# Some remarks on compactifications of commutative algebraic groups.


Autor(en): **Knop, F. / Lange, H.**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **60 (1985)**

PDF erstellt am: **16.08.2024**

Persistenter Link: https://doi.org/10.5169/seals-46329

Comment. Math. Helvetici 60 (1985) 497–507 0010–2571/85/040497–11$01.50 + 0.20/0


# Some remarks on compactifications of commutative algebraic groups

F. Knop and H. Lange

## Introduction

In the theory of transcendental numbers commutative algebraic groups play an important role. In fact if such a group $E$ is embedded in some projective space, its exponential map can be described by holomorphic functions. The values of these functions at algebraic points are good candidates for transcendency (cf. [8], [9]). In order to embed $E$ into a projective space, it is convenient to compactify $E$ and embed the compactification $\bar{E}$. The main method to compactify $E$ (due to Serre [7]) is the following:

Let $E$ be a connected commutative algebraic group over an algebraically closed field $k$. Then there is a canonical exact sequence

$$0 \to G \to E \xrightarrow{p} A \to 0 \tag{1}$$

with $G$ a connected linear group and $A$ an abelian variety over $k$. Given a projective $G$-variety $P$ over $\mathbb{P}^n$, every open $G$-equivariant immersion $G \to P$ induces in a natural way a compactification $\bar{E}$ of $E$, namely the fibre bundle $E(P) = E \times^G P$ with fibre $P$ over $A$, associated to the $G$-principal bundle $E \to A$ with respect to the given $G$-action of $P$ (cf. [4]).

In this note we shall study the following questions: Are there other compactifications (section 1)? Which are the $G$-equivariant embeddings of $\bar{E}$ into $\mathbb{P}^n$ (section 2)? How many compactifications of the above type are there (section 3)? And by forms of which degrees is the homogeneous ideal of $\bar{E}$ in $\mathbb{P}^n$ generated (section 4)?

To be more precise, in section 1 we shall prove that if the linear part $G$ of $E$ is a torus or one dimensional, then there are no other normal compactifications of $E$ then those described above (Theorem 1.1) and we give a counterexample to this statement in the non-normal case. In section 2 we investigate the group $\mathrm{Pic}_G(E(P))$ of $G$-line bundles on the compactification $E(P)$ and show (Theorem 2.1) that it is isomorphic to $\mathrm{Pic}_G(P) \times \mathrm{Pic}(A)$. Hence every $G$-embedding of

$E(P)$ into projective space comes from a pair consisting of a projective $G$-embedding of $P$ and a projective embedding of $A$.

In section 3 we show that Serre's original compactifications of $E$ depend on the splitting of $G$ into one dimensional groups. Finally in section 4 we show that in the most important cases (with respect to applications in transcendence theory) the homogeneous ideal of $E(P)$ in $\mathbb{P}^n$ is generated by forms of degree $\leq \dim A + 3$.

## 1. Compactifications of $E$

Let $E$ denote a connected commutative algebraic group over a field $k$ (algebraically closed for simplicity) with canonical exact sequence (1). Let $X$ be an $E$-variety (not necessarily projective) and $i: E \to X$ an equivariant open immersion. Let $\bar{G} = \overline{Gi(0)}$ be the closure of the orbit of $i(0)$ under $G$ in $X$. Then there is a canonical $E$-equivariant morphism

$$\psi: E(\bar{G}) \to X$$

defined by $\psi(h, x) = h \cdot x$ for $h \in E$ and $x \in \bar{G} \subset X$. Here $E(G)$ denotes as usual the fibre bundle $E \times^G \bar{G}$ associated to the principal bundle. It is the compactification of $E$ corresponding to the embedding $G \to \bar{G}$ (if $\bar{G}$ is proper over $k$, c.f. [4]). The following theorem in particular gives conditions under which the compactifications of type $E(G)$ are the only $E$-equivariant compactifications of $E$.

