the euclidean ball centered at q of radius r. It follows from (1) that $a_i \geq (1+|b|^2)^{-1/2}$, where b is the vector-valued second fundamental form. In particular, since C covers the compact set L, the coefficients a_i are bounded from below on C by a constant $\delta > 0$. Let now $\alpha:[a, b] \to C \cap B(r)$ be a path starting at q. From

$$\int_a^b \left(\sum_{i=1}^n a_i^2 \alpha_i'^2\right)^{1/2} dt \geq \delta \int_a^b \left(\sum_{i=1}^n \alpha_i'^2\right)^{1/2} dt$$

we get $C \cap B(r) \supset C^{\delta r}$. Denoting by $\operatorname{Vol}(A)$ the g-volume of a set $A \subset \mathbb{R}^n$ we have

$$\operatorname{Vol} C^{\delta r} \leqq \operatorname{Vol} C \cap B(r) \leq \operatorname{Vol} B(r) = \int_{B(r)} \left(\prod_{i=1}^n a_i\right) dx_1 \cdots dx_n = O(r^n),$$

by (2). This shows $\operatorname{Vol} C^s = O(s^n)$. We claim that $\operatorname{Vol} C^s$ must actually grow exponentially. Let μ denote the maximum length of the loops associated to the set L. The end point of the lift through q of a word of length at most s belongs to $C^{\mu s}$. Hence, for small enough $\varepsilon > 0$, $C^{\mu s+\varepsilon}$ contains at least as many disjoint balls of radius ε as there are words of length $\leq s$. Since $\pi_1(M^n)$ is free on two generators the number of words grows exponentially with the length. Hence $\operatorname{Vol} C^s$ must grow exponentially with s and we have established a contradiction.

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Received: August 7, 1984.