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# The growth of entire and harmonic functions along asymptotic paths

JOHN ROSSI[1] and ALLEN WEITSMAN

## 1. Introduction

In a recent paper of Lewis and the two authors [5], the following generalization of a theorem of Huber [4] is proved.

THEOREM A. *Let f be a transcendental entire function. Then there exists a path Γ from* 0 *to* ∞ *such that*

$$\lim_{\substack{z\to\infty\\ z\in\Gamma}} \frac{\log|f(z)|}{\log|z|}=\infty \tag{1.1}$$

$$l(\Gamma(z))\le|f(z)|^{\varepsilon(z)}\quad(0\le\varepsilon(z)\to 0,\ z\to\infty) \tag{1.2}$$

*where* $l(\Gamma(z))$ *is the length of* Γ *from* 0 *to z and*

$$\int_\Gamma \frac{1}{|f|^\lambda}|dz|<\infty\quad(\text{for all } \lambda>0). \tag{1.3}$$

In [7], one of the authors has proved.

THEOREM B. *Let f be an entire function such that for some* $K>0$ *at least one of the level curves* $|f|=K$ *tends to* ∞. *Then there exists a path* Γ *from* 0 *to* ∞ *such that*

$$\log|f(z)|>|z|^{1/2-\varepsilon(z)} \tag{1.4}$$

*and*

$$l(\Gamma(z))\le(\log|f(z)|)^{c+2+\varepsilon(z)} \tag{1.5}$$

*where* $c>0$ *is an absolute constant and* $0\le\varepsilon(z)\to 0$ *as* $z\to\infty$.

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[1] Research carried out as a NATO Postdoctoral Fellow at Imperial College, London.

In this paper we prove

THEOREM 1. *Let f be as in Theorem* B. *Then there exists a path $\Gamma$ from* 0 *to* $\infty$ *such that* (1.4) *holds and*

$$\int_\Gamma (\log|f|)^{-(2+\lambda)}\,|dz|<\infty \quad (\text{for all } \lambda>0). \tag{1.6}$$

Whereas (1.1) and (1.2) imply (1.3), we note that because of the presence of $c$, (1.4) and (1.5) do not imply (1.6). The constant $c$ is a by-product of the proof of Theorem B. We use a totally different approach in proving Theorem 1.

COROLLARY 1. *Let $u$ be a nonconstant harmonic function in* $\mathbb{C}$. *Then there exists a path $\Gamma$ from* 0 *to* $\infty$ *such that* (1.4) *and* (1.6) *hold with* $\log|f|$ *replaced by* $u$.

The proof of Corollary 1 is immediate from Theorem 1. Indeed, if $u$ is any such harmonic function and $v$ is its harmonic conjugate in $\mathbb{C}$ then $f=e^{u+iv}$ is transcendental and entire with $u=\log|f|$. Clearly by the harmonicity of $u$ every level curve of $|f|=1$ $(u=0)$ extends to $\infty$.

We also prove

THEOREM 2. *Let $f$ be an entire function of order $\rho\leqq\infty$ such that for some $K>0$ the set $\{z:|f|>K\}$ contains at least two components. Then there exists a path $\Gamma$ from* 0 *to* $\infty$ *such that*

$$\log|f(z)|>|z|^{[\rho/(2\rho-1)]-\varepsilon(z)} \quad (0\leq\varepsilon(z)\rightarrow 0 \text{ as } z\rightarrow\infty) \tag{1.7}$$

*and*

$$\int_\Gamma (\log|f|)^{-[(2\rho-1)/\rho]+\lambda}\,|dz|<\infty \quad (\text{for all } \lambda>0). \tag{1.8}$$

(We note that by hypothesis and an easy application of the Ahlfors, Denjoy, Carleman method, $\rho\geq 1$ and thus $(2\rho-1)/\rho\geq\frac{1}{2}$.)

Examples in Eremenko [3 p. 681] show that $\varepsilon(z)$ cannot be replaced by 0 in (1.4) and (1.6).

By modifying his examples slightly, we can find an entire function $f$ of order $\rho$, $1\leq\rho\leq\infty$ such that

$$\int_\Gamma (\log|f(z)|)^{-(2\rho-1)/\rho}\,|dz|=\infty$$

for every path $\Gamma$ on which $|f|>1$. This shows that (1.5) and (1.7) are "sharp" independent of (1.4) and (1.6).

