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Cohomology with free coefficients of the fundamental group of a graph of groups

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Let G be an HNN extension $H*_A$ with base group H, associated subgroup $A \subseteq H$, and monomorphism $\tau: A \to H$. Consider the Mayer-Vietoris sequence with $\mathbb{Z}G$ coefficients

$$\cdots \to H^q(G, \mathbb{Z}G) \to H^q(H, \mathbb{Z}G) \xrightarrow{\alpha} H^q(A, \mathbb{Z}G) \to \cdots$$

(*cf.* [1], [2], or [6]). We will be interested in the case where H and A are assumed to be of type FP_n for some n. [Recall that a group K is said to be of type FP_n if the $\mathbb{Z}K$-module $\mathbb{Z}$ with trivial K-action admits a projective resolution which is finitely generated in dimensions $\leq n$.] Bieri ([1], Theorem 6.6) showed in this case that the map α is a split monomorphism for $q \leq n$, provided A and $\tau(A)$ are of finite index in H, and he deduced under these hypotheses that G is a duality group if H and A are duality groups. He proved similar results for amalgamated free products in which the amalgamated subgroup is of finite index in both free factors.

In this paper we generalize Bieri's results by (a) dropping the finite index hypotheses and (b) allowing G to be the fundamental group of an arbitrary finite graph of groups of type FP_n. There is a Mayer–Vietoris sequence analogous to that above, and we give an interpretation (in dimensions $\leq n$) of α and its kernel and cokernel in terms of the tree X associated to G [9]. This leads to a short exact sequence for computing $H^*(G, \mathbb{Z}G)$, involving the compactly supported cohomology of X with coefficients in the system $\{H^*(G_\sigma, \mathbb{Z}G_\sigma)\}$, where G_σ ranges over the vertex and edge groups. See Theorem 2.2 for the precise statement.

We are able to deduce, among other things, sufficient conditions weaker than those of Bieri for α to be a monomorphism. In the HNN case, for instance, we prove

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THEOREM 0.1. *Let H and A be of type FP_n and let the restriction map $H^q(H, \mathbb{Z}H) \to H^q(A, \mathbb{Z}H)$ be a monomorphism for some $q \leq n$. Then $\alpha : H^q(H, \mathbb{Z}G) \to H^q(A, \mathbb{Z}G)$ in the Mayer–Vietoris sequence is a monomorphism.*

This holds, in particular, if A is of finite index in H, e.g., if $H = A$; $\tau(A)$, however, is allowed to be arbitrary. A concrete example of this situation is given in [5], where we use Theorem 0.1 to show that a certain interesting group F of type FP_∞ has $H^q(F, \mathbb{Z}F) = 0$ for all q.

Finally, in case $(H:A) < \infty$ and $G = H *_A$ as above, we obtain a result (Theorem 3.3) relating properties of $H^q(G, \mathbb{Z}G)$ to corresponding properties of $H^{q-1}(H, \mathbb{Z}H)$. In particular (i) if H is an m-dimensional duality group, then G is an $(m+1)$-dimensional duality group, and, (ii) if H is of type FP_n, $q \leq n$ and $1 < (H:A) < \infty$, then $H^{q-1}(H, \mathbb{Z}H)$ $\mathbb{Z}$-free implies $H^q(G, \mathbb{Z}G)$ $\mathbb{Z}$-free.

The paper is organized as follows. §1 contains some general observations about $H^*(G, \mathbb{Z}G)$ as a functor of G. These results might be well-known, but we know of no reference for them. In §2 we apply the results of §1 to the Mayer–Vietoris sequence discussed above. In particular, Theorem 2.2 falls out immediately. §3 contains examples, including Theorem 3.2 which implies Theorem 0.1, above. Finally, an appendix contains a direct proof via normal forms of Theorem 0.1 for the benefit of the reader who is not familiar with the theory of graphs of groups.

Some of the results of this paper were announced in [4].

§1. Preliminaries: Functorial properties of $H^*(G, \mathbb{Z}G)$

Let $D^*(G) = H^*(G, \mathbb{Z}G)$. We want D^* to be a functor.

Recall that group cohomology is contravariant with respect to group homomorphisms and covariant with respect to coefficient module homomorphisms. It will be convenient to formalize this as in [2], §III.8, by viewing $H^*(-,-)$ as a contravariant functor on the following category $\mathcal{U}$: the objects are pairs (G, M), where G is a group and M is a left G-module; a morphism $(G, M) \to (G', M')$ is a pair $(u : G \to G', v : M' \to M)$, where u is a group homomorphism and v is an abelian group homomorphism such that $v(u(g)m') = gv(m')$ for $g \in G$, $m' \in M'$. Equivalently, v is a G-module homomorphism when M' is regarded as a G-module via u.

Let $\mathcal{C}$ be the category of groups and monomorphisms. There is a covariant functor $d : \mathcal{C} \to \mathcal{U}$ taking G to $(G, \mathbb{Z}G)$ and $i : H \to G$ to $d(i) : (H, \mathbb{Z}H) \to (G, \mathbb{Z}G)$ given by $d(i) = (i, (i^{-1})^0)$, where $(i^{-1})^0(g) = i^{-1}(g)$ if $g \in i(H)$ and $(i^{-1})^0(g) = 0$ otherwise. [Here and throughout this section we use a superscript 0 to denote the "extension by zeroes" to $\mathbb{Z}G$ of a map defined on a subset of G.] We now set $D^*(i) = d(i)^* : D^*(G) \to D^*(H)$; in other words, letting $\mathcal{A}b$ be the category of abelian groups, $D^* : \mathcal{C} \to \mathcal{A}b$ is the composition $\mathcal{C} \xrightarrow{d} \mathcal{U} \xrightarrow{H^*} \mathcal{A}b$, a contravariant functor.

Some familiar concepts fit into this framework.

EXAMPLE 1.1. Let $i : H \to G$ be an inclusion with $(G:H) < \infty$. Then $D^*(i) : D^*(G) \to D^*(H)$ is an isomorphism; in fact, it is the usual Shapiro's Lemma isomorphism. This follows from the description of the latter given in [2], §III.8, exercise 2.

