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Parametrization of Möbius groups acting in a disk

MIKA SEPPÄLÄ and TUOMAS SORVALI

Introduction

We consider groups G generated by hyperbolic Möbius transformations acting in the upper half-plane H or in the unit disc D. It is an interesting problem to find invariants which determine G up to conjugation. For properly discontinuous groups G this is the same thing as parametrizing the Teichmüller space of the Riemann surface H/G. In our considerations the discontinuity of G plays no role. Consequently the results hold for rather general groups G.

The proofs are elementary. We use multipliers of transformations of G to parametrize G. In the considerations we suppose that G is finitely generated even though the methods and computations could be easily generalized for infinitely generated groups.

To formulate the main result suppose that the hyperbolic transformations $g_1, h_1, \ldots, g_s, h_s$ generate G and satisfy certain technical conditions described in Theorem 2.2. Assume that the commutator $c_s = h_s \circ g_s^{-1} \circ h_s^{-1} \circ g_s$ is hyperbolic and suppose that a relation is given such that c_s has a known representation in the group generated by $g_1, h_1, \ldots, g_{s-1}, h_{s-1}$. Then the multipliers of the following $6s-4$ transformations

g_j, h_j and $g_j \circ h_1$ for $j = 1, \ldots, s$,

$h_j \circ h_1$ for $j = 2, \ldots, s$,

$g_j \circ g_1$ and $h_j \circ g_1$ for $j = 2, \ldots, s-1$ and

$g_s \circ h_s$

determine G up to conjugation (Corollary to Theorem 2.2). Further, the multipliers of $g_s \circ h_s$ and $g_s \circ h_1$ can both have only two different values for fixed values of the other multipliers.

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Note that we allow here, in fact, conjugation by Möbius-transformations mapping the upper half-plane (or the unit disk) onto its complement. Then the conjugacy class of G becomes uniquely determined by the multipliers of the above transformations.

For discontinuous groups G this result can be used to parametrize the Teichmüller space T_S of $S = H/G$ globally by the multipliers of the above $6s-4$ transformations. This is the same thing as parametrizing T_S by the lengths of $6s-4$ closed geodesics. Irwin Kra has shown that such a parametrization is not even locally possible by $6s-6$ fixed curves only. Corollary to Theorem 2.2 gives a set of $6s-4$ closed curves whose lengths parametrize the Teichmüller space globally. Together with Kra's result this implies that the minimal number of curves parametrizing the Teichmüller space is either $6s-4$ or $6s-5$.

Deleting the transformations $g_s \circ h_1$ and $g_s \circ h_s$ from the above list we obtain a set of $6s-6$ Möbius transformations whose multipliers parametrize an open set of the Teichmüller space. A remarkable fact is that such a parametrization can be obtained by elementary computations concerning the group G and that the discontinuity of G or the geometry of H/G plays no role here.

We start with showing how certain basic results about the geometry of a Riemann surface H/G are actually implied by elementary computations concerning hyperbolic Möbius transformations with intersecting axes. The multipliers of such transformations with a hyperbolic commutator satisfy an inequality (1.6) that, in the case of a Fuchsian group G, implies an inequality between the lengths of closed intersecting geodesics. The considerations here are of the same nature as those in [A], II.3.3, but the result is somewhat stronger.

The parametrization of G is achieved by considering a suitably normalized set of generators of G. That normalization is explained in Section 2 (e.g. Fig. 3). It allows us to avoid huge technical difficulties. For Fuchsian groups a usual standard set of generators satisfies the conditions of the normalization.

Using this set of generators we first parametrize certain free subgroups of G. This is done in detail in Theorem 2.1. These considerations should be contrasted with those in [S–S] where a similar parametrization was obtained for the Teichmüller space of a Riemann surface H/G. There we applied rather strong results of the geometry of H/G. The present paper reflects, in our opinion, the nature of these things more truly. Many of the results which, a priori, look rather deep are actually simple consequences of elementary computations.

