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Lefschetz fixed point theorem for intersection homology

MARK GORESKY[1] and ROBERT MACPHERSON[2]

§1. Introduction

In this paper we give some formulas for the Lefschetz numbers in intersection in homology of a class of "placid" self maps of singular spaces. A placid self map $f: X \to X$ induces a self homomorphism

$$f_{*_i}: IH_i^{\bar{m}}(X) \to IH_i^{\bar{m}}(X)$$

on the intersection homology $IH_i^{\bar{m}}(X)$ of X. Its (intersection) Lefschetz number $IL(f)$ is given as usual by the formula

$$IL(f) = \sum_i (-1)^i \text{ Trace } (f_{*_i})$$

We show that both the graph of f and the diagonal carry fundamental classes in the intersection homology of $X \times X$, and that the Lefschetz number $IL(f)$ is the intersection number of these two classes. This is exactly the procedure originally used by Lefschetz to study fixed points in manifolds using ordinary homology [L]. So the results of this paper can be viewed as an addition to the series of theorems which show that the intersection homology of a singular space behaves like the ordinary homology of a smooth variety (see [CGM], [GM 4]).

Let X be an n dimensional Witt space. (See §2 for the definition of a Witt space. Any complex analytic variety of pure dimension k is a $2k$ dimensional Witt space.) A self map f of X is called placid if X can be stratified so that the dimension of the inverse image of each stratum is at most the dimension of that stratum. For example, all maps of manifolds are placid and all flat maps of algebraic varieties are placid. The induced map

$$f_{*_i}: IH_i^{\bar{m}}(X) \to IH_i^{\bar{m}}(X)$$

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on the middle intersection homology groups $IH_i^{\bar{m}}(X)$ of X may be defined by observing that with appropriate stratification the image of an allowable cycle in the sense of [GM 1] is still allowable.

Our approach hinges on the fact that the graph $G(f)\subset X\times X$ and the diagonal $\Delta\subset X\times X$ carry fundamental classes $[G(f)]$ and $[\Delta]$ in $IH_n^{\bar{m}}(X\times X)$, where n is the dimension of X. This fact is rather surprising since $G(f)$ and Δ are usually *not* allowable as cycles for middle intersection homology. For example, if $p\in X$ is an isolated singularity, then Δ goes through the stratum $p\times p$, but allowable n-cycles in $X\times X$ must miss all zero dimensional strata. Our solution to this difficulty is to show that $G(f)$ and Δ can be "moved slightly" to allowable cycles in a unique way (up to homology). We make this precise in two ways:

1) There is a larger perversity $\bar{p}$ (see §3) for which Δ and $G(f)$ are allowable as cycles for $IH_n^{\bar{p}}(X\times X)$ and for which $IH_n^{\bar{m}}(X\times X)\to IH_n^{\bar{p}}(X\times X)$ is an isomorphism (Proposition 4.2). Then $[G(f)]$ and $[\Delta]$ are the $IH_n^{\bar{m}}(X\times X)$ elements which map to these cycles.

2) Let $N(G(f))$ and $N(\Delta)$ be regular neighborhoods of $G(f)$ and Δ. Then there are unique classes $[G(f)]\in IH_n^{\bar{m}}(N(G(f)))$ and $[\Delta]\in IH_n^{\bar{m}}(N(\Delta))$ that restrict to their usual fundamental classes in the nonsingular parts of $N(G(f))$ and $N(\Delta)$. (See Theorem 5.2.)

Like any two intersection homology n-cycles in a $2n$ dimensional Witt space, $[G(f)]$ and $[\Delta]$ have an intersection number $[G(f)]\cdot[\Delta]$

THEOREM I (§6). $IL(f)=[G(f)]\cdot[\Delta]$

One corollary of this theorem is that $IL(f)$ can naturally be expressed as a sum of contributions from the connected components of the fixed point set of f. In particular, if f has no fixed points, then $IL(f)=0$. This corollary is not new; it follows from the Lefschetz fixed point theorem of Verdier [GI]. Verdier's formula for $IL(f)$ is less explicitly computable than Theorem I above, but it is more general: it treats an arbitrary complex of sheaves S instead of the complex $IC^{\cdot}$ which gives rise to intersection homology, and it treates an arbitrary self map f provided with a lift to S. (The lift may not be naturally induced by f.)

Probably there is no single ultimate Lefschetz fixed point theorem. There will always be a trade-off between explicitness and generality. In this vein, we give two more formulas for $IL(f)$ which are still more computable, but which apply only to special cases. If $p\in X$ is an isolated fixed point of f, then the cycles Δ and $G(f)$ intersect the link $\mathscr{L}$ of (p, p) in $X\times X$ in disjoint $n-1$ cycles which we denote by Δ_L and $G_L(f)$.

THEOREM II (§9). *If p is an isolated fixed point of f, then the contribution of p to the Lefschetz number* $IL(f)$ *is the linking number of* $[\Delta_L]$ *and* $[G_L(f)]$ *in* $\mathscr{L}$.

These linking numbers make sense because $IH_{n-1}(\mathscr{L}) = IH_n(\mathscr{L}) = 0$ (§8). (A more explicit version of Theorem II for the case of algebraic curves is due to C. Alibert [A].)

If p is an isolated fixed point of f, we call f contracting at p if there is a conical open neighborhood N of p such that f takes the closure of N into N. For example, if all the eigenvalues of the self map of the Zariski tangent space to X at p have absolute value less than 1, then f will be contracting at p. If f is contracting at p, then there is an induced map

$$\bar{f}_{*_i} : IH_i^{\bar{m}}(X, X-p) \to IH_i^{\bar{m}}(X, X-p)$$

THEOREM III (§12). *If p is an isolated fixed point of f and f is contracting at p, then the contribution of p to the Lefschetz number $IL(f)$ is equal to*

$$\sum_i (-1)^i \operatorname{Trace}(\bar{f}_{*_i}).$$

Theorem III is proved by an explicit geometric computation of the linking numbers of Theorem II. The class of $[G_L(f)]$ is decomposed into its Kunneth components, each of which bounds a chain in $\mathscr{L}$. Half of these chains meet $|\Delta_{\mathscr{L}}|$ and the other half do not, for geometric reasons stemming from the fact that f is contracting. The intersection numbers with $[\Delta_{\mathscr{L}}]$ of the half that meet it are the terms of the summation formula of Theorem III. This is explained in detail in §11 and §12.

In Section 14, we give generalizations of the above theorems to the case where the self map f is replaced by a placid self correspondence. This may be of interest in the treatment of Hecke correspondences. We also treat the case of intersection homology with perversities other than the middle one (§15).

We conclude the introduction with some speculations as to the form that a more general Lefschetz fixed point theorem than the ones we state here might take for a compact complex analytic variety X. Suppose that the self map f is provided with a lift to a complex of sheaves S (i.e., we are given a map of complexes from $f^*(S)$ to S). Then this data induces a global Lefschetz number $L(f, S)$, the alternating sum of the traces of the maps induced on hypercohomology, and a local trace number $\operatorname{Tr}(p)$ at any fixed point p, the alternating sum of the traces of the maps included on the stalk cohomology at p. We also note that f induces a self map on the normal cone C to the fixed point set X^f of f.

CONJECTURE: *If the fixed point set of C contains only X^f, then the Lefschetz number $L(X, S)$ is the Euler characteristics of X^f with coefficients in the constructible function* $\operatorname{Tr}(p)$.

(See [Mac] for the definition of the Euler characteristic of a variety with coefficients in a constructible function.) Note that this conjecture is compatible with Theorem III above, since the stalk cohomology of $\mathbf{IC}^{\cdot}$ at p is just the intersection homology of N. It is also compatible by analogy with Deligne's calculation of the Lefschetz number of the Frobenius map [DB].