Barth, Brannan and Hayman [2, Theorem 2] show that $\varepsilon(z)$ cannot be replaced by 0 in (1.4) where $\log|f|=u$ is harmonic. Brannan has pointed out in private communication that their example can be modified to show that (1.5) is also "sharp" for harmonic functions. Specifically one can construct a harmonic function $u$ such that

$$\int_\Gamma u(z)^{-2}\,|dz|=\infty$$

for all paths $\Gamma$ where $u>0$.

## 2. Preliminary lemmas

Let $D$ be an unbounded regular plane domain. We let $\theta^*(r)=\infty$ if $\{|z|=r\}\subseteq D$. Otherwise we let $r\theta^*(r)$ equal the length of the longest arc in the intersection of $\{|z|=r\}$ and $D$. Recall that a set $G$ has log density one if $(\log r)^{-1}\int_{G\cap[1,r]} dt/t\to 1$ as $r\to\infty$. We state

LEMMA 1. *Let $D$ be as above and suppose*

$$\inf_G \varlimsup_{\substack{r\to\infty\\ r\in G}} \theta^*(r)=\frac{\pi}{\alpha}\quad (\tfrac{1}{2}\le\alpha<\infty) \tag{2.1}$$

*where the* inf *is taken over all sets $G$ of log density one. Then there exists $v>0$ harmonic in $D$ such that for all $z\in D$*

$$v(z)\ge|z|^{\alpha-\varepsilon(|z|)}\quad (0\le\varepsilon(|z|)\to 0\quad\text{as}\quad |z|\to\infty). \tag{2.2}$$

We remark that without the log density statement, (2.2) was proved in [2] with $\alpha=\frac{1}{2}$.

Before we prove Lemma 1 we need the following lemma which asserts that the inf in (2.1) is attained.

LEMMA 2. *There exists a set $G$ of log density one such that*

$$\varlimsup_{\substack{r\to\infty\\ r\in G}} \theta^*(r)=\frac{\pi}{\alpha}. \tag{2.3}$$

*Proof.* Let l.m. $(E)=\int_E dt/t$ for any measurable set $E\subseteq[0,\infty)$. By (2.1) we may find $G_n$, $n=1,2,\ldots$ such that

$$\theta^*(r)\le\frac{\pi}{\alpha}+\frac{1}{n} \tag{2.4}$$

and

$$\text{l.m.}\,(G_n\cap[1,r])\ge\left(1-\frac{1}{n}\right)\log r \tag{2.5}$$

provided $r\in G_n$, $r\ge r_n$. We may choose $r_n$ so large that

$$\frac{1}{n-1}\log r_n\ge\log r_{n-1},\qquad n=2,3,\ldots \tag{2.6}$$

Define $G=\bigcup_{n=1}^{\infty}G_n\cap[r_n,r_{n+1}]$. To see that log dens $G=1$, choose $\varepsilon>0$ and let $N$ be such that $3/N<\varepsilon$. Suppose $r\in G$ and $r_n\le r<r_{n+1}$ for some $n\ge N+1$. We have by (2.5) and (2.6)

$$\begin{aligned}
\text{l.m.}\,(G\cap[1,r])&\ge\text{l.m.}\,(G_{n-1}\cap[r_{n-1},r_n])+\text{l.m.}\,(G_n\cap[r_n,r])\\
&\ge\left(1-\frac{1}{n-1}\right)\log r_n-\log r_{n-1}+\left(1-\frac{1}{n}\right)\log r-\log r_n\\
&=-\frac{1}{n-1}\log r_n-\log r_{n-1}+\left(1-\frac{1}{n}\right)\log r\\
&\ge-\frac{2}{n-1}\log r_n+\left(1-\frac{1}{n}\right)\log r\\
&\ge\left(1-\frac{3}{n-1}\right)\log r\\
&\ge\left(1-\frac{3}{N}\right)\log r\\
&\ge(1-\varepsilon)\log r.
\end{aligned}$$

Since $\varepsilon$ was arbitrary $G$ has log density one.

Furthermore given $\varepsilon>0$, there exists $N$ such that $1/N<\varepsilon$ and if $r\ge r_N$ we have by (2.4) and the definition of $G$ that $\theta^*(r) \leq (\pi/\alpha) - \varepsilon$. This implies

$$\overline{\lim_{\substack{r\to\infty\\ r\in G}}}\, \theta^*(r) \leq \frac{\pi}{\alpha}. \tag{2.7}$$

Since $G$ has log density one, (2.7) and (2.1) imply (2.4). Lemma 2 is now proved.