Suppose H and H' are subgroups of a group G and suppose g is an element of G such that $gHg^{-1} \subseteq H'$. Suppose M is a G-module and N (resp. N') is an H-submodule (resp. H'-submodule) such that $g^{-1}N' \subseteq N$. Then there is a map $(c_g, \lambda_{g^{-1}}) : (H, N) \to (H', N')$, where $c_g(h) = ghg^{-1}$ for $h \in H$ and $\lambda_{g^{-1}}(n') = g^{-1}n'$ for $n' \in N'$.

EXAMPLE 1.2. If we set $H = H' = G$ and $M = M' = N' = \mathbb{Z}G$, one checks that $d(c_g) = (id, \rho_g) \circ (c_g, \lambda_{g^{-1}}) : (G, \mathbb{Z}G) \to (G, \mathbb{Z}G)$, where $\rho_g(x) = xg$. The map $(c_g, \lambda_{g^{-1}})$ induces id on $H^*(G, \mathbb{Z}G)$ (cf. [2], III.8.3). So $D^*(c_g) = (id, \rho_g)^*$. Thus the left conjugation action of G on itself induces, by contravariance of $D^*(-)$, the usual right action of G on $H^*(G, \mathbb{Z}G)$ coming from the right-multiplication action of G on $\mathbb{Z}G$.

We wish to study $H^*(H, \mathbb{Z}G)$, where $H \subseteq G$, in the context of the functor D^*. More generally, if S is a G-set, i.e., a set with a left G-action, let G_s be the isotropy subgroup of G at $s \in S$, and let S_0 be a set of representatives for S mod G. Then we wish to study $\bigoplus_{s \in S_0} H^*(G_s, \mathbb{Z}G)$ functorially.

For $s \in S$ and $g \in G$, the isomorphism $c_g : G_s \to G_{gs}$ induces an isomorphism $D^*(c_g) : D^*(G_{gs}) \to D^*(G_s)$. Let

$$D^*_G(S) = \bigoplus_{s \in S} D^*(G_s)$$

and

$$\bar{D}^*_G(S) = \prod_{s \in S} D^*(G_s).$$

These are right G-modules in a natural way, via the isomorphisms $D^*(c_g)$. [In case S has only one element, for instance, this is the G-module structure on $D^*(G)$ discussed in Example 1.2. In the general case $D^*(c_g)$ is induced by $d(c_g) = (id, \rho_g) \circ (c_g, \lambda_{g^{-1}})$ where (in the terms preceding Example 1.2) $H = G_s$, $H' = G_{gs}$,

$M=\mathbb{Z}G$, $N=\mathbb{Z}G_s$ and $N'=\mathbb{Z}G_{gs}g$.] In the rest of this section, we will show that $D_G^*(S)$ is functorially isomorphic to $\bigoplus_{s\in S_0} H^*(G_s, \mathbb{Z}G)$ under suitable finiteness hypotheses – see 1.6 below.

First, we look at a special case. Let H be a subgroup of G and let $S=G/H=\{gH \mid g\in G\}$. Then $D_G^*(G/H)=\bigoplus D^*(gHg^{-1})$ where g ranges over a set of representatives for G/H. For each coset representative g there are morphisms

$$(H, \mathbb{Z}G) \xrightarrow{(c_g, \lambda_{g^{-1}})} (gHg^{-1}, \mathbb{Z}[gHg^{-1}])$$

$$(H, \mathbb{Z}G) \xleftarrow{(c_{g^{-1}}, \lambda_g^0)} (gHg^{-1}, \mathbb{Z}[gHg^{-1}])$$

inducing

$$\phi_g : D^*(gHg^{-1}) \to H^*(H, \mathbb{Z}G)$$
$$\psi_g : H^*(H, \mathbb{Z}G) \to D^*(gHg^{-1})$$

and, hence, morphisms of abelian groups

$$D_G^*(G/H) \xrightarrow{\phi} H^*(H, \mathbb{Z}G) \xrightarrow{\psi} \bar{D}_G^*(G/H).$$

[Recall that, according to the convention mentioned above, $\lambda_g^0 : \mathbb{Z}G \to \mathbb{Z}[gHg^{-1}]$ is given by $g' \mapsto gg'$ if $g' \in Hg^{-1}$ and $g' \mapsto 0$ otherwise.] Our main interest here is in the map ϕ, but ψ is useful because it enables one to compute ϕ^{-1} in case ϕ is an isomorphism. Note that ϕ_g and ψ_g depend only on the class of g in G/H because of the invariance of $H^*(H, -)$ under H-conjugation (cf. [2], III.8.3).

PROPOSITION 1.3. *$\psi \circ \phi$ is the canonical inclusion of the sum in the product. ϕ is a monomorphism for any H and is an isomorphism in dimensions $\leq n$ if H is of type FP_n. ϕ and ψ are morphisms of right G-modules, where $H^*(H, \mathbb{Z}G)$ has the usual right G-action coming from the right action of G on $\mathbb{Z}G$.*

Proof. The first sentence is checked in $\mathcal{U}$. It follows that ϕ is a monomorphism. The left H-module $\mathbb{Z}G$ decomposes as $\bigoplus \mathbb{Z}Hg^{-1}$, g ranging over coset representatives. The inclusions associated with this decomposition define ϕ' in the following diagram of abelian groups, which clearly commutes:

$$\begin{array}{ccc} & \bigoplus H^*(H, \mathbb{Z}Hg^{-1}) & \\ {\scriptstyle \oplus(c_g, \lambda_{g^{-1}})} \nearrow \approx & & \searrow {\scriptstyle \phi'} \\ \bigoplus H^*(gHg^{-1}, \mathbb{Z}[gHg^{-1}]) & \xrightarrow[\phi]{} & H^*(H, \mathbb{Z}G) \\ \| & & \\ D_G^*(G/H) & & \end{array}$$