More generally, we conjecture that the Lefschetz number of an arbitrary self correspondence on a compact complex analytic variety lifted to a complex of sheaves is given by the Euler characteristic of the fixed point set with coefficients in some constructible function whose value at a point p of the fixed point set is determined by local data.

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§2. Conventions

Unless otherwise specified, the spaces X and Y will be compact oriented subanalytic (or P.L.) pseudomanifolds and all maps between them will be subanalytic (or P.L.). By *intersection homology* we shall mean intersection homology, with coefficients in the rational numbers $\mathbf{Q}$, and with compact support.

If no perversity is specified, we mean the middle perversity $\bar{m}(c) = \left[\frac{c-2}{2}\right]$, so we write

$$IH_i(X) = IH_i^{\bar{m}}(X)$$

We shall also assume that X is a Witt space ([S], [GM 2]) which means that there exists a stratification of X such that for each stratum of odd codimension c,

$$IH_{(c-1)/2}(L; \mathbf{Q}) = 0$$

where L is the link of the stratum.

For example every complex analytic variety is a Witt space since it can be stratified by even dimensional strata.

It follows ([GM 2]) that

$$IH_*^{\bar{m}}(X; \mathbf{Q}) \cong IH_*^{\bar{n}}(X; \mathbf{Q})$$

where $\bar{n}$ is the "upper middle" perversity,

$$n(c)=\left[\frac{c-1}{2}\right].$$

and this intersection homology group satisfies Poincaré duality.

We refer to [GM 1] for the definitions of $(\bar{m}, i)$-allowability of cycles, geometric intersections etc., and to [GM 2] for Deligne's construction of the sheaf of intersection chains. We will use the sign conventions of Dold [D] VIII §13.

We shall also use "relative" intersection homology $IH_i^{\bar{m}}(X, Y)$ where X is compact and Y is open in X or else Y is a collared boundary of X. This group is represented by $(\bar{m}, i)$-allowable chains in X whose boundary lies in Y. (Alternatively it is the hypercohomology with closed support of the restriction of the intersection homology sheaf to $X-Y$). The relative Poincaré duality theorem states that the intersection pairing

$$IH_i^{\bar{m}}(X, Y)\times IH_{n-i}^{\bar{m}}(X-Y)\to \mathbf{Q}$$

is nondegenerate.

§3. Intersection homology of products and joins

The Lefschetz fixed point theorem requires a knowledge of the global and local (intersection) homological properties of $X\times X$, since the graph of a self-map of X lies in $X\times X$. The link of a point (p, q) in $X\times X$ is the join of the link of p in X and the link of q in X, and a neighborhood of (p, q) is the cone over this join. (This is shown in §7 below). We study the intersection homology of products and joins in this preliminary section.

We recall from [GM 2] §6.3 that it is X and Y are Witt spaces then the natural homomorphism

$$\bigoplus_{p+q=i} IH_p^{\bar{m}}(X)\otimes IH_q^{\bar{m}}(Y)\to IH_i^{\bar{m}}(X\times Y)$$

is an isomorphism. Furthermore, $X\times Y$ is again a Witt space. (This is proved in [S], and also follows from Proposition 3.1 below.) So the natural homomorphism $IH_*^{\bar{m}}(X\times Y)\to IH_*^{\bar{n}}(X\times Y)$ is an isomorphism (here $\bar{n}(c)=[(c-1)/2]$).

PROPOSITION 3.1. *Suppose X and Y are compact Witt spaces of dimensions* *r and s. Let $X * Y$ denote their join. Then if $r = 2R$ or $2R+1$ and $s = 2S$ or $2S+1$, i.e., $R = \left[\frac{r}{2}\right]$, $S = \left[\frac{s}{2}\right]$,*

$$IH_{R+S}(X * Y) = IH_R(X) \otimes IH_S(Y)$$

and

$$IH_{R+S+1}(X * Y) = IH_{R+S+2}(X \times Y) = 0.$$

Proof. The intersection homology (with *compact* supports) of the open cone is

$$IH_i(\mathring{c}X) = \begin{cases} IH_i(X) & \text{for } i \le R \\ 0 & \text{for } i > R. \end{cases}$$

$IH_k(X * Y)$ may be calculated from the Mayer–Vietoris sequence for the covering $\mathring{c}X \times Y$, $X \times \mathring{c}Y$ of $X * Y$, which (using the Kunneth formula) is:

$$\longrightarrow \bigoplus_{i+j=k} A_{ij} \xrightarrow{\phi_k} \Bigl(\bigoplus_{\substack{i+j=k\\ i \le R}} A_{ij}\Bigr) \oplus \Bigl(\bigoplus_{\substack{i+j=k\\ j \le S}} A_{ij}\Bigr) \longrightarrow IH_k(X * Y) \longrightarrow$$

where A_{ij} denotes $IH_i(X) \otimes IH_j(Y)$. Note that ϕ_k is surjective for $k \ge R+S+1$ and injective for $k \le R+S+1$.

The following technical lemma will be needed in the next section when we show that the diagonal and the graph of a placid map lift canonically to the middle intersection homology of $X \times X$.

Suppose X and Y are stratified pseudomanifolds. Stratify $X \times Y$ by the product stratification. Define the following (stratum-dependent) perversities on $X \times Y$,

$$\bar{p}(A \times B) = \begin{cases} \left[\dfrac{\operatorname{cod}(A) + \operatorname{cod}(B) - 1}{2}\right] & \text{if } \operatorname{cod}(A) \ne \operatorname{cod}(B) \\[2ex] \left[\dfrac{\operatorname{cod}(A) + \operatorname{cod}(B)}{2}\right] & \text{if } \operatorname{cod}(A) = \operatorname{cod}(B) \ne 0 \end{cases}$$

$$\bar{q}(A \times B) = \begin{cases} \dfrac{\operatorname{codim} A}{2} - 1 & \text{if } \operatorname{codim} B = 0 \\[2ex] \dfrac{\operatorname{codim} B}{2} - 1 & \text{if } \operatorname{codim} A = 0 \\[2ex] \dfrac{\operatorname{codim} A}{2} + \dfrac{\operatorname{codim} B}{2} - 2 & \text{otherwise.} \end{cases}$$

where A is a stratum of X, B is a stratum of Y, and [ ] denotes the "integer part" function. Note that $\bar{q} \leqq \bar{m} \leqq \bar{n} \leq \bar{p}$.

COROLLARY 3.2. *The natural maps*

$$IH_*^{\bar{q}}(X\times Y) \to IH_*^{\bar{m}}(X\times X) \to IH_*^{\bar{n}}(X\times Y) \to IH_*^{\bar{p}}(X\times Y)$$

are isomorphisms, if X and Y are Witt spaces.

Proof. In fact the natural map of sheaves $\mathbf{IC}_{\bar{q}} \to \mathbf{IC}_{\bar{m}} \to \mathbf{IC}_{\bar{n}} \to \mathbf{IC}_{\bar{p}}$ are quasi-isomorphisms. The middle arrow is a quasi-isomorphism since $X \times Y$ is a Witt space, as we noted at the beginning of this section.

If two perversites $\bar{a}$ and $\bar{b}$ differ minimally, i.e., if they agree except for their values on one stratum $\mathcal{S}$ where $b(\mathcal{S}) = a(\mathcal{S})+1$, then to check that $\mathbf{IC}_{\bar{a}}^{\cdot} \to \mathbf{IC}_{\bar{b}}^{\cdot}$ is a quasi isomorphism it suffices to check the vanishing of a certain intersection homology group of the link of $\mathcal{S}$ (see [GM 2], §5.5). If they differ by more, this technique can be applied inductively, one stratum at a time, starting with the smallest ones. We proceed to check the relevant vanishing.