*Proof of Lemma* 1. We denote by $\eta_i(r)$, $i = 1, 2, \ldots$ any nonnegative sequences such that $\eta_i(r) \to 0$ as $r \to \infty$. Then with $G$ as in Lemma 2, we have

$$\theta^*(r) \leq \frac{\pi}{\alpha - \eta_1(r)} \quad (r \in G). \tag{2.8}$$

Also if $a \in E$ where $E$ is compact in $\mathbb{C}$ and $|a| \geq 1$

$$\text{l.m.}\,(G \cap [|a|, r] \geq [1 - \eta_2(r)] \log \frac{r}{|a|} \tag{2.9}$$

uniformly in $E$.

By (2.8) we have

$$\int_{G\cap[|a|,\, r]} \frac{\eta_1(t)}{t}\, dt \leq \eta_3(r) \log \frac{r}{|a|}. \tag{2.10}$$

uniformly in $E$.

Let $D_R$ be any component of $D \cap \{|\zeta| < R\}$. Pick $z \in D_R$ with $|z| < R/4$ and let $\omega_R(z)$ be the harmonic measure of $\{|\zeta| = R\} \cap \partial D_R$ with respect to $z$ and $D_R$. Then by an inequality found in [8, p. 116] we have

$$\omega_R(z) \leq 9\sqrt{2} \exp\left\{-\pi \int_{2|z|}^{R/2} \frac{dt}{t\theta^*(t)}\right\}. \tag{2.11}$$

By (2.8)–(2.11) we have for $z \in E$ compact in $\mathbb{C}$

$$\omega_R(z) \leq K\left(\frac{|z|}{R}\right)^{\alpha - \eta_4(R)} \tag{2.12}$$

where $K$ is a constant depending only on $E$.

Let $\phi(r)$ be any convex increasing function of $\log r$ such that

$$\frac{\log \phi(r)}{\log r} \to \alpha \quad (r \to \infty) \tag{2.13}$$

and

$$\phi(2r) \le \frac{r^{\alpha - \eta_4(r)}}{(\log r)^2}. \tag{2.14}$$

We now employ a technique similar to the one used in Lemma 1 of [2]. Let $D_R$ be as above. Then there exists a unique function $v_R(z)$ harmonic in $D_R$, continuous in $\bar{D}_R$ such that for $z \in \partial D_R$

$$v_R(z) = \phi(|z|). \tag{2.15}$$

Let $R_n = 2^n$, $n = 0, 1, 2, \dots$ and define $D_{R_n}$ as before making sure that $D_{R_{n+1}} \supseteq D_{R_n}$. Let $\omega_{n,\nu}$, $n \ge \nu$ be the harmonic measure in $D_{R_n}$ of the portion of $\partial D_{R_n}$ in $\{|\zeta| \ge R_\nu\}$. Then for all $z \in D_{R_n}$, $|z| \le R_\nu/4$, we have

$$\omega_{n,\nu}(z) \le \omega_{R_\nu}(z). \tag{2.16}$$

Choose $R_k$ to be the smallest radius greater than $4|z|$. Then for $z \in D_{R_n} \cap \{|z| \le R_k/4\}$, $n \ge k$, we have by (2.12), (2.16) the definition of $\phi$, and the fact that $|z| \ge R_k/8$,

$$\begin{aligned} v_{R_n}(z) &\le \phi(R_k) + \sum_{\nu=k}^{n-1} \phi(R_{\nu+1})\omega_{n,\nu}(z) \\ &\le \phi(8|z|) + k\,|z|^\alpha \sum_{\nu=k}^{n-1} \phi(R_{\nu+1}) R_\nu^{-\alpha+\eta_4(R_\nu)} \\ &\le \phi(8|z|) + k\,|z|^\alpha \left(1 + \sum_{\nu=k}^{\infty} \frac{1}{(\nu+1)^2}\right) \\ &\le k_1 |z|^\alpha \end{aligned} \tag{2.17}$$

where $k_1 > 0$ is a constant depending only on the compact set $|z| \le R_K/4$.

Since $\phi$ is a convex function of $\log r$, we have that $\phi(|z|) - v_{R_n}(z)$ is subharmonic in $D_{R_n}$ and equal to 0 on $\partial D_{R_n}$. Thus for $z \in D_{R_n}$ we have

$$v_{R_n}(z) \ge \phi(|z|). \tag{2.18}$$

Also if $m \geq n$ and $z \in D_{R_n}$ we have

$$v_{R_m}(z) \geq v_{R_n}(z). \tag{2.19}$$

By (2.17)–(2.19), $v_{R_n}$ is an increasing sequence of harmonic functions uniformly bounded on compact sets. By Harnack's Theorem $v(z) = \lim_{n\to\infty} v_{R_n}(z)$ is harmonic in $D$. Thus (2.2) follows easily from (2.13) and (2.18).