THEOREM 1.1. (a) *$\psi$ is birational and proper, and in particular surjective.*
(b) *Suppose* (i) *$X$ is normal and* (ii) *$\bar{G}$ consists of finitely many $G$-orbits. Then $\psi$ is an isomorphism.*

Condition (ii) is always fulfilled if $G$ is a torus or one dimensional (cf. [5]).

*Proof.* 1. Since $\psi$ restricted to the open subset $E \times^G G$ of $E(\bar{G})$ is an isomorphism onto the open subset $E$ of $X$, $\psi$ is birational.

2. Let $\pi: E(\bar{G}) \to A$ denote the canonical projection map. We claim that $(\pi, \psi): E(\bar{G}) \to A \times X$ is a closed immersion.

Since $E \to A$ is faithfully flat, it suffices to show that $id_E \times_A (\pi, \psi): E \times_A E(\bar{G}) \to E \times_A (A \times X)$ is a closed immersion. Now $\phi_1: E \times \bar{G} \to E \times_A (E \times^G \bar{G}): (h, x) \to (h, (h, x))$ is an isomorphism (with inverse map $\phi_1^{-1}(h, (h', x)) = (h, (h^{-1}h')x)$; note that $h^{-1}h' \in G$). If $\phi_2: E \times_A (A \times X) \to E \times X$ denotes the canonical isomorphism and $\phi_3: E \times X \to E \times X$ the isomorphism

$(h, x) \to (h, h^{-1}x)$, then it suffices to show that the composed map

$$\phi_3\phi_2(id_E \times_A (\pi, \psi))\phi_1 : E \times \bar{G} \to E \times X$$

is a closed immersion. But

$$\begin{aligned}\phi_3\phi_2(id_E \times_A (\pi, \psi))\phi_1(h, x) &= \phi_3\phi_2(id_E \times_A (\pi, \psi))(h, (h, x))\\ &= \phi_3\phi_2(h, (\pi(h), hx))\\ &= \phi_3(h, hx) = (h, x)\end{aligned}$$

which proves the assertion. As composition of the closed immersion $E(\bar{G}) \to A \times X$ and the projection $A \times X \to X$ the map $\psi$ is proper. (Note that if $X$ is itself proper over $k$, assertion 2 follows immediately from EGA II, 5.4.3).

3. To complete the proof of the theorem, by Zariski's Theorem (EGA III 4.4.9) it suffices to show that $\psi$ has finite fibres under the assumptions (i) and (ii).

Suppose $x \in X$. Since $\psi$ is surjective there are points $h \in E$ and $x_0 \in \bar{G}$ with $x = hx_0$. Since $\psi$ is $E$-equivariant, it suffices to show that $\psi^{-1}(x_0)$ is finite. We have

$$\begin{aligned}\psi^{-1}(x_0) &= \{(h, x) \in E(\bar{G}) \mid hx = x_0\}\\ &= \{(h, h^{-1}x_0) \in E(\bar{G}) \mid h^{-1}x_0 \in \bar{G}\}.\end{aligned}$$

Denote $E_0 = \{h \in E \mid h^{-1}x_0 \in \bar{G}\}$. $E_0$ in general is not a subgroup of $E$, however it is $G$-stable. Obviously we have a bijection $E_0/G \xrightarrow{\sim} \psi^{-1}(x_0)$. Let $E_{x_0}$ denote the stabilizer of $x_0$ in $E$ . $E_0$ is also $E_{x_0}$-stable and we claim:

Suppose $h, \bar{h} \in E_0$. Then $h^{-1}x_0$ and $\bar{h}^{-1}x_0$ are in the same $G$-orbit of $\bar{G}$ if and only if $h\bar{h}^{-1} \in GE_{x_0}$. In fact for $g \in G$ we have

$$g\bar{h}^{-1}x_0 = h^{-1}x_0 \Leftrightarrow gh\bar{h}^{-1}x_0 = x_0 \Leftrightarrow gh\bar{h}^{-1} \in E_{x_0} \Leftrightarrow h\bar{h}^{-1} \in GE_{x_0}.$$