For the right hand arrow above, if codim A = codim $B = C$, the relevant group is $IH_{C-1}(L_A * L_B)$, where L_A, L_B are the links of A and B. If $C-1=2R+1=2S+1=R+S+1$, this is zero by §3.1. If $C-1=2R=2S=R+S$, this is $IH_{(C-1)/2}(L_A) \otimes IH_{(C-1)/2}(L_B)$ which is zero since X and Y are Witt spaces.

For the left hand side, let dim $L_A = 2R$ or $2R+1$ and dim $L_B = 2S$ or $2S+1$. In case dim $L_A = 2R$ and dim $L_B = 2S$ there are two offending groups (since then $\bar{q}(A \times B)$ and $\bar{m}(A\times B)$ differ by 2) which are $IH_{R+S+1}(L_A * L_B)$ and $IH_{R+S+2}(L_A * L_B)$. Otherwise there is just one, which is $IH_{R+S+2}(L_A * L_B)$. By §3.1, these are zero.

§4. Placid maps and their graphs

A subanalytic map $f: X \to Y$ between two subanalytic pseudo-manifolds will be called placid if there exists a subanalytic stratification of Y such that for each stratum S in Y we have

$$\operatorname{codim}_X f^{-1}(S) \geq \operatorname{codim}_Y (S)$$

Examples of placid maps are: branched coverings, smooth maps and flat maps (in algebraic geometry), Thom (A_f) mappings, and all subanalytic maps between smooth manifolds.

PROPOSITION 4.1. *Suppose $f: X \to Y$ is a placid map. Then pushforward of chains and pullback of generic chains induces homomorphisms on intersection homology,*

$$f_*: IH_i^{\bar{m}}(X) \to IH_i^{\bar{m}}(Y)$$
$$f^*: IH_i^{\bar{m}}(Y) \to IH_{i+\dim(X)-\dim(Y)}^{\bar{m}}(Y).$$

Proof. Stratify X so that $f^{-1}(S)$ is a union of strata in X whenever S is a stratum of Y. (This can be done since f is subanalytic.) A simple dimension count shows that for any $(\bar{m}, i)$-allowable chain ξ in X, the image $f(|\xi|)$ is also $(\bar{m}, i)$-allowable. This defines f_*. (Notation here as in [GM 1].)

To construct f^*, observe by McCrory [Mc] that any homology class in $IH_i^{\bar{m}}(Y)$ can be represented by a chain ξ with the property that for any stratum S of Y and for any stratum $A \subset f^{-1}(S)$ of X, the map

$$f \mid A: A \to S$$

is dimensionally transverse to $|\xi| \cap S$. It follows that $f^{-1}(|\xi|)$ is $(\bar{m}, i+\dim(X)-\dim(Y))$-allowable. A similar remark applies to homologies betwen two cycles ξ_1 and ξ_2. We thus obtain a homomorphism f^* on intersection homology.

Remark. The preceding calculation shows that for any perversity $\bar{p}$ and for any coefficient ring R we obtain homomorphisms

$$f_*: IH_i^{\bar{p}}(X, R) \to IH_i^{\bar{p}}(Y; R)$$
$$f^*: IH_i^{\bar{p}}(Y; R) \to IH_{i+\dim(X)-\dim(Y)}^{\bar{p}}(X; R)$$

whenever $f: X \to Y$ is placid.

Remark. If X and Y are topological pseudomanifolds of dimension n and m not necessarily subanalytic, then a placid map can be defined as a continuous map for which there exist stratifications ([GM 2], §1.1) $\{X_i\}$ and $\{Y_i\}$ such that $f^{-1}(X_{n-i}) \subset Y_{m-i}$ for all i. Then Deligne's construction gives a map f^* $\mathbf{IC}_Y^{\cdot} \to \mathbf{IC}_X^{\cdot}[m-n]$ and hence gives the induced maps f_* and f^*. The map f_* can also be constructed from King's singular construction [K], f^* is its adjoint.

DEFINITION. The intersection homology Lefschetz number $IL(f)$ associated to any placid self-map $f: X^r \to X^r$ *is*

$$IL(f) = \sum_{i=0}^{r} (-1)^i \operatorname{Trace}(f_*: IH_i^{\bar{m}}(X; \mathbb{Z}) \to IH_i^{\bar{n}}(X; \mathbb{Z}))$$

This number is an integer. (Note that the two stratifications of X cannot be taken to be the same).

PROPOSITION 4.2. *If* $f: X^r \to Y^s$ *is placid map between two compact oriented Witt spaces, then the graph of* f *determines a canonical homology class* $[G(f)] \in IH_r^{\bar{m}}(X \times Y; \mathbf{Q})$.

Proof. Projection to the first factor $G(f) \to X$ is a homomorphism and determines an orientation of $G(f)$ from the orientation of X. Chooose a stratification of Y with respect to which f is placid. Choose a stratification of X such that $f^{-1}(S)$ is a union of strata in X whenever S is a stratum of Y. Let $\bar{p}$ denote the (stratum-dependent) perversity of §3,

$$\bar{p}(A \times B) = \begin{cases} \dfrac{\operatorname{cod}(A) + \operatorname{cod}(B) - 1}{2} & \text{if } \operatorname{cod}(A) \neq \operatorname{cod}(B) \\ \operatorname{cod}(A) & \text{if } \operatorname{cod}(A) = \operatorname{cod}(B) \neq 0 \end{cases}$$

We claim that the graph $G(f)$ is $(\bar{p}, r)$-allowable in $X \times Y$ and thus determines a canonical class in $IH_r^{\bar{p}}(X \times Y)$. (This together with Lemma 3.2 will complete the proof of the proposition). We must show that for each stratum $A \times B$,

$$\dim(G(f) \cap A \times B) \leq r - \operatorname{cod}(A) - \operatorname{cod}(B) + \bar{p}(A \times B).$$

This is obvious if $\operatorname{codim}(A) = \operatorname{codim}(B)$ for in this case

$$\dim(G(f) \cap A \times B) \leq \dim(A)$$

If $\operatorname{codim}(A) > \operatorname{codim}(B)$ then $\bar{p}(A \times B) \geq \operatorname{cod}(B)$. Since $G(f)$ is a graph we have

$$\dim(G(f) \cap A \times B) \leq \dim(A) \leq r - \operatorname{cod}(A) + \bar{p}(A \times B) - \operatorname{cod}(B).$$

If $\operatorname{codim}(A) < \operatorname{codim}(B)$ then $\bar{p}(A \times B) \geq \operatorname{cod}(A)$. Since f is placid, we have

$$\begin{aligned} \dim(G(f) \cap A \times B) &\leq \dim(A \cap f^{-1}(B)) \leq \dim f^{-1}(B) = r - \operatorname{codim} f^{-1}(B) \\ &\leq r - \operatorname{codim}(B) \leq r - \operatorname{cod}(B) + p(A \times B) - \operatorname{cod}(A) \end{aligned}$$

as desired.

§5. Intersection homology of a neighborhood of a cycle

In this section, we establish a geometric characterization of the middle intersection homology class $[G(f)]$ of the graph $G(f)$ of a placid map f. Let $\bar{l}$ denote the logarithmic perversity

$$\bar{l}(c) = \begin{cases} 0 & \text{if } \ c \le 2 \\ \left[\frac{c}{2}\right] & \text{if } \ c > 2 \end{cases}$$

Suppose X is an n-dimensional Witt space and ξ is a j-dimensional subanalytic cycle in X which is $(\bar{l}, j)$-allowable. For example, the graph $G(f)$ of a placid map in $X \times X$ is $(\bar{l}, n)$ allowable since $\bar{l} \ge \bar{p}$. Let $N(\xi)$ be a regular neighborhood of $|\xi|$ (i.e., a subanalytic neighborhood with stratified and collared boundary $\partial N(\xi)$ such that $\partial N(\xi) \to X$ is a normally nonsingular inclusion and such that there exists a deformation retraction $f: N(\xi) \to |\xi|$).