## 3. Proof of Theorem 1 when $f$ has no zeros

We assume first that $f$ has no zeros. Then every level curve of $\log|f| = 1$ extends to $\infty$. Thus if $D$ is any component of $\{z : \log|f| > 1\}$, $D$ is simply connected and contains no full circle $|z| = r$ for $r \geq r_0$. Thus we may find a function $v$ harmonic in $D$ satisfying (2.2) for $\alpha = \frac{1}{2}$. Now let $z_0 \in D$. We can find $\delta > 0$ such that

$$\log|f(z_0)| - \delta v(z_0) > 1. \tag{3.1}$$

Define $w = \delta v$ and let $w^*$ be the harmonic conjugate of $w$ in $D$. Then $\phi = e^{w+iw^*}$ is analytic in $D$ with no zeros such that

$$\log|\phi| = w \tag{3.2}$$

satisfies (2.2) (for possibly another $\varepsilon(z)$).

Set $F = f/\phi$ in $D$. By (2.2), (3.1) and (3.2) $\log F$ has boundary values on $\partial D$ not exceeding 1 and is greater than 1 at $z_0 \in D$. Thus every component $\mathcal{F}_R$, $R \geq 1$ of $\{z : |F| > R\}$ is nonempty and contained in $D$.

To construct our path $\Gamma$ we will use extremal length arguments in each $\mathcal{F}_R$. We define extremal length as in [1. p. 11]. Let $\mathcal{G}$ be a family of curves. The extremal length $\lambda(\mathcal{G})$ of $\mathcal{G}$ is defined as

$$\lambda(\mathcal{G}) = \sup_\rho \frac{L^2(\rho)}{A(\rho)}$$

where

$$L(\rho) = \inf_{\gamma \in \mathcal{G}} \int_\gamma \rho\,|dz|, \qquad A(\rho) = \iint_{\mathbb{C}} \rho^2\,dx\,dy$$

and $\rho \geq 0$ ranges over all measurable functions for which $A(\rho) \neq 0, \infty$.

To get the construction started let $R_0 > e$ be such that $F' \neq 0$ when $|F| = R_0$ and take a component $\mathcal{F}_{R_0} \subseteq D$ with $\zeta_0 \in \partial\mathcal{F}_{R_0}$ arbitrarily chosen. It follows from the Cauchy–Riemann equations that $\arg F$ is then monotone on $\partial\mathcal{F}_{R_0}$ so that for some $\eta > 0$ a branch of the function $w = \log F$ maps a neighborhood of an arc of $\partial\mathcal{F}_{R_0}$ containing $\zeta_0$ univalently to a neighborhood of a segment

$$T_0 = \{w = \log R_0 + \mathrm{iv} : \psi_0 - \eta \leq v \leq \psi_0 + \eta\}$$

with the arc of $\partial\mathcal{F}_{R_0}$ and the segment $T_0$ corresponding. By replacing $F$ by $F^K$ where $K$ is a sufficiently large positive integer we may assume that $\eta$ is arbitrarily large. This modification of $F$ will in no way affect our method and so we asume that $\eta = e$ in the definition of $T_0$.

Recall the function $\varepsilon(r)$ in (2.2). Fix $\lambda_0 > 0$ such that

$$(2+4e)\sum_{j=0}^{\infty}\left(2\pi\int_0^{\infty}\frac{r\,dr}{[e^j - 1 + \log R_0 + r^{\frac{1}{2}-\varepsilon(r)}]^{4+2\lambda_0}}\right)^{1/2} \leq 1. \tag{3.4}$$

This is possible since the left side of (3.4) converges for every $\lambda_0 > 0$.