Similar questions have been considered by many authors. We would especially like to mention the work of H. Helling [H]. His methods are closely related to those of this paper and his results were a starting point of this investigation.

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1. A collar inequality

Let g be a hyperbolic Möbius transformation. Denote by $a(g)$ and $r(g)$ its *attracting* and *repelling* fixed points, respectively. Then its *multiplier* $k(g)$ is given by the cross-ratio

$$k(g) = (g(z), z, r(g), a(g))$$

for any $z \in \hat{\mathbb{C}}$ not fixed by g.

For the values $k > 0$ let

$$f(k) = \sqrt{k} + 1/\sqrt{k}$$

and denote

$$f(g) = f(k(g)).$$

Then $f(g) > 2$. For a parabolic g set $k(g) = 1$ and $f(g) = 2$.

If

$$z \to \frac{az+b}{cz+d}, \; ad - bc = 1,$$

is any transformation conjugate to g, then

$$f(g) = |a + d|.$$

Let (g, h) be a pair of hyperbolic transformations of the unit disc D onto itself. Suppose that g and h have no common fixed points. Denote

$$k_1 = k(g)$$
$$k_2 = k(h)$$
$$t = (r(g), r(h), a(h), a(g)).$$

Then a calculation yields

$$f(g \circ h) = |tf(k_1 k_2) + (1 - t)f(k_1/k_2)|.$$

Divide the pairs (g, h) in the classes $\mathcal{H}$, $\mathcal{P}$ and $\mathcal{E}$ as follows:

$$(g, h) \in \mathcal{H} \Leftrightarrow f(g \circ h) = tf(k_1 k_2) + (1-t)f(k_1/k_2) \geq 2$$
$$\Leftrightarrow t \geq t_1 = (2 - f(k_1/k_2))/(f(k_1k_2) - f(k_1/k_2)) \tag{1.1}$$
$$(g, h) \in \mathcal{P} \Leftrightarrow f(g \circ h) = -tf(k_1 k_2) - (1-t)f(k_1/k_2) \geq 2$$
$$\Leftrightarrow t \leq t_2 = (-2 - f(k_1/k_2))/(f(k_1k_2) - f(k_1/k_2)) \tag{1.2}$$
$$(g, h) \in \mathcal{E} \Leftrightarrow t_2 < t < t_1. \tag{1.3}$$

Suppose that k_1 and k_2 are kept fixed but t is let to vary. If t increases from $-\infty$ to t_2, then $(g, h) \in \mathcal{P}$ and $f(g \circ h)$ decreases from ∞ to 2. If $t_2 < t < t_1$ then $(g, h) \in \mathcal{E}$ and $g \circ h$ is elliptic. If t increases from t_1 to ∞, then $(g, h) \in \mathcal{H}$ and $f(g \circ h)$ increases from 2 to ∞.

If we conjugate by a Möbius transformation sending $r(h) \to 1$, $a(h) \to 0$, $a(g) \to \infty$ and $r(g) \to t$, we get Fig. 1. Here the open interval $]t_2, t_1[$ separates the classes $\mathcal{P}$ and $\mathcal{H}$. Note that the pairs (g, h) and (h, g) belong to the same class.

We state two theorems (cf. [S–S], §1).

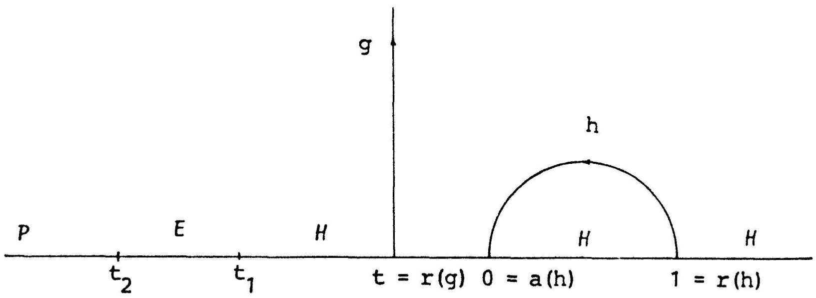

Figure 1.