LEMMA 5.1. *The rank of $IH_j^{\bar{m}}(N(\xi))$ is no greater than the number of connected components of the nonsingular part of $|\xi|$.*

Proof. The nonsingular part of $|\xi|$ is contained in the nonsingular part of X. Choose one point in each of the k connected component of the nonsingular part of $|\xi|$ and for each of these k points choose an $n-j$ dimensional disk in X which intersects $|\xi|$ transversally. Take $N(\xi)$ sufficiently small that $\partial N(\xi)$ intersects each of these disks transversally. We define a homomorphism

$$\phi: IH_j^{\bar{m}}(N(\xi)) \to \mathbf{Q}^k$$

by assigning to any $(\bar{m}, j)$-allowable cycle η its intersection number with each of these k disks. We claim ϕ is an injection. We shall sketch the argument since it is essentially the same as that in [S] (where he assumes that $|\xi|$ is $(\bar{m}, j)$-allowable but the same proof works when $|\xi|$ is $(\bar{l}, j)$-allowable): Suppose $\phi(\eta) = 0$. Using a carefully chosen deformation retraction $r: N(|\xi|) \to |\xi|$ it is possible to deform $|\eta|$ into $|\xi|$. The resulting $j+1$ dimensional chain $\tilde{\eta}$ is $(\bar{m}, j+1)$-allowable because its intersection with each stratum S has dimension no more than max (dim $|\eta| \cap S + 1$, dim $|\xi| \cap S$). Furthermore this chain $\tilde{\eta}$ does not contain any component of the nonsingular part of $|\xi|$ since $\phi(\eta) = 0$. This implies that $\partial\tilde{\eta} = \eta$, so the intersection homology class represented by η is 0.

Remark. The rank of $IH_{n-j}^{\bar{m}}(N(\xi), \partial N(\xi))$ is also bounded by the number of components of the nonsingular part of $|\xi|$ because the intersection pairing

$$IH_j^{\bar{m}}(N(\xi)) \times IH_{n-j}^{\bar{m}}(N(\xi), \partial N(\xi)) \to \mathbf{Q}$$

is nondegenerate.

If we combine the preceding lemma with Proposition 4.2, we obtain the following

THEOREM 5.2. *Let $f: X^r \to Y^s$ be a placid map between two compact oriented Witt spaces and let $N(G(f))$ be a regular neighborhood of the graph of f in $X \times Y$. Then its homology class*

$$[G(f)] \in IH_r(N(G(f)))$$

is uniquely characterized by the following condition: For every nonsingular point $p \in X$ the intersection number

$$[p] \times [Y] \cdot j_*[G(f)] = +1$$

where $j_*: IH_r(N(G(f))) \to IH_r(X \times Y)$ is induced by inclusion, and $[p] \times [Y]$ is the (intersection) homology class represented by the oriented cycle $\{p\} \times Y$.

Proof. By Lemma 5.1, there is at most one such class. But the cycle represented by the graph lifts canonically to $IH_r^{\bar{m}}(X \times Y)$ and satisfies the above equation, so there is at least one such class.

§6. Intersections in $X \times Y$

In this section, we calculate the fundamental class $[G(f)]$ of the graph of f in $IH_n(X \times X)$, and we prove the first version of the Lefschetz fixed point theorem: that the Lefschetz number $IL(f)$ is the intersection number $[G(f)] \cdot [\Delta]$.

Suppose $f: X^r \to Y^s$ is a placid map between Witt spaces. Let $[G(f)] \in IH_r^{\bar{m}}(X \times Y; \mathbf{Q})$ denote the homology class represented by the graph of f (via Theorem 4.2). If $[\xi] \in IH_i^{\bar{m}}(X; \mathbf{Q})$ and $[\eta] \in IH_j^{\bar{m}}(Y; \mathbf{Q})$, let $[\xi] \otimes [\eta]$ denote their cross product in $IH_{i+j}^{\bar{m}}(X \times Y)$ and let $[G(f)] \cdot ([\xi] \otimes [\eta])$ denote the image of the class $[G(f)] \otimes ([\xi] \otimes [\eta])$ under the intersection product

$$IH_r^{\bar{m}}(X \times X) \otimes IH_{i+j}^{\bar{m}}(X \times Y) \to IH_{i+j-s}^{\bar{t}}(X \times Y)$$

where $\bar{t}$ is the top perversity $\bar{t}(c)=c-2$. Let π_X and π_Y denote the (placid) projections of $X\times Y$ to the first and second factors respectively. Using the sign conventions of Dold [D] VIII. 13, (see appendix) we have:

LEMMA 6.1.

$$(\pi_Y)_*([G(f)]\cdot([\xi]\otimes[\eta]))=(-1)^{s(r-i)}f_*([\xi])\cdot[\eta]\in IH^{\bar{t}}_{i+j-s}(Y)$$
$$(\pi_X)_*([G(f)]\cdot([\xi]\otimes[\eta]))=(-1)^{s(r-i)}[\xi]\cdot f^*([\eta])\in IH^{\bar{t}}_{i+j-s}(X)$$

In particular, if $[\xi]$ *and* $[\eta]$ *are classes in integral intersection homology then these two products will also be integral classes.*

Proof. Choose stratifications of X and Y so that $f^{-1}(S)$ is a union of strata in X whenever S is a stratum in Y. Choose geometric cycles $\xi\in IC^{\bar{m}}_i(X)$ and $\eta\in IC^{\bar{m}}_j(Y)$ representing $[\xi]$ and $[\eta]$, so that the support $|\eta|$ of η is dimensionally transverse to f (as described in §4) and also to $f(|\xi|)$. Let $\bar{p}$ and $\bar{q}$ denote the (stratification-dependant) perversities defined in §3. Then the graph $G(f)$ is a $(\bar{p}, r)$-allowable chain in $X\times X$ and the cycle $|\xi|\times|\eta|$ is a $(\bar{q}, i+j)$-allowable chain in $X\times X$. (This is a simple dimension count). By [GM 1] we have a commutative diagram, where the top and left side are the isomorphisms from §3.2 and §3.3,

$$\begin{array}{ccc} IH^{\bar{m}}_r(X\times Y)\otimes IH^{\bar{q}}_{i+j}(X\times Y) & \to & IH^{\bar{m}}_r(X\times Y)\otimes IH^{\bar{m}}_{i+j}(X\times Y) \\ \downarrow & & \downarrow \\ IH^{\bar{p}}_r(X\times Y)\otimes IH^{\bar{q}}_{i+j}(X\times Y) & \to & IH^{\bar{t}}_{i+j-s}(X\times Y) \end{array}$$

It follows that the classes $[G(f)]$ and $[\xi]\otimes[\eta]$ may be multiplied in the lower left hand corner, where they have transverse representatives $G(f)$ and $|\xi|\times|\eta|$. Since $\bar{p}+\bar{q}\leq\bar{t}$ we obtain their product by orienting the intersection $G(f)\cap|\xi|\times|\eta|$ ([GM 1]). Thus $(\pi_X)_*([G(f)]\cdot[\xi]\otimes[\eta])$ is represented by the cycle $\pi_X(G(f)\cap(|\xi|\times|\eta|))=|\xi|\cap f^{-1}(|\eta|)$. Similarly $(\pi_Y)_*([G(f)]\cdot[\xi]\otimes[\eta])$ is represented by the cycle $f(|\xi|)\cap|\eta|$. This completes the proof.

Remark. The integrality statement is explained by the fact that the bottom line of the diagram can be defined with integer coefficients.