Hence there is an injective map of $E_0/GE_{x_0}$ into the set of $G$-orbits of $\bar{G}$. It follows from assumption (ii) that $E_0/GE_{x_0}$ is a finite set. It remains to show that $GE_{x_0}/G = p(E_{x_0}) \subseteq A$ ($p : E \to A$ the canonical map) is finite. But by the following lemma $E_{x_0}$ is a linear group. Hence $p(E_{x_0})$ is finite as a linear subgroup of an abelian variety. This completes the proof of Theorem 1.1. It remains to show:

LEMMA 1.2. *Let $E$ be an algebraic group acting effectively on a variety $X$ (i.e. $E \to \mathrm{Aut}(X)$ is injective). Then for every $x \in X$ the stabilizer $E_x$ of $x$ in $E$ is linear.*

*Proof.* We have the following canonical inclusions

$$E_x \hookrightarrow \operatorname{Aut} \mathcal{O}_{X,x} \hookrightarrow \operatorname{Aut} \hat{\mathcal{O}}_{X,x} = \varprojlim_n \operatorname{Aut} (\mathcal{O}_{X,x}/m_x^n).$$

Hence there is an integer $n$, such that the canonical map $E \to \operatorname{Aut} (\mathcal{O}_{X,x}/m_X^n)$ is injective. Since $\operatorname{Aut} (\mathcal{O}_{X,x}/m_x^n)$ as a subgroup of the group of automorphisms of a finite dimensional vector space is linear, the same is true for $E_x$.

EXAMPLE 1.3. We want to give an example showing that Theorem 1.1 is not correct without the assumption that $X$ is normal.

Let $l \neq \operatorname{char} k$ be a prime number, $k$ algebraically closed, $E$ as above with canonical exact sequence (1) with the additional assumption that $G = \mathbb{G}_m$. Let $\bar{G} = \mathbb{P}^1$ be the canonical compactification and $\Gamma \subseteq A$ be a subgroup with $\Gamma \simeq \mathbb{Z}/l\mathbb{Z}$. The exact sequence

$$0 \to G \to p^{-1}(\Gamma) \to \Gamma \to 0$$

splits (cf. [2] Theorem 16.2). Since $\operatorname{Hom}(\Gamma, G) \neq 0$, there are 2 sections $s_1, s_2 : \Gamma \to p^{-1}(\Gamma) \subseteq E$ such that $s_1(\Gamma) \cap s_2(\Gamma) = \{0\}$ and $s_1(\Gamma) \cap G = s_2(\Gamma) \cap G = \{0\}$.

Define $E_1 = E/s_1(\Gamma)$ and $E_2 = E/s_2(\Gamma)$. Then the sequences

$$0 \to G \to E_i \to A/\Gamma \to 0$$

are exact. Moreover for $i = 1, 2$ there is a natural $E$-equivariant morphism $\phi_i : E(\bar{G}) \to E_i(\bar{G})$.
Let

$$X := \operatorname{Im} (\phi_1, \phi_2) \subseteq E_1(\bar{G}) \times E_2(\bar{G}).$$

Then $i : E \to E(\bar{G}) \to X$ is an open imbedding, since $s_1(\Gamma) \cap s_2(\Gamma) = \{0\}$. On the other hand $i \mid \bar{G}$ is an isomorphism, since $s_1(\Gamma) \cap G = s_2(\Gamma) \cap G = \{0\}$. But $A \simeq E(\infty) \subseteq E(\bar{G})$ is mapped onto $A/\Gamma$. It follows $p(E_{x_0}) = \Gamma \neq \{0\}$, where $x_0$ denotes the image of $(0, \infty) \in E(\bar{G})$ in $X$, which means that $\psi : E(\bar{G}) \to X$ is not an isomorphism.