If H is of type FP_n, ϕ' is onto in dimensions $\leq n$ [1, p. 9] hence also ϕ. We have a commutative diagram

$$\begin{array}{ccc} D^*(gHg^{-1}) & \xrightarrow{D^*(c_g)} & D^*(H) \\ \downarrow{\scriptstyle \phi_g} & & \downarrow{\scriptstyle \phi_1} \\ H^*(H, \mathbb{Z}G) & \xrightarrow{(\mathrm{id}, \rho_g)^*} & H^*(H, \mathbb{Z}G) \end{array}$$

from which it follows that ϕ is a morphism of G-modules. A similar argument works for ψ. $\|$

We can apply 1.3 to general G-sets by decomposing them into orbits. If S_0 is a set of representatives for the G-set S, there is a monomorphism of right G-modules $\Phi: D_G^*(S) \to \bigoplus_{s \in S_0} H^*(G_s, \mathbb{Z}G)$; if each G_s is of type FP_n, then Φ is an isomorphism in dimensions $\leq n$.

Next we wish to consider the effect on $D_G^*(-)$ of maps between G-sets. Let S and T be G-sets and let $f: S \to T$ be a map commuting with the G-action. It is easy to construct an induced map $f^*: \bar{D}_G^*(T) \to \bar{D}_G^*(S)$ by using the inclusions $i_s: G_s \to G_{f(s)}$ $(s \in S)$ and the induced maps $D^*(i_s): D^*(G_{f(s)}) \to D^*(G_s)$. Namely, given $(u_t)_{t \in T}$ with $u_t \in D^*(G_t)$, set $f^*((u_t)) = (v_s)$, where $v_s = D^*(i_s)(u_{f(s)}) \in D^*(G_s)$. In case f^* carries $D_G^*(T)$ into $D_G^*(S)$, we will also write f^* for the induced map $D_G^*(T) \to D_G^*(S)$. This in fact happens under suitable finiteness hypotheses, as we will see below.

The crucial case to understand is that where $S = G/H$ and $T = G/K$, where $H, K \subseteq G$. In this case $f: G/H \to G/K$ is necessarily given by $f(gH) = gg_0K$ for some g_0 such that $H \subseteq g_0Kg_0^{-1}$. Let $\gamma: (H, \mathbb{Z}G) \to (K, \mathbb{Z}G)$ be the map $(c_{g^{-1}}, \lambda_{g_0})$. Note that γ^* in the following proposition does not depend on the choice of g_0.

PROPOSITION 1.4. *The diagram*

$$\begin{array}{ccc} H^*(K, \mathbb{Z}G) & \xrightarrow{\psi} & \bar{D}_G^*(G/K) \\ \downarrow{\scriptstyle \gamma^*} & & \downarrow{\scriptstyle f^*} \\ H^*(H, \mathbb{Z}G) & \xrightarrow{\psi} & \bar{D}_G^*(G/H) \end{array}$$

commutes. In any dimension where $\phi: D_G^*(G/H) \to H^*(H, \mathbb{Z}G)$ *is an isomorphism,* f^* *carries* $D_G^*(G/K)$ *into* $D_G^*(G/H)$, *and the resulting diagram*

$$\begin{array}{ccc} D_G^*(G/K) & \xrightarrow[\approx]{\phi} & H^*(K, \mathbb{Z}G) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \gamma^*} \\ D_G^*(G/H) & \xrightarrow[\phi]{\approx} & H^*(H, \mathbb{Z}G) \end{array}$$

also commutes.

Proof. For any $g \in G$ there is a commutative diagram

$$\begin{array}{ccc} (K, \mathbb{Z}G) & \xleftarrow{(c_{(gg_0)^{-1}},\, \lambda^0_{gg_0})} & (gg_0Kg_0^{-1}g^{-1}, \mathbb{Z}[gg_0Kg_0^{-1}g^{-1}]) \\ {\scriptstyle\gamma}\big\uparrow & & \big\uparrow{\scriptstyle d(i)} \\ (H, \mathbb{Z}G) & \xleftarrow[(c_{g^{-1}},\, \lambda^0_g)]{} & (gHg^{-1}, \mathbb{Z}[gHg^{-1}]), \end{array}$$

where i is an inclusion map. The first assertion of the propositions now follows at once from the definitions, and the second assertion follows from the first. ||

An important special case of 1.4 is that where $K = G$ and $g_0 = 1$. One obtains, in particular:

COROLLARY 1.5. *Let $H \subseteq G$ be a subgroup of type FP_n. Then there is a commutative diagram*

$$\begin{array}{ccc} D^*(G) & = & H^*(G, \mathbb{Z}G) \\ \big\downarrow & & \big\downarrow{\scriptstyle res} \\ D^*_G(G/H) & \xrightarrow[\Phi]{\approx} & H^*(H, \mathbb{Z}G) \end{array}$$

in dimensions $\leq n$, where res is the usual restriction map and the left hand vertical arrow has as components the maps $D^*(i_g): D^*(G) \to D^*(gHg^{-1})$ induced by the inclusions $i_g: gHg^{-1} \to G$ $(g \in G/H)$. In particular, for any $d \in D^q(G)$ $(q \leq n)$, $D^*(i_g)(d) = 0$ for almost all $g \in G/H$. ||

By decomposing a general G-set into orbits and applying 1.3 and 1.4 we get the following result, which will be needed in the next section. Let $f: S \to T$ be a map of G-sets. Let S_0 (resp. T_0) be a set of representatives for S (resp. T) mod G. For each $s \in S_0$, let $f_0(s)$ be the element of T_0 which is equivalent to $f(s)$ mod G, and choose $g_s \in G$ such that $f(s) = g_s f_0(s)$. Then $G_s \subseteq G_{f(s)} = g_s G_{f_0(s)} g_s^{-1}$. Let $\gamma_s: (G_s, \mathbb{Z}G) \to (G_{f_0(s)}, \mathbb{Z}G)$ be the map $(c_{g_s^{-1}}, \lambda_{g_s})$.