We now use the method of Lefschetz ([L]) to compute the homology class of the graph of a placid map. Let $\mu_1,\ldots,\mu_\alpha$ be a basis for $IH^{\bar{m}}_*(X;\mathbf{Q})$ with dual basis $\mu_1^*,\ldots,\mu_\alpha^*$ and let $P_{ij}=\varepsilon(\mu_i\cdot\mu_j)$ where $\varepsilon: H_*(X)\to\mathbf{Q}$ is the augmentation (so $P_{ij}=0$ unless $\dim(\mu_i)+\dim(\mu_j)=\dim(X)$). Let $v_1,\ldots,v_\beta$ be a basis for

$IH_*^{\bar{m}}(Y; \mathbf{Q})$ with dual basis $v_1^*, \ldots, v_\beta^*$ and let $Q_{ij} = \varepsilon(v_i \cdot v_j)$. Let $F = (F_{ij})$ be the matrix of $f_*: IH_*^{\bar{m}}(X) \to IH_*^{\bar{m}}(Y)$ with respect to the bases $\{\mu_i\}$ and $\{v_j\}$. We shall use the symbol $|v|$ to denote the dimension of a homology class v.

PROPOSITION 6.2. *The homology class of the graph $G(f)$ is given by*

$$[G(f)] = \sum_i \sum_j (-1)^{|\mu_i|(r-|\mu_i|)} F_{ij} \mu_i^* \otimes v_j$$

$$= \sum_i \sum_j (-1)^{|\mu_i|(r+s)} T_{ij} \mu_i \otimes v_j^*$$

where T is the matrix $T = P^{-1}FQ$.

Remark. If we specialize to the case $f =$ identity: $X \to X$ and $\mu_i = v_i$ for all i, then this second formula becomes

$$[\Delta] = \sum_i \mu_i \otimes \mu_i^*$$

Proof. By Poincaré duality it suffices to show that both sides of the equation have the same product with each class $\mu_i \otimes v_j^*$ where $|\mu_i| + |v_j^*| = s$. By the preceding lemma,

$$[G(f)] \cdot (\mu_i \otimes v_j^*) = (\pi_Y)_*([G(f)] \cdot (\mu_i \otimes v_j^*))$$

$$= (-1)^{s(r-|\mu_i|)} \sum_p F_{ip} v_p \cdot v_j^*$$

$$= (-1)^{sr-|\mu_i|} \sum_p F_{ip} v_j^* \cdot v_p = (-1)^{sr-|\mu_i|} F_{ij}$$

which coincides with

$$\sum_i \sum_j (-1)^{|\mu_i|(r-|\mu_i|)} F_{ij} (\mu_i^* \otimes v_j) \cdot (\mu_i \otimes v_j^*).$$

These formulas give the following result:

THEOREM I. *Suppose $f: X^r \to X^r$ is a placid self map of a Witt space. Let $[\Delta] \in IH_r(X \times X; \mathbf{Q})$ denote the homology class represented by the diagonal. Then* *the Lefschetz number* $IL(f)$ *is given by*

$$IL(f)=[G(f)]\cdot[\Delta]$$

§7. Local geometry of $X\times Y$

In this section the points in a join $A * B$ of two spaces will be denoted by triples (a, t, b) where $a\in A$, $t\in[0,1]$, $b\in B$, and where we identify $(a, 0, b)\sim(a', 0, b)$ or $(a, 1, b)\sim(a, 1, b')$. The cone $c(A)$ (with vertex v) over a space A is the join $A * \{v\}$. We will omit the third coordinate (which is always v) when denoting points in $c(A)$.

Suppose X is a pseudomanifold and L_1 is the link of a point $x_0\in X$. Then it is possible to choose a stratum preserving homeomorphism $h_1: c(L_1)\to U_1$ between the cone on L_1 and a neighborhood U_1 of X_0, such that h_1 (vertex) $=x_0$ and $h_1\,|\,L_1$ is the identity. This determines a 'radial distance" function $|\ |: U_1\to[0,1]$ by specifying that $|h_1(l,t)|=t$ for all $l\in L_1$ and $t\in[0,1]$.

A similar choice of homeomorphism

$$h_2: c(L_2)\to U_2$$

between the cone on L_2 and a neighborhood U_2 of a point $y_0\in Y$ determines an embedding

$$H: \text{cone}\,(L_1 * L_2)\to X\times Y$$

which is a homeomorphism onto the conical neighborhood

$$V=\{(x,y)\in U_1\times U_2\,|\,|x|+|y|\le 1\}$$

of (x_0, y_0) in $X\times Y$, by specifying

$$H((l_1,\Delta,l_2),t)=(h_1(l_1,t(1-\Delta)),h_2(l_2,t\Delta)).$$

We may think of Δ and t as "polar coordinates" on V, since

$$t(x,y)=|x|+|y| \quad\text{and}\quad \Delta(x,y)=|y|/(|x|+|y|).$$

We shall denote by $\mathscr{L}$ the boundary of this conical neighborhood, i.e.

$$\mathscr{L}=\partial V=\{(x,y)\in X\times Y\,|\,|x|+|y|=1\}$$

Thus, $\mathscr{L}$ is homeomorphic to the join $L_1 * L_2$. It contains two distinguished subsets, the "top"

$$T = \mathscr{L} \cap \{x_0\} \times Y = H(l_1, 1, l_2), 1)$$

and the "bottom"

$$B = \mathscr{L} \cap X \times \{y_0\} = H((l_1, 0, l_2), 1)$$

which are homeomorphic to L_1 and L_2 respectively.

§8. Local linking numbers

Suppose X is an n-dimensional Witt space. Fix a point $x \in X$. Let L denote the link of x in X and let $\mathscr{L}$ denote the link of (x, x) in $X \times X$. Let N be a regular neighborhood (in $\mathscr{L}$) of $\Delta_L = \Delta \cap \mathscr{L}$. We use "Alexander duality" to define a nondegenerate linking pairing

$$\mu : IH_{n-1}(N) \otimes IH_{n-1}(\mathscr{L} - N) \to \mathbf{Q}$$

by

$$\mu(a \otimes b) = \partial_*^{-1}(a) \cdot b$$

where ∂_* is the connecting homomorphism in the following diagram of dually paired exact sequences:

$$\begin{array}{ccccccc}
0 & & & & & & 0 \\
\| & & & & & & \| \\
IH_n(\mathscr{L}) & \longrightarrow & IH_n(\mathscr{L}, N) & \xrightarrow{\partial_*} & IH_{n-1}(N) & \longrightarrow & IH_{n-1}(\mathscr{L}) \\
 & & \times & & \times & & \\
IH_{n-1}(\mathscr{L}) & \longleftarrow & IH_{n-1}(\mathscr{L} - N) & \longleftarrow & IH_n(\mathscr{L}, \mathscr{L} - N) & \longleftarrow & IH_n(\mathscr{L}) \\
\| & & & & & & \| \\
0 & & & & & & 0
\end{array}$$

(The vanishing of the intersection homology groups on the ends was computed in §3.1).

We have already seen (§4.2, §5.2) that Δ_L determines a unique class $[\Delta_L] \in IH^{\bar{m}}_{n-1}(N)$ and that if X is normal then (§5.1) this group is one-dimensional. Thus we have proven:

PROPOSITION. *If X is normal, the linking number with the diagonal determines an isomorphism*

$$\mu_\Delta : IH_{n-1}^{\bar{m}}(\mathscr{L}-N) \to \mathbf{Q}.$$

§9. Local contributions to the Lefschetz number

Suppose $f: X \to X$ is a placid map with isolated fixed points. For each fixed point x, choose a conical neighborhood U of x and let $V = \text{cone}(\mathscr{L})$ be the corresponding conical neighborhood of (x, x) in $X \times X$. By choosing U sufficiently small we can guarantee that it contains no fixed points other than x, and that the graph $G(f)$ of f is transverse to $\mathscr{L}$. Orient the intersection $G_L(f) = G(f) \cap \mathscr{L}$ with the product orientation: it then determines a homology class

$$[G_L(f)] \in IH_{n-1}^{\bar{m}}(\mathscr{L}-N)$$

where N is a small regular neighborhood of $\Delta \cap \mathscr{L}$ in $\mathscr{L}$.