THEOREM 1.1. *For any $k_1 > 1$, $k_2 > 1$ and $k_3 > 1$ there exist up to conjugation unique pairs $(g, h) \in \mathcal{H}$ and $(g, h) \in \mathcal{P}$ such that $k_1 = k(g)$, $k_2 = k(h)$ and $k_3 = k(g \circ h)$.* □

THEOREM 1.2. *Deform $a(g)$, $r(g)$, $a(h)$, $r(h)$, $k(g)$ and $k(h)$ continuously. If g, h and $g \circ h$ stay hyperbolic during the deformation, then the class of (g, h) is not changed.* □

Theorem 1.2 shows that, in later applications, the class of any pair (g, h) can be kept fixed.

Theorem 1.1 shows that the numbers $k_1 > 1$, $k_2 > 1$ and $k_3 > 1$ can be given freely in the classes $\mathcal{P}$ and $\mathcal{H}$. We consider next the class $\mathcal{H}$, impose some natural restrictions on (g, h) and show that k_1 and k_2 cannot then be simultaneously ≈ 1.

Consider the commutator

$$c = h \circ g^{-1} \circ h^{-1} \circ g.$$

A straightforward calculation (cf. [L], p. 100) yields

$$f(c) = |t(1-t)(f(k_1^{\frac{1}{2}}) - 2)(f(k_2^{\frac{1}{2}}) - 2) - 2|.$$

Suppose that the axes of g and h intersect, i.e., $0 < t < 1$ (cf. Fig. 1). If c is hyperbolic, then either

$$f(c) = t(1-t)(f(k_1^{\frac{1}{2}}) - 2)(f(k_2^{\frac{1}{2}}) - 2) - 2 > 2 \tag{1.4}$$

or

$$f(c) = 2 - t(1-t)(f(k_1^{\frac{1}{2}}) - 2)(f(k_2^{\frac{1}{2}}) - 2) > 2$$

Since $0 < t < 1$, $t(1-t)$ is positive. Hence also $t(1-t)(f(k_1^{\frac{1}{2}}) - 2)(f(k_2^{\frac{1}{2}}) - 2) > 0$ which implies that (1.4) holds. Since $0 < t(1-t) \leqq \frac{1}{4}$ for $0 < t < 1$, it follows from (1.4) that

$$2 < f(c) \leqq \tfrac{1}{4}(f(k_1^{\frac{1}{2}}) - 2)(f(k_2^{\frac{1}{2}}) - 2) - 2$$

or

$$(f(k_1^{\frac{1}{2}}) - 2)(f(k_2^{\frac{1}{2}}) - 2) > 16$$

or

$$(\sqrt{k_1} - 1/\sqrt{k_1})(\sqrt{k_2} - 1/\sqrt{k_2}) > 4. \tag{1.5}$$

If $k_2 \to 1$ then $k_1 \to \infty$. We state our result as a theorem.

THEOREM 1.3. *Suppose that g and h are hyperbolic Möbius transformations of the unit disc D onto itself. If the axes of g and h intersect and the commutator* $h \circ g^{-1} \circ h^{-1} \circ g$ *is hyperbolic, then*

$$k(g) > 16/(k(h) - 1)^2. \tag{1.6}$$

Proof. Under the hypotheses of the theorem, (1.5) holds with $k_1 = k(g)$ and $k_2 = k(h)$. Since $0 < 1/\sqrt{k_j} < 1$, $j = 1, 2$, we get

$$\sqrt{k_1} > \frac{4}{(1/\sqrt{k_2})(k_2 - 1)} > \frac{4}{k_2 - 1}. \quad \square$$

To (1.6) analogous collar inequalities are proved e.g. in [K] and [A]. Retaining the assumptions of Theorem 1.3 we derive an equation satisfied by the multipliers

$$k_1 = k(g),$$
$$k_2 = k(h),$$
$$k_3 = k(g \circ h),$$
$$k_4 = k(h \circ g^{-1} \circ h^{-1} \circ g).$$