**2.** $\mathrm{Pic}_G\,(EP))$

Let $E$ be as in section 1 with canonical exact sequence (1). Let $P$ be a complete $G$-variety over $k$ and $i:G\to P$ an equivariant open immersion. Then $E(P)=E\times^G P$ is a compactification of $E$. Moreover if $L$ is a $G$-linearized line bundle in $P$, then $E(L)=E\times^G L$ is a $G$-linearized (even $E$-linearized) line bundle on $E(P)$ (cf. [4], Lemma 1.2). If as usual Pic ( ) (resp. $\mathrm{Pic}_G$ ( )) denotes the group of line bundles (resp. $G$-linearized line bundles), there is a canonical map

$$\Phi:\begin{cases}\mathrm{Pic}_G\,(P)\times\mathrm{Pic}\,(A)\to\mathrm{Pic}_G\,(E(P))\\ \qquad\qquad (L,M)\mapsto E(L)\otimes\pi^*M.\end{cases}$$

(Note that, whereas $E(L)$ is $E$-linearized, $\pi^*M$ is only $G$-linearized on $E(P)$). The aim of this section is to prove

THEOREM 2.1. *Given an exact sequence* (1) *and let $P$ denote a complete $G$-variety over $k$. Then the canonical map*

$$\Phi:\mathrm{Pic}_G\,(P)\times\mathrm{Pic}\,(A)\to\mathrm{Pic}_G\,(E(P))$$

*is an isomorphism of groups.*

*Proof.* We shall construct an inverse map $\Psi:\mathrm{Pic}_G\,(E(P))\to\mathrm{Pic}_G\,(P)\times\mathrm{Pic}\,(A)$. Suppose $N\in\mathrm{Pic}_G\,(E(P))$. Let $\pi:E(P)\to A$ denote the projection map. $\pi^{-1}(0)$ is canonically isomorphic to $P$. We identify both and consider the closed embedding $j:P\hookrightarrow E(P)$. Define $L=j^*N$ with the induced $G$-action. We claim that $E(L)^{-1}\otimes N\mid\pi^{-1}(a)\simeq\mathcal{O}_{\pi^{-1}(a)}$ (without $G$-action) for every point $a\in A$.

First of all $E(L)\mid\pi^{-1}(0)=L$ (even as $G$-line bundles) that is $E(L)^{-1}\otimes N\mid\pi^{-1}(0)\simeq\mathcal{O}_{\pi^{-1}(0)}$ with trivial $G$-action. Since $E(L)^{-1}\otimes N$ may be considered as a family of line bundles on $P$ parametrized by $A$, it suffices to show that the Picard variety $\mathrm{Pic}^0\,(P)$ of $P$ is zerodimensional, since then every deformation of a line bundle on $P$ is trivial. But $P$ is as compactification of a connected linear group *a* rational variety which implies $\dim\mathrm{Pic}^0\,(P)\le\dim H^1(P,G_P)=0$.

Applying Grauert's theorem (cf. [1], III, 12.9), we get that $M:=\pi_*(E(L)^{-1}\otimes N)$ is a line bundle on $A$. We claim that the natural map

$$\sigma:\pi^*M=\pi^*\pi_*(E(L)^{-1}\otimes N)\to E(L)^{-1}\otimes N$$

is an isomorphism of line bundles. Since $\sigma$ is a homomorphism of line bundles we have only to show that $\sigma$ is surjective.

For this is sufficient to show that for every point $a \in A$ there is a neighbourhood $U$ in $A$ such that $E(L)^{-1} \otimes N \mid \pi^{-1}(U)$ admits a nowhere vanishing section. But since $\pi_*(E(L)^{-1} \otimes N)$ is a line bundle on $A$, we may take for $U$ any trivializing open set in $A$. Hence $1 \times \sigma : E(L) \otimes \pi^* M \to N$ is an isomorphism of line bundles and it remains to show that it is compatible with the $G$-actions. Since any 2 $G$-linearizations of a given line bundle on $E(P)$ differ by a character on $G$, it suffices to check this on the restrictions to the fibre $\pi^{-1}(0)$. But we noted already that $E(L) \mid \pi^{-1}(0) \simeq N \mid \pi^{-1}(0)$ as $G$-line bundles.