PROPOSITION 1.6. *Suppose that G_s and G_t are of type FP_n for all $s \in S$ and $t \in T$, and suppose further that the inverse image under f of any G-orbit in T consists of only finitely many G-orbits in S. Then $f^*: D^*_G(T) \to D^*_g(S)$ is defined in dimensions $\leq n$ and there is a commutative diagram*

$$\begin{array}{ccc} D^*_G(T) & \xrightarrow[\Phi]{\approx} & \bigoplus_{t \in T_0} H^*(G_t, \mathbb{Z}G) \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow \\ D^*_G(S) & \xrightarrow[\Phi]{\approx} & \bigoplus_{s \in S_0} H^*(G_s, \mathbb{Z}G), \end{array}$$

where the unlabelled vertical map is given by $(u_t)_{t\in T_0}\mapsto(v_s)_{s\in S_0}$ *with* $v_s=\gamma_s^*(u_{f_0(s)})$. *All maps in this diagram are maps of right* G*-modules.* ||

§2. The Mayer–Vietoris sequence with $\mathbb{Z}G$ coefficients

Let G be the fundamental group of a finite graph of groups of type FP_n, and let X be the associated tree [9]. Recall that X is oriented and comes with an orientation-preserving left G-action. For any vertex or edge σ of X we denote by G_σ the isotropy subgroup of G at σ. By hypothesis, then, each G_σ is of type FP_n.

Let X_0 be the set of vertices of X and let X_1 be the set of positively oriented edges of X. Let Y_p $(p=0,1)$ be a set of representatives for X_p mod G. Then the Mayer–Vietoris sequence for computing $H^*(G,\mathbb{Z}G)$ has the form

$$\cdots\to H^q(G,\mathbb{Z}G)\to\prod_{v\in Y_0}H^q(G_v,\mathbb{Z}G)\xrightarrow{\alpha}\prod_{e\in Y_1}H^q(G_e,\mathbb{Z}G)\to\cdots$$

([6]; see also [2], §VII.9). It is a sequence of right G-modules. Note that the direct products here are in fact direct sums since Y_0 and Y_1 are finite. We wish to use 1.6 to interpret the map α.

Recall that α can be described as follows (cf. [2], §§VII.8 and VII.9). For any $e\in Y_1$ let $o(e)$ (resp. $t(e)$) be the origin (resp. terminal vertex) of e, as in [9]. Let $v_0(e)$ (resp. $v_1(e)$) be the element of Y_0 equivalent to $o(e)$ (resp. $t(e)$) mod G. Choose $g_i(e)\in G$ $(i=0,1)$ such that $o(e)=g_0(e)v_0(e)$ and $t(e)=g_1(e)v_1(e)$. The elements $g_0(e)$ induce maps $(c_{g_0(e)^{-1}},\lambda_{g_0(e)}):(G_e,\mathbb{Z}G)\to(G_{v_0(e)},\mathbb{Z}G)$ as before, which in turn induce a map $\beta:\bigoplus_{v\in Y_0}H^*(G_v,\mathbb{Z}G)\to\bigoplus_{e\in Y_1}H^*(G_e,\mathbb{Z}G)$. Similarly, $g_1(-)$ and $v_1(-)$ yield a map $\delta:\bigoplus_{v\in Y_0}H^*(G_v,\mathbb{Z}G)\to\bigoplus_{e\in Y_1}H^*(G_e,\mathbb{Z}G)$, and the map α that we are interested in is $\delta-\beta$.

The hypotheses of 1.6 are satisfied, so we have:

PROPOSITION 2.1. *The functions* $o,t:X_1\to X_0$ *induce maps* $o^*,t^*:D_G^*(X_0)\to D_G^*(X_1)$ *in dimensions* $\le n$. *The map* α *in the Mayer–Vietoris sequence is isomorphic to* t^*-o^* *in dimensions* $\le n$. ||

This result can be conveniently rephrased in terms of cohomology of X with compact supports. For each integer q we have a "coefficient system" $\mathcal{D}^q$ on X which associates to each vertex or edge σ the group $D^q(G_\sigma)$ and to each incidence relation "v is a vertex of e" the map $D^q(G_v)\to D^q(G_e)$ induced by the inclusion $G_e\to G_v$. We can therefore form, in the usual way, the cochain complex

$C^*(X, \mathcal{D}^q)$, with $C^p(X, \mathcal{D}^q) = \prod_{\sigma \in X_p} D^q(G_\sigma)$ $(p = 0, 1)$. Let $C^p_c(X, \mathcal{D}^q) = \bigoplus_{\sigma \in X_p} D^q(G_\sigma) = D^q_G(X_p)$. We say that $\mathcal{D}^q$ is *locally finite* if for each vertex v and each $d \in D^q(G_v)$ the image of d in $D^q(G_e)$ is zero for almost all edges e of which v is a vertex. In this case $C^*_c(X, \mathcal{D}^q)$ is a subcomplex of $C^*(X, \mathcal{D}^q)$, and we denote by $H^*_c(X, \mathcal{D}^q)$ the resulting cohomology groups. A restatement of 2.1, then, is:

THEOREM 2.2. *Let G be the fundamental group of a finite graph of groups of type FP_n. Then $\mathcal{D}^q$ is locally finite for $q \leq n$, and the map α in the Mayer–Vietoris sequence, above, is isomorphic to the coboundary map $C^0_c(X, \mathcal{D}^q) \to C^1_c(X, \mathcal{D}^q)$. Consequently,* $\ker \alpha \approx H^0_c(X, \mathcal{D}^q)$, $\operatorname{coker} \alpha \approx H^1_c(X, \mathcal{D}^q)$, *and the Mayer–Vietoris sequence yields a short exact sequence of G-modules*

$$0 \to H^1_c(X, \mathcal{D}^{q-1}) \to H^q(G, \mathbb{Z}G) \to H^0_c(X, \mathcal{D}^q) \to 0$$

for $q \leq n$, where the (right) G-module structure on $H^*_c(X, \mathcal{D}^q)$ is induced by the conjugation isomorphisms $D^*(c_g): D^*(G_{g\sigma}) \to D^*(G_\sigma)$. ||

Remark 2.3. The free coefficient module $\mathbb{Z}G$ can be replaced by an induced module $\mathbb{Z}G \otimes A$ in this and the previous section, and everything goes through without essential change. Dually, one can prove analogous results relating homology of G with coefficients in a coinduced module $\operatorname{Hom}(\mathbb{Z}G, A)$ to homology of X based on infinite chains.