From (1.1) we get

$$t = \frac{f(k_3) - f(k_1/k_2)}{f(k_1k_2) - f(k_1/k_2)}.$$

Hence by (1.4)

$$\frac{f(k_4) + 2}{(f(k_1^2) - 2)(f(k_2^2) - 2)} = t(1 - t) = \frac{(f(k_3) - f(k_1/k_2))(f(k_1k_2) - f(k_3))}{(f(k_1k_2) - f(k_1/k_2))^2}. \tag{1.7}$$

With respect to $f(k_3)$, the equation (1.7) is of second degree. This proves the following theorem:

THEOREM 1.4. *Suppose that g and h are hyperbolic Möbius transformations of the unit disc D onto itself such that the axes of g and h intersect and the commutator* $c = h \circ g^{-1} \circ h^{-1} \circ g$ *is hyperbolic. If* $k_1 = k(g)$, $k_2 = k(h)$ *and* $k_4 = k(c)$ *are known, then* $k_3 = k(g \circ h)$ *has two possible values except in the case*

$$f(k_4) + 2 = \tfrac{1}{4}(f(k_1^2) - 2)(f(k_2^2) - 2) \tag{1.8}$$

when k_3 *is uniquely determined.* □

By (1.4), the case (1.8) occurs if and only if $t = \frac{1}{2}$.

Denote by k_3 and k_3' and two possible values in Theorem 1.4. If

$$f(k_3) = tf(k_1k_2) + (1 - t)f(k_1/k_2), \tag{1.9}$$

then, by (1.7),

$$f(k_3') = (1 - t)f(k_1k_2) + tf(k_1/k_2). \tag{1.10}$$

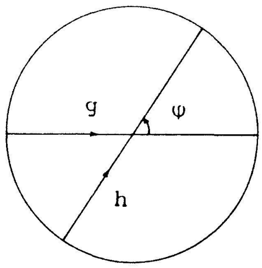

Figure 2. The angle between the axes of g and h is φ.

In the case (1.9), conjugate in such a way that $r(g) \to -1$, $a(g) \to 1$, $a(h) \to e^{i\varphi}$ and $r(h) \to -e^{i\varphi}$, $0 < \varphi < \pi$ (Fig. 2). It follows that $t = \cos^2(\varphi/2)$. If we denote by φ' the corresponding angle of (1.10), then

$$\cos^2 \frac{\varphi'}{2} = 1 - t = \sin^2 \frac{\varphi}{2}.$$

Hence $\varphi' = \pi - \varphi$. In both cases the acute angle between the axes of g and h is the same. We have proved the following corollary to Theorem 1.4:

COROLLARY. *The axes of g and h are orthogonal if and only if* (1.8) *holds. In other cases $k(g)$, $k(h)$ and $k(h \circ g^{-1} \circ h^{-1} \circ g)$ determine uniquely the acute angle between the axes of g and h.* □

Analogous results for Fuchsian groups representing compact Riemann surfaces have been applied e.g. in [S–S].

2. Parametrization of a Möbius group with a relation

Let $\mathcal{H} = \{g_1, h_1, g_2, h_2, \ldots, g_s, h_s\}$ be a set of hyperbolic Möbius transformations of D onto itself. Suppose that the cyclic order of the fixed points is given by Fig. 3 for any $\gamma \in \{g_2, h_2, \ldots, g_{s-1}, h_{s-1}\}$. Then $(g_1, h_1) \in \mathcal{H}$ and $(g_s, h_s) \in \mathcal{H}$.

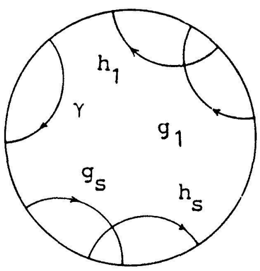

Figure 3.