Now define $\Psi : \mathrm{Pic}_G(E(P)) \to \mathrm{Pic}_G(P) \times \mathrm{Pic}(A)$ by $\Psi(N) = (L, M)$ with $L$ and $M$ as above. It is easy to see that the maps $\Phi$ and $\Psi$ are inverse to each other.

## 3. Serre-compactifications of $E$

In this section let $k$ denote the field of complex numbers. The group $G$ in the exact sequence (1) then is of the form

$$G = \prod_{i=1}^{r} \mathbb{G}_m \times \prod_{j=1}^{s} \mathbb{G}_a. \tag{2}$$

In [7] Serre constructed a compactification of $E$ as follows: Consider for each factor $\mathbb{G}_m$ and $\mathbb{G}_a$ of $G$ in (2) the natural embedding into $\mathbb{P}^1$. This gives a $G$-equivariant embedding of $G$ into $(\mathbb{P}^1)^{r+s}$. The compactification of $E$ is defined to be the associated bundle $X = E((\mathbb{P}^1)^{r+s})$. We want to show by an example that this compactification heavily depends on the splitting (2) of $G$.

Start with an abelian variety $A$ and $G = \mathbb{G}_m^2$. Let $G = \mathbb{G}_m \times \mathbb{G}_m$ be a given decomposition. The isomorphism $\Phi : G \to G$, $\Phi(a_1, a_2) = (a_1^{-1} a_2, a_2)$ yields another decomposition of $G$.

Now let $E_1 = \mathbb{G}_m \times A$ be the trivial bundle and $E_2 \to A$ be an arbitrary principal $\mathbb{G}_m$-bundle over $A$, whole associated line bundle $M_2$ on $A$ is algebraically equivalent to zero. By the theorem of Weil–Rosenlicht–Serre (cf. [6], p. 184 Théorème 6) $E = E_1 \times_A E_2$ is a group extension of $A$ by $G$.

If $(U_i, \alpha_{ij})_{i,j}$ is a description of $E_2$ by open sets $U_i$ in $A$ and transition morphisms $\alpha_{ij}$, then $(U_i, (1, \alpha_{ij}))_{ij}$ is a description of the principal $G$-bundle $E = E_1 \times_A E_2$ over $A$. On the other hand any principal $G$-bundle over $A$ may be considered as an element of $H^1(A, G)$. The element of $H^1(A, G)$ corresponding to $E$ over $A$ does not reflect the decomposition of $G$ which however the description $(U_i, (1, \alpha_{ij}))_{i,j}$ does. Applying the isomorphism $\Phi : G \to G$ we get

$$E \hat{=} (U_i, (1, \alpha_{ij}))_{ij} \hat{=} (U_i, \Phi((1, \alpha_{ij})))_{i,j} = (U_i, (\alpha_{ij}, \alpha_{ij}))_{i,j}$$

which means

$$E \simeq E_1 \times_A E_2 \simeq E_2 \times_A E_2.$$

If $X$ (resp. $\bar{X}$) denotes the compactification of $E$ corresponding to the given decomposition of $G$ (resp. the decomposition of $G$ given by applying $\Phi$), then we have according to the definitions

$$X \simeq P(\mathcal{O}_A^2) \times_A P(\mathcal{O}_A \oplus M_2)$$

and

$$\bar{X} \simeq P(\mathcal{O}_A \oplus M_2) \times_A P(\mathcal{O}_A \oplus M_2)$$

where $P(\cdot)$ denotes the projective bundle associated to the vector bundle $(\cdot)$.