THEOREM II. *The intersection number $[G(f)] \cdot [\Delta] = IL(f)$ is the sum of the linking numbers $\mu_\Delta([G_L(f)])$ taken over all the fixed points.*

Proof. The proof is standard, so we will only sketch it here. Let $x_1, \ldots, x_\alpha$ denote the fixed points of f. For each fixed point x_i choose a conical neighborhood U_i of (x_i, x_i) in $X \times X$ and let $\mathscr{L}_i = \partial U_i$ be the link of (x_i, x_i). Let N_i be a regular neighborhood of $\Delta \cap \mathscr{L}_i$ in $\mathscr{L}_i$ and let K_i be a relative cycle in $(\mathscr{L}_i, N_i)$ so that $[\partial K_i] = [\Delta \cap \mathscr{L}_i] \in IH_{n-1}(N_i)$. Find a chain $H_i \subset N_i$ which realizes this homology, i.e., so that $\partial H_i = \partial K_i - \Delta \cap \mathscr{L}_i$. Thus there is a cycle Δ' with support,

$$|\Delta'| = \Delta - \Big(\bigcup_i \Delta \cap U_i\Big) \cup \Big(\bigcup_i K_i \cup H_i\Big)$$

It is easy to see that Δ' is an $(\bar{m}, n)$-allowable cycle which is contained in a regular neighborhood of the diagonal in $X \times X$ and which only differs from the diagonal near the fixed points of f.

Since $[\Delta'] = [\Delta] \in IH_n^{\bar{m}}(X \times X)$ they have the same intersection number with the graph $G(f)$. However $[G(f)] \cap [\Delta'] = \sum [G(f) \cap \mathscr{L}_i] \cap [K_i]$ since the points of intersection of the graph of f with Δ' occur in the chains K_i. But this sum is precisely the definition of the local linking numbers of each $(G(f) \cap \mathscr{L}_i)$ with $(\Delta \cap \mathscr{L}_i)$.

§10. The local trace

Suppose $f: X^n \to X^n$ is a placid self map with an isolated fixed point $x \in X$. Let U_1 and U_2 be (conical) neighborhoods of x, with boundaries ∂U_1 and ∂U_2, such that

$$U_1 \subset f^{-1}(U_2 - \partial U_2)$$

If ξ is a compactly supported cycle in $IC_i(U_1)$ then $f_*(\xi)$ is a compactly supported cycle in $IC_i(U_2)$. Thus f determines a "local homorphism"

$$(f_*^x)_i : IH_i(U_1) \to IH_i(U_2)$$

The adjoint to f_* is a homomorphism

$$(f_x^*)_{n-i} : IH_{n-i}(U_2, \partial U_2) \to IH_{n-i}(U_1, \partial U_1)$$

which may be interpreted geometrically as assigning to almost every relative cycle $\xi \in IC_{n-i}(U_2, \partial U_2)$ the (appropriately oriented) relative cycle $f^{-1}(\xi) \cap U_1 \in IC_{n-i}(U_1, \partial U_1)$.

DEFINITION. The local trace $\mathrm{Tr}_x(f)$ of f at x is the sum

$$\mathrm{Tr}_x(f) = \sum_{i=0}^{n} (-1)^i \,\mathrm{Tr}\,(f_*^x)_i = (-1)^n \sum_{i=0}^{n} (-1)^i \,\mathrm{Tr}\,(f_x^*)_i.$$

Remark. Let L denote the link of the point x. Then

$$IH_i(U_1) = \begin{cases} 0 & \text{for } i \geq \left[\dfrac{n+1}{2}\right] \\ IH_i(L) & \text{for } i \leq \left[\dfrac{n-1}{2}\right] \end{cases}$$

$$IH_i(U_1, \partial U_1) = \begin{cases} IH_{i-1}(L) & \text{for } i \geq \left[\dfrac{n+3}{2}\right] \\ 0 & \text{for } i \leq \left[\dfrac{n+1}{2}\right] \end{cases}$$

Thus the local homomorphism $(f_*^x)_i$ corresponds to a homomorphism $IH_i(L) \to$

$IH_i(L)$ if $i \le \left[\frac{n-1}{2}\right]$ (and is 0 otherwise) while its adjoint $(f_x^*)_{n-i}$ corresponds to the homomorphism $IH_{n-i-1}(L) \to IH_{n-i-1}(L)$ if $i \le \left[\frac{n-1}{2}\right]$ (and is 0 otherwise).

§11. The local class of the graph

Let $\mathcal{L}$ denote the link of (x, x) in $X \times X$. Consider the $n-1$ dimensional cycle $G_L(f) = G(f) \cap \mathcal{L}$. Since $IH_{n-1}(\mathcal{L}) = 0$, the graph does not represent an interesting class in $\mathcal{L}$. However it is always contained in a subset $\mathcal{L} - N(T)$ which is homeomorphic to $c(L) \times L$. (Here, T is the "top" defined in §7, and $N(T)$ denotes a small regular neighborhood of T in $\mathcal{L}$.) We will now compute the homology class $[G_L(f)] \in IH_{n-1}(\mathcal{L} - N(T))$ which is represented by $G_L(f)$.

Observe that the intersection product (§2)

$$IH_{n-1}(\mathcal{L} - N(T)) \times IH_n(\mathcal{L}, N(T)) \to \mathbf{Q} \qquad (*)$$

is a nondegenerate pairing between

$$IH_{n-1}(\mathcal{L} - N(T)) \cong \bigoplus_{a \le [(n-1)/2]} IH_a(L) \otimes IH_{n-1-a}(L)$$

and

$$IH_n(\mathcal{L}, N(T) \cong \bigoplus_{a \ge [(n+1)/2]} IH_a(L) \otimes IH_{n-1-a}(L).$$

Suppose $a \ge \left[\frac{n+1}{2}\right]$, $\alpha \in IH_a(L)$, and $\beta \in IH_{n-1-a}(L)$. Let $\alpha \otimes \beta$ denote the corresponding class in $IH_n(\mathcal{L}, N(T))$.

LEMMA 11.1. *The product $G_L(f) \cdot (\alpha \otimes \beta)$ under the intersection pairing $(*)$ is equal to the product*

$$(-1)^{(n-1)(n-1-a)} \alpha \cdot f_x^*(\beta).$$

under the intersection pairing

$$IH_a(L) \otimes IH_{n-1-a}(L) \to \mathbf{Q}$$

Proof. Choose conical neighborhoods U_1 and U_2 such that $U \subset f^{-1}$ (interior (U_2)) as above. Let $L_1 = \partial U_1$ and $L_2 = \partial U_2$. Choose homeomorphisms $h_i : c(L_i) \to U_i$, to obtain H: cone $(L_1 * L_2) \to V$ as in §7. The homeomorphism H restricts to a homeomorphism, $H: L_1 \times L_2 \to \mathcal{L} = \partial V$. Choose cycle representatives ξ in L_1 and η in L_2 of the homology classes α and β, so that the cycle $\xi \times c(\eta)$ is dimensionally transverse to the cycle $G(f)$ in $U_1 \times U_2$.

It is easy to verify the following assertions:

(1) The class $\tilde{\beta} \in IH_{n-a}(U_2, \partial U_2)$ corresponding to $\beta \in IH_{n-a-1}(L_2)$ is represented by the chain $c(\xi)$.

(2) The class $\alpha \otimes \beta \in IH_n(\mathcal{L}, N(T))$ is represented by the chain $H(\xi * \eta) = \mathcal{L} \cap (c(\xi) \times c(\eta))$

(3) The class $G_L(f) \in IH_{n-1}(\mathcal{L} - N(T))$ is represented by the chain $G(f) \cap \mathcal{L}$.