Remark 2.4. With a little more effort, one can generalize the results of this section to the case of a G-CW-complex X in the sense of [2]. The analogue of the Mayer–Vietoris sequence above is the equivariant cohomology spectral sequence converging to $H^*_G(X, \mathbb{Z}G)$ [$= H^*(G, \mathbb{Z}G)$ if X is contractible], with E_1-term involving the groups $H^*(G_\sigma, \mathbb{Z}G)$ as σ ranges over the cells of X. If X is finite $\operatorname{mod} G$ and each isotropy group is of type FP_n, then there is an analogue of Theorem 2.2 which expresses the E_1-term in total degrees $\leq n$ as the cochain complex of X with compact supports and coefficients in systems $\mathcal{D}^q = \{D^q(G_\sigma)\}$. We have only treated the case where X is a tree, however, since the resulting low-dimensional cohomology groups $H^p_c(X, \mathcal{D}^q)$ $(p = 0, 1)$ are often easy to compute, and one obtains thereby concrete applications. We will illustrate this in the next section.

§3. Examples

We continue to denote by G the fundamental group of a finite graph of groups of type FP_n and by X the associated tree. To avoid trivialities, we will assume that

X is infinite. [If X is finite, then G is an amalgamated free product of a finite tree of groups, and the construction is trivial in the sense that one of the vertex groups is equal to the whole group G.] As our first illustration of Theorem 2.2 we will generalize results of Bieri ([1], Theorems 6.3 and 6.6 and Proposition 9.16(b)) on the cohomology of amalgamations and HNN extensions. In the amalgamation case Bieri required the amalgamated subgroup to be of finite index in both free factors, and in the *HNN* case he required both associated subgroups to be of finite index in the base group. Both of these cases are included in the following:

THEOREM 3.1. *Suppose for each edge e of X and each vertex v of e that* $(C_v : G_e) < \infty$.

(i) *The groups* $D^q(G_\sigma)$, *where* σ *ranges over the vertices and edges of X, are all canonically isomorphic to the group* $D^q = H^0(X, \mathcal{D}^q)$. *The latter admits a right G-module structure which, for all* σ, *is consistent with the usual action of* G_σ *on* $D^q(G_\sigma)$.

(ii) *The map* α *in the Mayer–Vietoris sequence for G with* $\mathbb{Z}G$*-coefficients is a* $\mathbb{Z}$*-split monomorphism for* $q \leq n$, *with cokernel G-isomorphic to* $H^1_c(X, \mathbb{Z}) \otimes D^q$ *(with the diagonal G-action).*

(iii) $H^q(G, \mathbb{Z}G) \approx H^1_c(X, \mathbb{Z}) \otimes D^{q-1}$ *for* $q \leq n$; *hence* $H^q(G, \mathbb{Z}G)$ *is* $\mathbb{Z}$*-isomorphic (non-canonically) to a direct sum of* $E-1$ *copies of* D^{q-1}, *where E is the number of ends of X (necessarily, E is 2 or* ∞*).*

(iv) *If the vertex and edge groups are duality groups then so is G. The vertex and edge groups all have the same dimension m and all have the G-module* $D = D^m$ *as dualizing module; G has dimension* $m+1$ *and dualizing module* $H^1_c(X, \mathbb{Z}) \otimes D$.

(Note that the FP_n hypothesis is irrelevant for (iv) since duality groups are known to be of type FP_∞ [3].)

Proof. Example 1.1 shows that the map $D^*(G_v) \to D^*(G_e)$ is an isomorphism whenever v is a vertex of e. The first assertion of (i) follows at once.

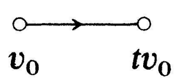
The G-action on $D^q = H^0(X, \mathcal{D}^q)$ required for the second assertion is induced as in 2.2 by the conjugation isomorphisms $D^*(c_g): D^*(G_{g\sigma}) \to D^*(G_\sigma)$; it is consistent with the usual action of G_σ on $D^q(G_\sigma)$ by Example 1.2. To prove (ii) and (iii), note that $C^*_c(X, \mathcal{D}^q) \approx C^*_c(X, \mathbb{Z}) \otimes D^q$. Since $H^0_c(X, \mathbb{Z}) = 0$ and $H^1_c(X, \mathbb{Z})$ is free abelian of rank $E-1$, (ii) and (iii) follow easily from Theorem 2.2. Turning now to (iv), recall (cf. [1] or [2]) that a group H is a duality group if and only if (a) H is of type FP_∞ and of finite cohomological dimension; and (b) there is a unique integer m such that $D^m(H) \neq 0$, and this $D^m(H)$ is $\mathbb{Z}$-torsion-free. The integer m in (b) is then the dimension of H, and $D^m(H)$ is the dualizing module of H. Suppose now that the G_σ all satisfy (a) and (b). It is then well-known that G

satisfies (a) (see, for instance, [1], proof of Propositions 2.13 and 6.1). Next note that the G_σ all have the same dimension m and the same dualizing module $D = D^m$ by (i). And (iii) shows that $D^q(G) = 0$ for $q \neq m+1$ and that $D^{m+1}(G) \approx H^1_c(X, \mathbb{Z}) \otimes D$, which is $\mathbb{Z}$-torsion-free. Thus G satisfies (b), whence the first assertion of (iv); the rest of (iv) has been proved along the way. ||

Next we wish to concentrate on the *HNN* case but, as promised in the introduction, drop the finite index hypothesis. Let G be an *HNN* extension $H *_A$ with respect to $\tau : A \xrightarrow{\approx} B$, where A and B are subgroups of H; thus G is obtained from H by adjoining a new generator t and relations $t^{-1}at = \tau(a)$ for all $a \in A$.