With $\psi_0$ as chosen, we let $Q_0$ be the square in the $w$-plane defined by $Q_0 = \{(s, t_0) : \log R_0 < s < 2e + \log R_0,\ \ \psi_0 - e < t < \psi_0 + e\}$. Set $\gamma = \gamma_{t_0} = \{(s, t_0) : \log R_0 \leq s < s'\}$ where $t_0$ ranges between $\psi_0 - e$ and $\psi_0 + e$ and $s' \leq \log R_0 + 2e$. The point $s'$ is chosen to be $\log R_0 + 2e$ if the inverse $h(w)$ of $\log F$ can be uniquely continued on $\gamma_{t_0}$ from $\log R_0$ to $\log R_0 + 2e$. Otherwise $s'$ is chosen so that $(s', t_0)$ is the first point on the horizontal segment $\gamma_{t_0}$ where $h$ cannot be continued uniquely. Since $s' > \log R_0$ and since $h$ cannot tend to $\partial\mathcal{F}_{R_0} \subseteq D$ this can only happen if either there exists a point $z_1 \in \mathcal{F}_{R_0}$ such that $\log F(z_1) = (s', t_0)$ and $F'(z_1) = 0$ or if $h \to \infty$ as $w \to (s', t_0)$.

By taking unions over all such horizontal segments and their preimages in the $z$-plane, we obtain a measurable set $\mathcal{F} \subseteq \mathcal{F}_{R_0}$ which maps 1–1 under $\log F$ to a subset $\tilde{Q}_0$ of $Q_0$. Let $\mathcal{G}$ be the family of *all* horizontal segments in $Q_0$ connecting both sides of $Q_0$. Since $Q_0$ is a square this implies [1, p. 12] that $\lambda(\mathcal{G}) = 1$. Furthermore since the curves in $\tilde{\mathcal{G}}$ are no "longer" than those in $\mathcal{G}$, we have in the notation of [1, p. 12] that $\tilde{\mathcal{G}} < \mathcal{G}$ and so $\lambda(\tilde{\mathcal{G}}) \leq 1$. Let $\tilde{C}$ be the collection of the images under $h$ of those curves in $\tilde{\mathcal{G}}$ which extend all the way across $Q_0$. Then $\tilde{C} = h(\tilde{\mathcal{G}}) - C_1 - C_2$ where $C_1$ are the curves which run into points where $F' = 0$ and $C_2$ are the unbounded curves. But the number of curves in $C_1$ is countable and the curves in $C_2$ extend to $\infty$. Thus it is easy to see [6, Theorems 2.13 and 2.14] that $\lambda(\tilde{C}) = \lambda(h(\tilde{\mathcal{G}}))$. Since $(\log F)' \neq 0$ on $h(\tilde{\mathcal{G}})$, it is easy to show that $\lambda(h(\tilde{\mathcal{G}})) = \lambda(\tilde{\mathcal{G}})$. This gives

$$\lambda(\tilde{C}) \leq 1. \tag{3.5}$$

On $\tilde{C}$ we take (in (3.3)) $\rho=\rho_0=(\log|f|)^{-2-\lambda_0}$ and $\rho=0$ off $\tilde{C}$. Clearly $A(\rho)\neq 0$. To show that $A(\rho)\neq\infty$ we have by (2.2), (3.2) and the fact that the union of the $\tilde{C}$ lies in $\mathcal{F}\subseteq\mathcal{F}_{R_0}$

$$\begin{aligned} A(\rho_0) &\leq \iint_{\mathcal{F}} (\log|f|)^{-4-2\lambda_0} r\,dr\,d\theta \\ &\leq \iint_{\mathcal{F}} (\log R_0+\delta v)^{-4-2\lambda_0} r\,dr\,d\theta \\ &\leqq 2\pi\int_0^\infty (\log R_0+r^{\frac{1}{2}-\varepsilon(r)})^{-4-2\lambda_0} r\,dr \\ &<\infty. \end{aligned} \tag{3.6}$$

Let us define for $R$ and $\lambda$ positive

$$K(R,\lambda)=\left(2\pi\int_0^\infty (R+r^{\frac{1}{2}-\varepsilon(r)})^{-4-2\lambda} r\,dr\right)^{1/2}. \tag{3.7}$$

Thus it follows by (3.3), (3.6) and (3.7) that there exists in $\tilde{C}$ a curve $\tilde{\beta}_0\subseteq\mathcal{F}_{R_0}$ that joins a point $z\in\partial\mathcal{F}_{R_0}$ to $\partial\mathcal{F}_{e^{2e}R_0}$ for some component $\mathcal{F}_{e^{2e}R_0}\subseteq\mathcal{F}_{R_0}$ of the set $\{z:|F|>e^{2e}R_0\}$. Furthermore

$$\int_{\tilde{\beta}_0} (\log|f|)^{-2-\lambda_0}|dz|\leq 2K(\log R_0,\lambda_0). \tag{3.8}$$