We claim that $X$ and $\bar{X}$ are not isomorphic in general. For this it suffices to compute the canonical line bundles $K_X$ and $K_{\bar{X}}$. We get

$$K_X = M_2 \otimes_A \mathcal{O}_{P(\mathcal{O}_A^2)}(-2) \otimes_A \mathcal{O}_{P(\mathcal{O}_A \oplus M_2)}(-2)$$

and

$$K_{\bar{X}} = M_2^2 \otimes_A \mathcal{O}_{P(\mathcal{O}_A \oplus M_2)}(-2) \otimes_A \mathcal{O}_{P(\mathcal{O}_A \oplus M_2)}(-2)$$

which obviously are nonisomorphic for $M_2 \neq \mathcal{O}_A$. (Consider $X$ and $\bar{X}$ as projective bundles over $P(\mathcal{O}_A \oplus M_2)$).

## 4. Projective embeddings

We want to study the projective embeddings of the compactifications $\bar{E}$ of $E$, and in particular the question: by forms of which degrees is the homogeneous ideal of $\bar{E}$ in $\mathbb{P}^N$ generated? For this we need a slight generalization of a criterium of Mumford (cf. [5], pp. 39–40) which we shall prove first.

If $X$ is any projective variety embedded in $\mathbb{P}^N$ by the complete linear system of a very ample line bundle $L$, denote by $I := \oplus_{k \geq 0} I_k$ its homogenous ideal in $\mathbb{P}^N$. For any $i \geq 1$ define

$$\mathcal{R}(L^i, L) = \operatorname{Ker}\,(H^0(L^i) \otimes H^0(L) \to H^0(L^{i+1}))$$

and

$$\mathcal{R}_i(L) = \operatorname{Ker}(H^0(L)^i \to H^0(L^i)).$$

If moreover for a vector space $V$ we denote by $S^n(V)$ its $n$-th symmetric product, we have

LEMMA 4.1. *If $L$ is normally generated on $X$, then for any $k \geq 1$ the following conditions are equivalent*

(1) *The canonical map $H^0(L) \otimes \mathcal{R}(L^i, L) \to \mathcal{R}(L^{i+1}, L)$ is surjective for every $i \geq k$.*

(2) *The canonical map $I_{k+1} \otimes S^{i-k} H^0(L) \to I_{i+1}$ is surjective for every $i \geq k$.*

In other words: condition (1) is equivalent to the fact that the ideal $I$ is generated by forms of degree $\leq k+1$. For $k = 1$ this is just Mumford's result.

*Proof.* We shall prove only the implication (1)$\Rightarrow$(2), since we do not need the converse. Consider the condition (2′): The canonical map

$$\phi = \sum_v \phi_v : \bigoplus_v \mathcal{R}_{k+1}(L) \otimes H^0(L)^{i-k} \to \mathcal{R}_{i+1}(L)$$

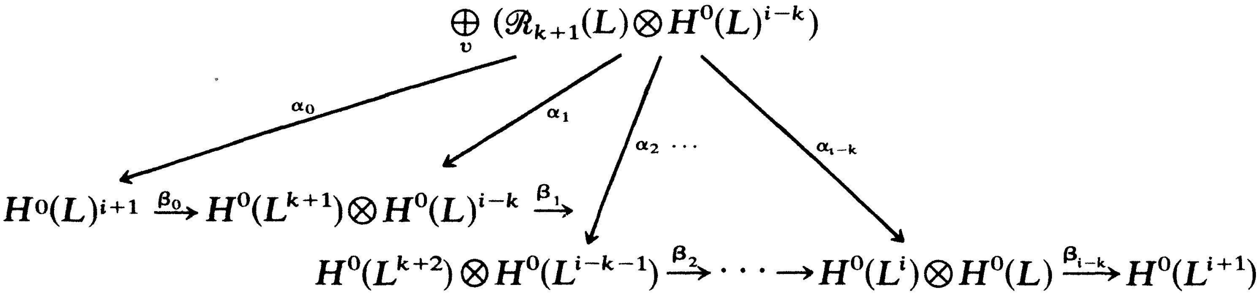
is surjective, where the direct sum is to be taken over all $v = (v_1, \ldots, v_{k+1})$ with $1 \leq v_1 < \cdots < v_{k+1} \leq i+1$. Here the map $\phi_v$ on the direct summand with index $v = (v_1, \ldots, v_{k+1})$ is given by