Consider the chain

$$
\begin{aligned}
&\partial[(U_1 \times U_2 - V) \cap G(f) \cap (c(\xi) \times c(\eta))] \\
&\quad = \partial(U_1 \times U_2 - V) \cap G(f) \cap (c(\xi) \times c(\eta)) \\
&\quad = (-1)^n G(f) \cap (\xi \times c(\eta)) - \mathcal{L} \cap G(f) \cap (c(\xi) \times c(\eta)) \\
&\quad = (-1)^n G(f) \cap (\xi \times c(\eta)) - (-1)^n G_L(f) \cap (\xi * \eta)
\end{aligned}
$$

By the same argument as in §6.1, the first term is the number $(-1)^{n(n-a-1)} \alpha \cdot f^*(\tilde{\beta})$ which equals $(-1)^{(a+1)}(-1)^{n(n-a-1)} \alpha \cdot f_x^*(\beta)$. We have shown this is homologous to $(-1)^n G_L(f) \cdot (\alpha \otimes \beta)$ as desired.

§12. Contracting fixed points

DEFINITION. *A fixed point $x \in X$ of a placid self map $f: X \to X$ is contracting if there exists a (conical) distinguished neighborhood U of x which contains no fixed points other than x, such that $\bar{U} \subset f^{-1}$ (interior (U)).*

THEOREM III. *Suppose $x \in X$ is an isolated contracting fixed point of f. Then the local contribution at x to the Lefschetz number of f is precisely the local trace of f at x.*

Proof. By §9, we must compute the intersection number of $[G_L(f)] \in IH_{n-1}(\mathcal{L} - N(\Delta))$ and the unique class $K \in IH_n(\mathcal{L}, N(\Delta))$ such that $\partial_*(K) = [\Delta_L]$. We will instead view this intersection as taking place in the "lower half" of $\mathcal{L}$.

For simplicity of notation we now identify $\mathcal{L}$ with the join $L * L$. Since f is contracting, there exists $\varepsilon > 0$ so small that the graph $G_L(f)$ does not intersect the "middle section" of the join,

$$M=\{(l_1, t, l_2)\in L * L \mid |t-\tfrac{1}{2}|<\varepsilon\}$$

which we may assume contains the regular neighborhood $N(\Delta_L)$ of the diagonal $\Delta \cap \mathscr{L}$. (see §7 and §8 for notation.)

Let $\tilde{T}=\{(l_1, t, l_2)\in L * L \mid t>\tfrac{1}{2}-\varepsilon\}$. Corresponding to the inclusions

$$N(\Delta)\subset M\subset \tilde{T}\subset \mathscr{L}$$

we have a diagram of groups

$$IH_{n-1}(\mathscr{L}-\tilde{T})\xrightarrow{j_*} IH_{n-1}(\mathscr{L}-M)\xrightarrow{i_*} IH_{n-1}(\mathscr{L}-N(\Delta))$$

which are dual to the groups

$$\begin{array}{ccccc}
IH_n(\mathscr{L}, \tilde{T}) & \xleftarrow{j^*} & IH_n(\mathscr{L}, M) & \xleftarrow{i^*} & IH_n(\mathscr{L}, N(\Delta)) \\
\downarrow & & \downarrow & & \downarrow \\
IH_{n-1}(\tilde{T}) & \longleftarrow & IH_{n-1}(M) & \longleftarrow & IH_{n-1}(N(\Delta)) \\
\| & & \| & & \| \\
\bigoplus_{a=[(n+1)/2]}^{n-1} IH_a(L)\otimes IH_{n-1-a}(L) & & \bigoplus_{a=0}^{n-1} IH_a(L)\otimes IH_{n-1-a}(L) & & \mathbb{Z}
\end{array}$$

(The vertical arrows are isomorphisms since $IH_n(\mathscr{L})=IH_{n-1}(\mathscr{L})=0$ by §3. The calculation of $IH_{n-1}(\tilde{T})$ appears in §3).

Let $\{e_1, \ldots, e_r\}$ be a basis for $IH_*(L)$, with dual basis $\{e_1^*, \ldots, e_r^*\}$. The local homomorphism

$$f_x^*: IH_*(L)\rightarrow IH_*(L)$$

may be expressed as a matrix (f_{ij}) with respect to the basis $\{e_1, \ldots, e_r\}$. The intersection number of $[G_L(f)]$ with K can be computed as follows (using the fact $\partial_* i^*(K)=[\Delta_L]$ which was calculated in §6):

$$\begin{aligned}
i_* j_*[G_L(f)]\cdot K &= [G_L(f)]\cdot j^* i^*\,\partial_*^{-1}[\Delta_L]\\
&= [G_L(f)]\cdot \partial_*^{-1} j^* i^*[\Delta_L]\\
&= [G_L(f)]\cdot \partial_*^{-1} j^* i^*[\Delta_L]\\
&= [G_L(f)]\cdot \partial_*^{-1} \sum_{|e_i|=[(n+1)/2]}^{n-1} e_i\otimes e_i^*\\
&= \sum_{|e_i|=[(n+1)/2]}^{n-1} (-1)^{(n-1)(n-1-|e_i|)} e_i\cdot f_x^*(e_i^*)\\
&= \sum_{|e_i|=[(n+1)/2]}^{n-1} (-1)^{(n-1)(n-1-|e_i|)} \sum_{j=1}^{r} e_i\cdot f_{ji} e_j^*\\
&= (-1)^{n-1} \sum_{|e_i|=[(n+1)/2]}^{n-1} (-1)^{|e_i|} f_{ii}\\
&= \mathrm{Tr}_x\,(f)
\end{aligned}$$

§13. Nonexpanding fixed points

Suppose $U_1\subset U_2$ are (compact) conical neighborhoods of an isolated fixed point x_0 of a placid self map $f: X\to X$, such that

$$U_1\subset f^{-1}(U_2-\partial U_2)$$

Let $h: c(\partial U_2)\to U_2$ be a stratum preserving homeomorphism (as considered in §7) between the cone over ∂U_2 and the neighborhod U_2. For each $x\in U_2$ we define the *ray* containing x to be the set of points

$$\{h(h^{-1}(x), t)\mid t\in[0, 1]\}$$

PROPOSITION 13.1. *Suppose U_2 contains no fixed points other than x_0, and suppose that for each $x\in U_2$ either* (a) $f(x)\in U_1-\partial U_1$ *or* (b) *$f(x)$ does not lie on the ray containing x. Then the local contribution at x_0 to the Lefschetz number of f is equal to the local trace of f at x_0.*

Proof. Modify the map $f: X\to X$ near x_0 by composing it with a contraction which preserves rays. The new self map is contracting and assumption (b) implies that it has no new fixed points. However the new map has the same local linking number at x_0 and the same local trace at x_0 as did the original map f. Theorem III implies that these numbers are equal.

§14. Correspondences

We define a *placid correspondence* C between n-dimensional Witt spaces X and Y to be an n-dimensional compact oriented pseudomanifold $C \subset X \times Y$ such that each of the projections

$$\pi_X : C \to X \quad \text{and} \quad \pi_Y : C \to Y$$

is placid. (Notice that a map $f: X \to Y$ is placid iff its graph is a placid correspondence). Such a correspondence determines homomorphisms on intersection homology

$$(\pi_Y)_*(\pi_X)^* : IH_i^{\bar{m}}(X) \to IH_i^{\bar{m}}(Y)$$
$$(\pi_X)_*(\pi_Y)^* : IH_i^{\bar{m}}(Y) \to IH_i^{\bar{m}}(X)$$

If C_1 and C_2 are two such correspondences we define the Lefschetz number $IL(C_1, C_2)$ to be the alternating sum of traces of the induced map

$$(\pi_X^2)_*(\pi_Y^2)^*(\pi_Y^1)_*(\pi_X^1)^* : IH_*^{\bar{m}}(X) \to IH_*^{\bar{m}}(X)$$

It is an integer.