We let $\tilde{\beta}_0$ correspond to $\gamma_{t_0}$ in $Q_0$. Then a similar procedure is applied to the rectangle $S_0=\{(s,t):e+\log R_0<s<2e+\log R_0,\ \ t_0<t<t_0+2e^2\}$ in the $w$ plane where the bottom of $S_0$ corresponds to half of $\tilde{\beta}_0$ under a branch $h$ of $(\log F)^{-1}$. Here we consider the family of vertical segments $\gamma=\gamma_{s_0}=\{(s_0,t):t_0-e^2<t<t_0+e^2\}$ in $S_0$. As before we obtain a family $\tilde{\mathcal{G}}$ whose union is mapped 1–1 onto a set $\xi\subseteq\mathcal{F}_{e^eR_0}$. Since $S_0$ is a rectangle of length $2e^2$ and width $e$ we obtain with $\tilde{C}$ as before

$$\lambda(\tilde{C})=\lambda(\tilde{\mathcal{G}})\leq\lambda(\mathcal{G})=\frac{2e^2}{e}=2e.$$

So in $\mathcal{E}$ we again get a curve $\tilde{\alpha}_0$ whose image $\gamma_{s_0}$ under $\log F$ is a vertical segment joining the two sides of $S_0$ and

$$\int_{\tilde{\alpha}_0} (\log|f|)^{-2-\lambda_0}|dz|\leq 4eK(\log R_0,\lambda_0). \tag{3.9}$$

We now cut $\tilde{\beta}_0$ off where it joins $\tilde{\alpha}_0$ at $\log R_1 (\geq \log R_0 + e)$ and obtain the first piece $\beta_0 \subseteq \tilde{\beta}_0$ of our curve $\Gamma$. With $\lambda_0$ still fixed we continue with the square

$$Q_1 = \{(s, t) : \log R_1 < s < 2e^2 + \log R_1,\ t_0 < t < t_0 + 2e^2\}$$

and obtain a curve $\tilde{\beta}_1$ on which $F' \neq 0$ joining $\tilde{\alpha}_0$ to the boundary of a component $\mathscr{F}_{e^{2e^2}R_1} \subseteq \mathscr{F}_{R_1}$ of the set $\{z : |F| > e^{2e^2} R_1\}$. Then (3.6) becomes

$$\int_{\tilde{\beta}_1} (\log |f|)^{-2-\lambda_0} |dz| \leq 2K(\log R_1, \lambda_0).$$

We now cut $\tilde{\alpha}_0$ off where it joins $\tilde{\beta}_1$ and obtain the second piece $\alpha_0$ of $\Gamma$. Let $\tilde{\beta}_1$ correspond to $\gamma_{t_1}$ in $Q_1$ and define the rectangle

$$S_1 = \{(s, t) : e^2 + \log R_1 < s < 2e^2 + \log R_1,\ t_1 < t < t_1 + 2e^3\}.$$

Again we find that the extremal length of the vertical lines joining the two sides of $S_1$ is $2e$. So we again obtain a curve $\tilde{\alpha}_1$ such that

$$\int_{\tilde{\alpha}_1} (\log |f|)^{-2-\lambda_0} |dz| \leq 4eK(\log R_1, \lambda_0).$$

This process is continued yielding a curve $\beta_0 \cup \alpha_0 \cup \beta_1 \cup \alpha_1 \cup \cdots \cup \beta_n \cup \tilde{\alpha}_n$ extending from $\partial \mathscr{F}_{R_0}$ to the boundary of a component $\mathscr{F}_{R_n}$ where

$$\log R_n \geq e^n - 1 + \log R_0 \qquad n = 0, 1, 2, \ldots \tag{3.10}$$

Our construction yields

$$\int_{\tilde{\beta}_i} (\log |f|)^{-2-\lambda_0} |dz| \leq 2K(\log R_j, \lambda_0)$$

and

$$\int_{\tilde{\alpha}_i} (\log |f|)^{-2-\lambda_0} |dz| \leq 4eK(\log R_j, \lambda_0).$$

Adding these contributions and taking into account (3.4), (3.7) and (3.10) we obtain

$$\int_{\beta_0\cup\alpha_0\cup\cdots\cup\beta_n\cup\tilde{\alpha}_n} (\log|f|)^{-2-\lambda_0}|dz| \le (2+4e)\sum_{j=0}^{\infty} K(\log R_j, \lambda_0)$$

$$\le (2+4e)\sum_{j=0}^{\infty} K(e^j - 1 + \log R_0, \lambda_0)$$

$$\le 1$$

independent of $n$. We keep $\lambda_0$ fixed until $N$ is so large that

$$(2+4e)\sum_{j=0}^{\infty}\left(2\pi\int_0^{\infty}\frac{r\,dr}{(e^j - 1 + \log R_N + r^{1/2-\varepsilon(r)})^{4+\lambda_0}}\right)^{1/2} \le \tfrac{1}{2}. \tag{3.11}$$