THEOREM 3.2. *Suppose that H and A are of type FP_n and that the restriction map $H^q(H, \mathbb{Z}H) \to H^q(A, \mathbb{Z}H)$ is a monomorphism for some $q \leq n$. Then $H^0_c(X, \mathcal{D}^q) = 0$. Consequently, the map $\alpha : H^q(H, \mathbb{Z}G) \to H^q(A, \mathbb{Z}G)$ in the Mayer–Vietoris sequence is a monomorphism, and there is an isomorphism $H^q(G, \mathbb{Z}G) \approx H^1_c(X, \mathcal{D}^{q-1})$.*

Proof. Recall that the tree X in this case has a "fundamental" edge e_0 of the form



with the following properties: (a) every positively oriented edge of X is equivalent mod G to e_0; (b) every vertex of X is equivalent mod G to v_0; and (c) the isotropy subgroups of G at v_0 and e_0 are given by $G_{v_0} = H$ and $G_{e_0} = A$. These properties imply: (d) the positively oriented edges of X starting at v_0 are given by $(ge_0)_{g \in H/A}$ and the positively oriented edges of X ending at v_0 are given by $(gt^{-1}e_0)_{g \in H/B}$.

In view of Corollary 1.5, the restriction map $H^q(H, \mathbb{Z}H) \to H^q(A, \mathbb{Z}H)$ for $q \leq n$ can now be identified with the map $D^q(G_{v_0}) \to \bigoplus_e D^q(G_e)$ whose components are induced by the inclusions $G_e \to G_{v_0}$, where e ranges over the positively oriented edges starting at v_0. Our hypothesis that this map is a monomorphism can therefore be restated as the following property of the vertex v_0: For every non-zero $d \in D^q(G_{v_0})$ there is a positively oriented edge e starting at v_0 such that the image of d in $D^q(G_e)$ is non-zero. Since every vertex of X is equivalent to v_0 mod G, it follows that *every* vertex has this same property.

It is now immediate that $H^0_c(X, \mathcal{D}^q) = 0$. Indeed, an element of $H^0_c(X, \mathcal{D}^q)$ is a compatible family $d = (d_v)_{v \in X_0}$, where $d_v \in D^q(G_v)$ and $d_v = 0$ for almost all v. ["Compatible" means that if v and w are the vertices of an edge e then d_v and d_w have the same image in $D^q(G_e)$.] Suppose there is a non-zero such family, and let v be a vertex such that $d_v \neq 0$. By the previous paragraph, we can then find a positively oriented edge starting at v such that d is also non-zero at the terminal vertex of the edge. Repeating this argument, we obtain an edge path of positively oriented edges

$$\circ\!\longrightarrow\!\circ\!\longrightarrow\!\circ\!\longrightarrow \cdots,$$

with $d \neq 0$ at every vertex. Since X is a tree, this path contains infinitely many distinct vertices, contradicting the fact that $d_v = 0$ almost everywhere. ||

Finally, we specialize still further to the case where $(H:A) < \infty$; B, however, is allowed to be arbitrary.

THEOREM 3.3. *Let G be an HNN extension as above with H and A of type FP_n and $(H:A) < \infty$. Then* $\operatorname{res}: H^*(H, \mathbb{Z}H) \to H^*(A, \mathbb{Z}H)$ *is a monomorphism. Hence* 3.2 *applies and $H^q(G, \mathbb{Z}G) \approx H_c^1(X, \mathcal{D}^{q-1})$ for $q \leq n$. Moreover:*

(i) *If $D^{q-1}(H)$ is $\mathbb{Z}$-torsion-free for some $q \leq n$ then $D^q(G)$ is $\mathbb{Z}$-torsion-free.*

(ii) *If $(H:A) > 1$ and $D^{q-1}(H)$ is $\mathbb{Z}$-free for some $q \leq n$, then $D^q(G)$ is $\mathbb{Z}$-free.*

(iii) *If H is an m-dimensional duality group, then G is an $(m+1)$-dimensional duality group.*

Proof. Example 1.1 shows that $D^*(G_v) \xrightarrow{\approx} D^*(G_e)$ for every vertex v of X and every positively oriented edge e starting at v. Arguing as in the proof of Theorem 3.2, one shows that res is a monomorphism. Now fix $q \leq n$ and set $\mathcal{D} = \mathcal{D}^{q-1}$ and $D_\sigma = D^{q-1}(G_\sigma)$. (i)–(iii) will be based on the following computation:

LEMMA 3.4. *If $(H:A) > 1$ then there is a subset S of X_1 such that $H_c^1(X, \mathcal{D})$ is $\mathbb{Z}$-isomorphic to $\bigoplus_{e \in S} D_e$. If $(H:A) = 1$ then there are subsets W_k of X_0 $(k \geq 0)$ such that $H_c^1(X, \mathcal{D})$ is $\mathbb{Z}$-isomorphic to the direct limit of a system of the form*

$$\cdots \to \bigoplus_{v \in W_k} D_v \to \bigoplus_{v \in W_{k+1}} D_v \to \cdots.$$

(i) and (ii) of the theorem follow immediately from the lemma; and (iii) then follows from (i) and what was proved earlier, via the usual criterion for duality (cf. proof of 3.1). It remains to prove the lemma.

The definition of $H_c^1(X, \mathcal{D})$ shows that the latter is the abelian group generated by the groups D_e $(e \in X_1)$, subject to relations of the following form for each $v \in X_0$: for each $d \in D_v$, $\sum_e \pm \rho_{v,e}(d) = 0$, where e ranges over the elements of X_1 having v as a vertex and $\rho_{v,e}: D_v \to D_e$ is induced by the inclusion $G_e \to G_v$. [The sign above depends on whether v is the initial vertex or the terminal vertex of e.] The set of relations of this form associated to a given v will be denoted R_v. Recall that $\rho_{v,e}$ is an isomorphism if v is the initial vertex of e. Hence if we choose for a given v one $e \in X_1$ starting at v, then R_v can be viewed as expressing the elements of this D_e in terms of the others.