$$a_1 \otimes \cdots \otimes a_{k+1} \otimes b_1 \otimes \cdots \otimes b_{i-k}$$

$$\mapsto b_1 \otimes \cdots \otimes \overset{\overset{v_1}{\vee}}{a_1} \otimes \cdots \otimes \overset{\overset{v_2}{\vee}}{a_2} \otimes \cdots \otimes \overset{\overset{v_{k+1}}{\vee}}{a_{k+1}} \otimes \cdots \otimes b_{i-k}$$

that is $a_i$ is inserted in the $v_i$-th place. It is easy to see that (2′) is just the desymmetrization of (2). Hence it suffices to show that (1) implies (2′).

Consider the commutative diagram




with $\alpha_0 = \phi$, considered as a map into $H^0(L)^{i+1}$, with $\beta_j$ the canonical maps, and

$\alpha_j = \beta_j \cdot \alpha_{j-1}$. We have to show:

$$\operatorname{Ker}(\beta_{i-k} \cdot \cdots \cdot \beta_1 \cdot \beta_0) \subseteq \operatorname{Im} \alpha_0.$$

Since $L$ is normally generated, $\beta_j$ is surjective for every $j = 0, \ldots, i-k$ (c.f. [5]). Hence it suffices to show

$$\operatorname{Ker}(\beta_j) \subseteq \operatorname{Im} \alpha_j \quad \text{for} \quad j = 0, \ldots, i-k.$$

This is true for $j = 0$ by definition of the maps. For $j = 1, \ldots, i-k$ we have

$$\operatorname{Ker}(\beta_j) = \mathcal{R}(L^{k+j}, L) \otimes H^0(L^{i-k-j}).$$

By restriction to a suitable direct summand of $\bigoplus_v (\mathcal{R}_{k+1}(L) \otimes H^0(L)^{i-k})$ and omission of some tensor factors $H^0(L)$ we see, that it suffices to show that the canonical map

$$\tilde{\alpha}_j : H^0(L)^j \otimes \mathcal{R}_{k+1}(L) \to \mathcal{R}(L^{k+j}, L)$$

is surjective for $j = 1, \ldots, k-i$. But $\tilde{\alpha}_j$ factorizes canonically as follows

$$\begin{array}{ccc}
H^0(L)^j \otimes \mathcal{R}_{k+1}(L) & \xrightarrow{\tilde{\alpha}_j} & \mathcal{R}(L^{k+j}, L) \\
{\scriptstyle 1\otimes\gamma_j}\searrow & & \nearrow{\scriptstyle \delta_j} \\
& H^0(L)^j \otimes \mathcal{R}(L^k, L) &
\end{array}$$

$\delta_j$ is surjective according to the assumption (1). To show the surjectivity of $\gamma_j$ consider the diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & 0 & & \mathcal{R}(L^k, L) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker}\phi & \longrightarrow & H^0(L)^k \otimes H^0(L) & \xrightarrow{\phi} & H^0(L^k) \otimes H^0(L) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi} & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathcal{R}_{k+1}(L) & \longrightarrow & H^0(L)^{k+1} & \longrightarrow & H^0(L^{k+1}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow & & & & \\
 & & \operatorname{Coker}\psi & & 0 & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

According to the serpent lemma $\mathcal{R}(L^k, L) \simeq \mathrm{Coker}\ \psi$ canonically, and under this isomorphism $\pi$ identifies with $\gamma_j$ which completes the proof of the lemma.