The methods of this paper also work for correspondences: C_i determines a canonical intersection homology class $[C_i] \in IH_n^{\bar{m}}(X \times Y; \mathbf{Q})$ and the main theorems become:

THEOREM I′. *The Lefschetz number $IL(C_1, C_2)$ of two correspondences is equal to the intersection product*

$$[C_1] \cdot [C_2]$$

of the intersection homology classes represented by C_1 and C_2.

THEOREM II′. *Suppose C_1 intersects C_2 in finitely many points (x_1, y_1), $(x_2, y_2), \ldots, (x_\alpha, y_\alpha)$. Let $\mathcal{L}_i$ denote the full link (in $X \times Y$) of (x_i, y_i) and let*

$C_j^i = C_j \cap \mathcal{L}_i$ be the intersection of C_j with this link. Then by Alexander duality there is a well defined linkiing number $\mu(C_1^i, C_2^i)$ (see §8) and the Lefschetz number $IL(C_1, C_2)$ is equal to the sum

$$\sum_{i=1}^{\alpha} \mu(C_1^i, C_2^i)$$

of these local linking numbers.

§15. Other perversities

It is possible (in a somewhat artificial way) to extend the results of this paper to other perversities. This situation is amusing when the fixed points of f are isolated because the associated local intersection numbers of the graph of f with the diagonal will change as the perversity changes.

If $f: X \to X$ is placid we have an induced homomorphism

$$(f_*)_i : IH_i^{\bar{p}}(X) \to IH_i^{\bar{p}}(X)$$

and an associated Lefschetz number

$$IL^{\bar{p}}(f) = \sum (-1)^i \text{ Trace } (f_*)_i$$

for any perversity $\bar{p}$. Let $\bar{q}$ be the complementary perversity, $q(c) = c - p(c)$. Stratify $X \times X$ with a product stratification and consider the (stratification dependant) perversities

$$pq(A \times B) = p(A) + q(B)$$

$$qp(A \times B) = q(A) + p(B)$$

whenever A and B are strata in X. These are also dual perversities so the intersection of dimensionally transverse geometric cycles ([GM 1]) determines a nondegenerate pairing

$$IH_n^{pq}(X \times X) \otimes IH_n^{qp}(X \times X) \to \mathbf{Q}.$$

Furthermore the graph of f canonically determines a homology class $[G(f)] \in$

$IH_n^{pq}(X \times X)$ and the diagonal determines a homology class $[\Delta] \in IH_n^{qp}(X \times X)$. We obtain

THEOREM I″. *The Lefschetz number $IL^{\bar{p}}(f)$ is equal to the product $[G(f)] \cdot [\Delta]$ under the intersection pairing described above.*

At an isolated fixed point x we can again define a linking pairing: let $\mathscr{L} = L * L$ be the link of (x, x) in $X \times X$, let N be a regular neighborhood in $\mathscr{L}$ of $\Delta_L = \Delta \cap \mathscr{L}$. These spaces inherit a stratification from the product stratification of $X \times X$, and a calculation as in §3.1 gives $IH_n^{pq}(\mathscr{L}) = IH_n^{qp}(\mathscr{L}) = IH_{n-1}^{pq}(\mathscr{L}) = IH_{n-1}^{qp}(\mathscr{L}) = 0$. Therefore we can again use Alexander duality to define a non-degenerate linking pairing

$$\mu^{\bar{p}} : IH_{n-1}^{qp}(N) \otimes IH_{n-1}^{pq}(L * L - N) \to \mathbf{Q}$$

by $\mu^{\bar{p}}(a \otimes b) = \partial_*^{-1}(a) \cdot b$ where ∂_* is the connecting homomorphism

$$\partial_* : IH_n^{qp}(\mathscr{L}, N) \to IH_{n-1}^{qp}(N).$$

The cycles $G_L(f) = G(f) \cap \mathscr{L}$ and Δ_L canonically determine homology classes

$$[G_L(f)] \in IH_{n-1}^{pq}(\mathscr{L} - N)$$
$$[\Delta_L] \in IH_{n-1}^{qp}(N)$$

THEOREM II″. *If $f : X \to X$ has isolated fixed points then the Lefschetz number $IL^{\bar{p}}(f)$ is equal to the sum of the linking numbers*

$$\mu^{p}([\Delta_L], [G_L(f)])$$

at each of the fixed points.

As in §10 we obtain a local trace and the same proof as in §12 gives:

THEOREM III″. *If x is a contracting fixed point of a placid self map then this local linking number is equal to the trace of the induced homomorphism on the stalk cohomology.*

§16. Appendix: sign conventions

In this section we summarize the sign conventions of Dold [D] VIII. 13 for the intersection products. Let X^r and Y^s be oriented pseudomanifolds and let $f: X \to Y$ be a placid map which induces homomorphisms f_*: $IH_i^{\bar{m}}(X) \to IH_i^{\bar{m}}(Y)$ and $f^*: IH_i^{\bar{m}}(Y) \to IH_{i+r-s}^{\bar{m}}(X)$. We will use the symbol μ to denote the map from cohomology

$$\mu: H^i(X) \to IH_{r-i}^{\bar{m}}(X)$$

and we will use a period to denote the intersection product

$$IH_*^{\bar{m}}(X) \otimes IH_*^{\bar{m}}(X) \to H_*(X).$$

We will use a "cap" to denote multiplication by cohomology class, and a symbol $\otimes$ to denote the intersection homology cross product

$$IH_*^{\bar{m}}(X) \otimes IH_*^{\bar{m}}(Y) \to IH_*^{\bar{m}}(X \times Y).$$

1. $a \cdot b = (-1)^{(r-i)(r-j)} b \cdot a$ for all $a \in IH_i^{\bar{m}}(X)$, $b \in IH_j^{\bar{m}}(X)$.
2. $\mu(A) \cdot \mu(B) = \mu(A \cup B) = A \cap \mu(B)$ for all $A, B \in H^*(X)$
3. $f_*(f^*(a) \cdot b) = a \cdot f_*(b)$ for all $a \in IH_*^{\bar{m}}(Y)$, $b \in IH_*^{\bar{m}}(X)$
4. $f_*(f^*(A) \cap b) = A \cap f_*(b)$ for all $A \in H^*(Y)$, $b \in IH_*^{\bar{m}}(X)$
5. $a \times b = (-1)^{r(s-j)}(\pi_X^*(a) \cdot \pi_Y^*(b))$ for all $a \in IH_*^{\bar{m}}(X)$, $b \in IH_j^{\bar{m}}(Y)$
6. $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^{(s-j)(r-i)}(a_1 \cdot a_2) \otimes (b_1 \cdot b_2)$ for all $a_1 \in IH_*(X)$, $a_2 \in IH_i(X)$, $b_1 \in IH_j(Y)$, $b_2 \in IH_*(Y)$.

Orienting the graph $G(f)$ so that $(\pi_X)_*[G(f)] = [X]$ we have; for all $a \in IH_i(X)$ and $b \in IH_j(Y)$ the following equalities:

7. $(\pi_Y)_*([G(f)] \cdot \pi_X^*(a)) = (-1)^{s(r-i)} f_*(a)$
8. $(\pi_Y)_*([G(f)] \cdot a \times b) = (-1)^{s(r-i)} f_*(a) \cdot b$
9. $(\pi_X)_*([G(f)] \cdot \pi_Y^*(b)) = (-1)^{r(s-j)} f^*(b)$
10. $(\pi_X)_*([G(f)] \cdot a \times b) = (-1)^{s(r-i)} a \cdot f^*(b)$

The choices of sign in formulae #2 and #5 determine all the other signs.

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Dept. of Mathematics
Northeastern University
Boston MA. 02115

Dept. of Mathematics
Brown University
Providence RI 02912

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