Now let V_k be the set of vertices of X whose distance from our fundamental vertex v_0 is k. Let M_k be the abelian group generated by those D_e whose vertices are in $\bigcup_{i \le k} V_i$, subject to the relations R_v for $v \in \bigcup_{i \le k-1} V_i$. Then we have a direct system

$$\cdots \to M_k \to M_{k+1} \to \cdots,$$

and $H_c^1(X, \mathcal{D})$ is the direct limit. [Note: M_k is in fact $H_c^1(X, X^k; \mathcal{D})$, where X^k is the full subgraph of X with vertex set $\bigcup_{i \ge k} V_i$.] For any $w \in V_{k+1}$ let $e(w)$ be the edge in X_1 connecting w to V_k. Then the passage from M_k to M_{k+1} consists of adjoining new generating groups $D_{e(w)}$ $(w \in V_{k+1})$ and new relations R_v $(v \in V_k)$.

Recall that each vertex of X is the initial vertex of precisely $(H:A)$ edges in X_1. For $v \in V_k$, all except possibly one of these terminates in V_{k+1}. [The exception is $e(v)$, if the latter starts at v.] So if $(H:A) > 1$ there must be at least one w in V_{k+1} such that $e(w)$ is oriented from v to w. In this case, then, R_v can be used to eliminate the generating group $D_{e(w)}$, and it follows that $M_{k+1} = M_k \oplus (\bigoplus_{e \in E_k} D_e)$ for some set E_k of edges joining V_k and V_{k+1}. Passing to the limit, we find $H_c^1(X, \mathcal{D}) = \bigoplus_{e \in S} D_e$, where $S = \bigcup_k E_k$.

Suppose now that $(H:A) = 1$, so that every vertex v is the initial vertex of exactly one edge in X_1. Then a reduced path in X necessarily consists of zero or more positively oriented edges followed by zero or more negatively oriented edges. In particular, the path from v_0 to a vertex $w \in V_{k+1}$ must end with a negatively oriented edge unless $w = t^{k+1}v_0$; hence $e(w)$ starts at w unless $w = t^{k+1}v_0$, in which case $e(w)$ is the edge $t^k e_0$ from $t^k v_0$ to $t^{k+1}v_0$. For $v = t^k v_0$, then, R_v can be used as above to eliminate the generating group $D_{e(w)}$, where $w = t^{k+1}v_0$. For $v \in W_k \equiv V_k - \{t^k v_0\}$, on the other hand, R_v expresses the elements of $\operatorname{Im}\{D_{e(v)} \to M_{k+1}\}$ in terms of the $D_{e(w)}$ for $w \in W_{k+1}$. If we now assume inductively that $M_k = \bigoplus_{v \in W_k} D_{e(v)}$, it follows easily that $M_{k+1} = \bigoplus_{w \in W_{k+1}} D_{e(w)}$. Since $D_{e(v)} \approx D_v$ for $v \in W_k$, this completes the proof. ||

Remark 3.5. It follows from the proof that the map $M_k \to M_{k+1}$, in case $H = A$, is equivalent to the direct sum of card(W_k) copies of the restriction map $H^{q-1}(H, \mathbb{Z}H) \to H^{q-1}(B, \mathbb{Z}H)$.

Remark 3.6. It is not clear to us whether Theorem 3.3(ii) can be improved to include the case $H = A$. In case $n = q = 2$ this can be done, provided the hypothesis that H and A be of type FP_2 is strengthened to "finitely presented." Then, combining results in [7] and [8], one gets $D^2(G)$ $\mathbb{Z}$-free when $H = A$.

Remark 3.7. As we write, there is no known example of a group G of type FP_n and an integer $q \le n$ for which $D^q(G)$ is not $\mathbb{Z}$-free. For the topological meaning of this problem, see [7].

Appendix: A proof of Theorem 0.1 using normal forms

The map $H^q(H, \mathbb{Z}G) \to H^q(A, \mathbb{Z}G)$ is denoted by α in §§1–3. Here it is convenient to denote it by α_*, reserving the letter α for a cochain map which induces it.

We recall the definition of a normal form in G. Let $\{Au \mid u \in U\}$ and $\{Bv \mid v \in V\}$ be the right cosets of A and B in H, where U and V are sets of coset representatives, both containing 1. Let t be the stable letter in $H *_A$ (with respect to $\tau: A \xrightarrow{\approx} B$). A *normal form* is a product $ht^{\varepsilon_1}w_1 \cdots t^{\varepsilon_n}w_n$ where (i) $h \in H$ and $\varepsilon_i = \pm 1$, (ii) if $\varepsilon_i = -1$, $w_i \in U$, (iii) if $\varepsilon_i = 1$, $w_i \in V$, and (iv) $t^{\varepsilon}1t^{-\varepsilon}$ does not occur. h is called the *initial element*. The normal form is *special* if its initial element is 1. The *length* of the above normal form is n. Each element of G can be written as a normal form uniquely. If $g_1, g_2 \in G$, the product g_1g_2 is *reduced* if its normal form is the product of the separate normal forms of g_1 and g_2.