In order to apply Lemma 4.1 suppose we are given an exact sequence (1). Let $P$ be a $G$-equivariant compactification of $G$. It induces a compactification $E(P)$ of $E$. Denote by $\pi: E(P) \to A$ the natural projection. In [4] the following result was proved (cf. [4], pp. 564–567).

THEOREM 4.2. *Let $L \in \mathrm{Pic}(P)$ be normally presented, $G$-linearized, $M \in \mathrm{Pic}(A)$ ample, generated by its global sections, and $F = E(L) \otimes \pi^* M$. Then for every $i \geq \dim A + 2$ the canonical map*

$$H^0(F) \otimes \mathcal{R}(F^i, F) \to \mathcal{R}(F^{i+1}, F)$$

*is surjective.*

If moreover $M$ is normally generated on $A$, the methods of [4], section 3 show, that $F = E(L) \otimes \pi^* M$ is normally generated on $E(P)$ and we may apply Lemma 4.1 to get:

COROLLARY 4.3. *Let $L \in \mathrm{Pic}(P)$ be normally presented, $G$-linearized, and $M \in \mathrm{Pic}(A)$ normally generated. Then $F = E(L) \otimes \pi^* M$ is normally generated on $E(P)$ and the homogeneous ideal of the corresponding projective embedding $E(P) \hookrightarrow \mathbb{P}^N$ is generated by forms of degree $\leq \dim A + 3$.*

The most important compactifications of $E$ are those where $P$ is a multiprojective space. Since for such a $P$ a line bundle is normally presented if and only if it is very ample (or even ample) (cf. [4] section 6) we get

COROLLARY 4.4. *Let $P$ be a multiprojective $G$-equivariant compactification of $G$, $L \in \mathrm{Pic}(P)$ very ample, $G$-linearized and $M \in \mathrm{Pic}(A)$ normally generated. Let $E(P) \hookrightarrow \mathbb{P}^N$ be the projective embedding associated to the line bundle $F = E(L) \otimes \pi^* M$. Then the homogeneous ideal of $E(P)$ in $\mathbb{P}^N$ is generated by its forms of degree $\leq \dim A + 3$.*

Since for any ample line bundle $M$ an $A$ the third power $M^3$ is normally generated, one can even give a bound in case $M$ is only very ample. We omit this.

## REFERENCES

[1] HARTSHORNE, R., *Algebraic Geometry*. Grad. Texts in Math. 52, Springer, New York, 1977.
[2] HUMPHREYS, J. E., *Linear Algebraic Groups*. Grad. Texts in Math. 21, Springer, New York, 1975.

[3] KEMPF, G., KNUDSEN, F., MUMFORD, D., and SAINT-DONAT, B., *Toroidal Embeddings I.* Lecture Notes in Math. 339, Springer, 1970.
[4] KNOP, F. and LANGE, H., *Commutative algebraic groups and intersection of quadrics.* Math. Ann. *267*, 555–571 (1984).
[5] MUMFORD, D., *Varieties defined by quadratic equations.* In: *Questions on algebraic varieties.* CIME, 29–100 (1977).
[6] SERRE, J.-P., *Groups algébriques et corps de classes.* Hermann, Paris, 1957.
[7] SERRE, J.-P., *Quelques propriétés de groupes algébriques commutatifs.* Astérisque 69–70, 191–202 (1978).
[8] WALDSCHMIDT, M., *Nombres transcendents et groupes algébriques.* Astérisque 69–70.
[9] WÜSTHOLZ, G., *Recent progress in transcendence theory*, Preprint (1984).

*Mathematisches Institut*
*Rheinsprung 21*
*CH-4051 Basel*

*Mathematisches Institut*
*Bismarckstraße 1 1/2*
*D-8520 Erlangen*