At this point we change $\lambda_0$ to $\lambda_0/2$ with (3.11) playing the role of (3.4). We then continue from the arc $\tilde{\alpha}_n$ where $|F| = R_N$ in place of the original arc $\gamma_0$ on $|F| = R_0$. In the general case we obtain a sequence

$$0 = N_0 < N_1 < \cdots < N_j$$

such that

$$\log R_{N_j} \ge e^{N_j - N_{j-1}} + \log R_{N_{j-1}} \qquad j = 1, 2, \ldots \tag{3.12}$$

The $N_j$ are chosen such that

$$(4+2e)\sum_{n=0}^{\infty} K(e^n - 1 + \log R_{N_j}, \lambda_0/(j+1)) \le 2^{-j} \tag{3.13}$$

with $\beta_{N_j}\cup\alpha_{N_j}\cup\cdots\cup\beta_{N_{j+1}}\cup\tilde{\alpha}_{N_{j+1}}$ extending from $\partial\mathcal{F}_{R_{N_j}}$ to $\partial\mathcal{F}_{R_{N_{j+1}}}$ and satisfying

$$\int_{\beta_{N_j}\cup\alpha_{N_j}\cup\cdots\cup\beta_{N_{j+1}}\cup\alpha_{N_{j+1}}} (\log|f|)^{-2-\lambda_0/(j+1)}|dz|$$

$$\le (4+2e)\sum_{n=0}^{\infty} K(e^n - 1 + \log R_{N_j}, \lambda_0/(j+1))$$

Let $\Gamma = \beta_0\cup\alpha_0\cup\cdots\cup\beta_k\cup\alpha_k\cup\cdots$ Then since $\log|f| > 1$ in $D$ and hence on $\Gamma$ we have

$$\int_\Gamma (\log|f|)^{-2-\lambda}|dz| < \int_\Gamma (\log|f|)^{-2-\lambda'}|dz|$$

if $\lambda > \lambda'$. Thus it follows from (3.13) and (3.14) that $\Gamma$ satisfies (1.6) for all $\lambda > 0$.

## 4. Proof of Theorem 1—general case

When $f$ has zeros the proof in §3 must be modified slightly. First of all by hypothesis there exists a component $D$ of $\{z : \log|f(z)| > \log K\}$ such that $\theta^*(r) \leq 2\pi$ for $r \geq r_0$, where we can assume $K > e$. Thus we can still find $v$ satisfying (2.2) and (3.1). Since $D$ is not necessarily simply connected, we can only define a local conjugate of $w = \delta v$ and so our function $F$ is now multivalued. However $|F|$ and $\log|F|$ are single valued and subharmonic in $\mathbb{C}$. Thus we see that $\mathcal{F}_R$ is again nonempty for all $R \geqq K$.

We then proceed as before taking $\gamma_0$ to be a level curve of $|F| = R_0$ extending to infinity, where $F' \neq 0$ and find a curve $\tilde{\beta}_0$. We remark that $\tilde{\beta}_0$ never intersects a level curve $|F| = R$, $R_0 \leq R \leq R + 2e$ which forms a loop. In fact inside such a loop $|F| < R$ so if $\beta_0^*$ is the portion of $\tilde{\beta}_0$ joining $R_0$ to $R$, $\beta_0^*$ must pass through some point $z_0$ where $|F(z)| > R$. This is impossible since $\beta_0^*$ is the image under $h$ of the horizontal segment beginning at $\log R_0$ and ending at $\log R$. Hence we can find an $\tilde{\alpha}_0$ as before. We now continue as in §3.

## 5. Proof of Theorem 2

To prove Theorem 2 we need the following.