We need the following commutative diagram for $q \le n$

$$\begin{array}{ccc} H^q(H, \mathbb{Z}H) \otimes_H \mathbb{Z}G & \xrightarrow[\approx]{\mu_*} & H^q(H, \mathbb{Z}G) \\ \downarrow{\scriptstyle \bar{\alpha}} & & \downarrow{\scriptstyle \alpha_*} \\ H^q(A, \mathbb{Z}H) \otimes_H \mathbb{Z}G & \xrightarrow{\nu_*} & H^q(A, \mathbb{Z}G). \end{array}$$

To set it up, we start with a free $\mathbb{Z}G$-resolution P_* of $\mathbb{Z}$, and define the morphisms in the following diagram

$$\begin{array}{ccc} \mathrm{Hom}_H(P_q, \mathbb{Z}H) \otimes_H \mathbb{Z}G & \xrightarrow{\mu} & \mathrm{Hom}_H(P_q, \mathbb{Z}G) \\ & & \downarrow \\ \mathrm{Hom}_A(P_q, \mathbb{Z}H) \otimes_H \mathbb{Z}G & \xrightarrow{\nu} & \mathrm{Hom}_A(P_q, \mathbb{Z}G) \end{array}$$

by: $\mu(f \otimes x) = \rho_x f$, $\nu(f \otimes x) = \rho_x f$ and $\alpha(f) = f - \lambda_t f \lambda_{t^{-1}}$. Here ρ_z and λ_z stand for right and left multiplication by z. It is well known (see [1]) that μ_* and ν_* are isomorphisms when $q \le n$, and that α_* (in the Mayer–Vietoris sequence) is induced by α. Define $\bar{\alpha}$ to get commutativity.

LEMMA A1. *Let $c \otimes x \in H^q(H, \mathbb{Z}H) \otimes_H \mathbb{Z}G$, where x is a special normal form.*

Then $\bar{\alpha}(c\otimes x)$ *has the form*

$$\operatorname{res}(c)\otimes x-\sum_i c_i\otimes tv_ix \quad (v_i\in V).$$

LEMMA A.2. *Let M be a right H-module. Suppose*

$$\sum_{x\in X} m_x\otimes x=\sum_{x\in X}\sum_{v\in Y_x} m_{x,v}\otimes tvx$$

in $M\otimes_H \mathbb{Z}G$, *where X is a non-empty finite set of special normal forms in G, and* Y_x *is a finite subset of V for each* $x\in X$. *Suppose every* $m_{x,v}\neq 0$. *Then some* $m_x=0$.

Theorem 0.1 follows easily from Lemmas A1 and A2. Suppose $0\neq\sum_{x\in X}c_x\otimes x \in\ker\bar{\alpha}$, where X is a finite non-empty set of normal forms, and each $c_x\neq 0\in H^q(H,\mathbb{Z}H)$. We may assume each x is special. By Lemma A1,

$$\sum_{x\in X}\operatorname{res}(c_x)\otimes x=\sum_{x\in X}\sum_{v\in Y_x} m_{x,v}\otimes tvx$$

as in the hypotheses of Lemma A2, which therefore implies $\operatorname{res}(c_x)=0$ for some x. This contradicts the fact that res is a monomorphism. It only remains to prove A1 and A2.

*Proof of Lemma A*1. Let S be the set of special normal forms; then $G=\bigcup\{Hs\mid s\in S\}$. $H^q(A,\mathbb{Z}G)=H^q(A,\bigoplus_S\mathbb{Z}H)$ can be canonically identified with $\bigoplus_S H^q(A,\mathbb{Z}H)$, since A is FP_n and $q\leq n$. If x' is such and $d\in H^q(A,\mathbb{Z}H)$, then the element of $\bigoplus_S H^q(A,\mathbb{Z}H)$ whose only non-zero entry is d in the Hx' position is mapped by ν_*^{-1} to $d\otimes x'$. Now, $tH=\bigcup_{v\in V}Atv\subset\bigcup_{v\in V}Htv$, so any element of $\mathbb{Z}tH$ lies in $\bigoplus_{v\in V}\mathbb{Z}Htv$. In the light of these remarks, if $\nu_*^{-1}\circ\alpha_*\circ\mu_*$ is applied to $c\otimes x$, one clearly obtains an element of the form stated. ||

*Proof of Lemma A*2. We will repeatedly use the cancellation principle that if $m_0\otimes g_0=\sum_{i=1}^s m_i\otimes g_i$ in $M\otimes_H\mathbb{Z}G$ with $m_0\neq 0$, then $g_0\in Hg_i$ for some i. In particular this results out card $X=1$ in the hypothesis of the lemma.

Suppose the Lemma is false. Pick a counter example for which card X is minimal (necessarily ≥ 2). Let $\bar{x}\in X$ be of maximal length. Let $X'=X\setminus\{x\}$.

$$m_{\bar{x}}\otimes\bar{x}+\sum_{x\in X'}m_x\otimes x=\sum_{x\in X'}\sum_{v\in Y_x}m_{x,v}\otimes tvx+\sum_{v\in Y_{\bar{x}}}m_{\bar{x},v}\otimes tv\bar{x}.$$

$m_{\bar{x}}\otimes\bar{x}$ must cancel with some $m_{x',v'}\otimes tv'x'$ where $x'\in X'$, $v'\in Y_{x'}$. $\bar{x}=h'tv'x'$

$(h' \in H)$. The latter is reduced, by maximal length. Thus $\bar{x} = tv'x'$, reduced. We claim $Y_{\bar{x}}$ is empty. Suppose there exists $\bar{v} \in Y_{\bar{x}}$. Applying the cancellation principle to the above equation we get

$$t\bar{v}\bar{x} = h''tv''x'' \quad (h'' \in H, x'' \in X', v'' \in Y_{x''}),$$

for the other possibility, $t\bar{v}\bar{x} = h''x''$, is ruled out by maximal length since $t\bar{v}\bar{x}$ $(= t\bar{v}tv'x')$ is reduced. If $h''tv''x''$ is reduced then $\bar{x} = x''$, a contradiction. If $h''tv''x''$ is not reduced then length$(x'')>$length$(h''tv''x'')\equiv$length$(t\bar{v}\bar{x})=$length$(\bar{x})+1$, a contradiction. The Claim is proved. Thus if we let $Y'_{x'} = Y_{x'} \backslash \{v'\}$, and $Y'_x = Y_x$ when $x \neq x'$, we get

$$\sum_{x \in X'} m_x \otimes x = \sum_{x \in X'} \sum_{v \in Y'_x} m_{x,v} \otimes tvx$$

and all $m_x \neq 0$. $X' \neq \phi$ and card $X' <$ card X, a contradiction. ||

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