LEMMA 3. *Let $f$ be entire of order $\frac{1}{2} < \rho < \infty$. If $D$ is any component of $\{z : |f(z)| > K\}$, $K > e$ then*

$$\sup_G \varliminf_{\substack{r\to\infty \\ r \in G}} \theta^*(r) \geq \frac{\pi}{\rho} \tag{5.1}$$

*where the* sup *is taken over all sets $G$ of log density one.*

*Proof.* Suppose on the contrary that the left side of (5.1) equals $\pi/\rho_1$, $\rho_1 > \rho$. As in Lemma 2 we may find a set $G$ of log density one where the sup on the left side of (5.1) is attained. Thus for $r \geq r_0$, $r \in G$

$$\theta^*(r) \leq \frac{\pi}{\rho_2} \quad (\rho_1 \geq \rho_2 > \rho). \tag{5.2}$$

Let $z \in D$ and choose $R$ such that $|z| < R/4$. With the notation of (2.11) we have

$$\omega_R(z) \le 9\sqrt{2} \exp\left\{-\rho_2 \int_{G\cap[2|z|,\,R/2]} \frac{dt}{t}\right\}$$
$$\le 9\sqrt{2} \exp\left\{-\rho_2(1-\varepsilon_m) \log\left(\frac{R}{|z|}\right)\right\}$$
$$= 9\sqrt{2}\left(\frac{|z|}{R}\right)^{\rho_2(1-\varepsilon_m)} \tag{5.3}$$

where (since log dens $G = 1$) $\varepsilon_m \to 0$ as $m \to \infty$. Thus by (2.12) we have for fixed $z \in D_R$

$$1 \le \log|f(z)| \le \log K + \log M(R, f)\omega_R(z)$$
$$\le K_1 \log M(R, f)\left(\frac{|z|}{R}\right)^{\rho_2(1-\varepsilon_m)}$$

where $K_1 > 0$ is constant. Then

$$\log M(R, f) \ge \frac{1}{K_1}\left(\frac{R}{|z|}\right)^{\rho_2(1-\varepsilon_m)}.$$

Since $z$ is fixed this implies that $f$ has order at least $\rho_2 > \rho$, a contradiction. Thus (5.1) holds and Lemma 3 is true.

*Proof of Theorem* 2. Let $D_1$ be a component of $\{|f| > K\}$ and suppose

$$\inf_{G} \overline{\lim_{\substack{r\to\infty\\ r\in G}}}\, \theta_1^*(r) = \frac{\pi}{\alpha} \quad (\tfrac{1}{2} \le \alpha < \infty) \tag{5.4}$$

where the inf is taken over all sets $G$, log dens $G = 1$ and $\theta_1^*$ corresponds to $\theta^*$ for $D_1$. Since there exists another component $D_2$ of $\{|f| > K\}$, (5.4) implies

$$\sup_{G} \underline{\lim_{\substack{r\to\infty\\ r\in G}}}\, \theta_2^*(r) \le 2\pi - \frac{\pi}{\alpha} \tag{5.5}$$

where $\theta_2^*$ corresponds to $\theta^*$ for $D_2$.

By Lemma 3 we must have

$$\frac{\pi}{\rho} \leq 2\pi - \frac{\pi}{\alpha}$$

or

$$\alpha \geq \frac{\rho}{2\rho - 1}. \tag{5.6}$$

By Lemma 1, (5.4) and (5.6) we may find a function $v$ harmonic in $D_1$ such that for all $z \in D_1$

$$v(z) \geq |z|^{[\rho/(2\rho-1)]-\varepsilon(|z|)} \quad (0 \leq \varepsilon(|z|) \to 0 \quad \text{as} \quad |z| \to \infty). \tag{5.7}$$

We now define $\phi$ and $F$ as in the proof of Theorem 1. The proof of Theorem 2 now follows in the same way as that of Theorem 1 using $\rho/(2\rho-1)$ instead of $\frac{1}{2}$.

## REFERENCES

[1] L. AHLFORS, *Lectures on quasiconformal mappings*, Van Nostrand, 1966.
[2] K. BARTH, D. BRANNAN and W. K. HAYMAN, *The growth of harmonic functions along an asymptotic path*, Proc. Lond. Math. Soc. *37* (1978) 363–384.
[3] A. EREMENKO, *Growth of entire and subharmonic functions on asymptotic curves*, Sibirsk Mat. Z. *21* (1980) 39–51, Eng Trans: Siberian Math. J. (1981) 673–683.
[4] A. HUBER, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv. *32* (1957) 13–72.
[5] J. LEWIS, J. ROSSI and A. WEITSMAN, *On the growth of subharmonic functions along paths*, Ark. Mat. *22* (1984) 109–119.
[6] M. OHTSUKA, *Dirichlet Problem, Extremal length and Prime Ends*, Van Nostrand, 1970.
[7] J. ROSSI, *The length of asymptotic paths of harmonic functions*, J. London Math. Soc. (to appear).
[8] M. TSUJI, *Potential theory in modern function theory*, Maruzen, 1959.

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