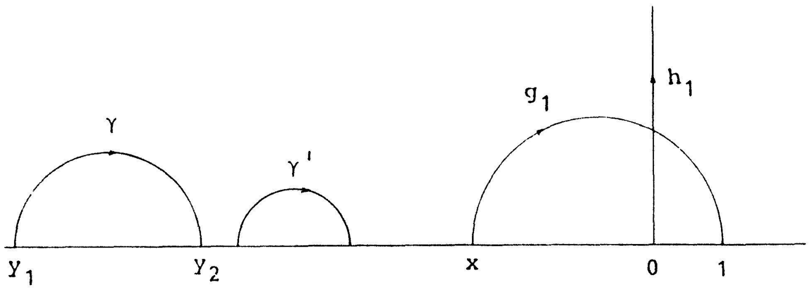

Figure 4.

Suppose the classes of (γ, g_1) and (γ, h_1) are fixed for all $\gamma \in \{g_2, h_2, \ldots, g_s, h_s\}$.

Conjugate to the upper half-plane H in such a way that $r(h_1) \to 0$, $a(h_1) \to \infty$, $a(g_1) \to 1$ and $r(g_1) \to x = 1 - 1/t < 0$ (Fig. 4).

Fix the multipliers $k_1 = k(g_1)$, $k_2 = k(h_1)$ and $k_3 = k(g_1 \circ h_1)$. Then, by Theorem 1.1, x is uniquely determined.

LEMMA 2.1. *Under the above assumptions, the numbers* $k(\gamma)$, $k(\gamma \circ g_1)$ *and* $k(\gamma \circ h_1)$ *determine* $\gamma \in \{g_2, h_2, \ldots, g_s, h_s\}$ *uniquely.*

Proof. Let γ' be another candidate for γ. By Theorem 1.1, the pairs (γ, g_1) and (γ', g_1) as well as the pairs (γ, h_1) and (γ', h_1) are conjugate. Hence we have Möbius transformations ψ and σ such that

$$\gamma' = \sigma \circ \gamma \circ \sigma^{-1},\ \sigma(0) = 0,\ \sigma(\infty) = \infty$$

$$\gamma' = \psi \circ \gamma \circ \psi^{-1},\ \psi(1) = 1,\ \psi(x) = x.$$

Moreover, ψ and σ are hyperbolic. Let y_1 and y_2 denote the fixed points of γ. Then

$$\psi(y_1) = \sigma(y_1)$$

$$\psi(y_2) = \sigma(y_2)$$

or

$$\frac{\psi(y_1)}{\psi(y_2)} = \frac{y_1}{y_2}$$

since $\sigma(z) = kz$, for $k = k(\sigma)$ or $k^{-1} = k(\sigma)$.

The next Lemma shows that $\psi = id$ and $\gamma' = \gamma$.

LEMMA 2.2. *Let* $\psi: H \to H$,

$$\psi(z) = \frac{(ak - r)z - ar(k - 1)}{(k - 1)z - kr + a},$$

$k > 0$, $k \neq 1$, *be a hyperbolic Möbius transformation with the fixed points* a *and* r *and with the multiplier* $\max(k, 1/k)$. *If there exist real numbers* y_1 y_2 *such that* $y_2\psi(y_1) = y_1\psi(y_2)$, *then*

$$k = \frac{r}{a}\frac{(y_1 - a)(y_2 - a)}{(y_1 - r)(y_2 - r)}. \quad \square \tag{2.1}$$

The proof of Lemma 2.2 is a direct calculation. To prove Lemma 2.1 assume that ψ is not the identity. Then we can apply, to this ψ, Lemma 2.2 with $a = 1$, $r = x < 0$, $y_1 < x$ and $y_2 < x$. Hence $k < 0$ which is impossible. Lemma 2.1 is hereby proved. $\square$

At this stage we have proved the following result:

THEOREM 2.1. *Let* $\mathcal{H} = \{g_1, h_1, \ldots, g_s, h_s\}$ *be a set of hyperbolic Möbius transformations of D onto itself. Suppose that the cyclic order of the fixed points is given by Fig. 3 and the classes of the pairs* (γ, g_1) *and* (γ, h_1) *are fixed for all* $\gamma \in \{g_2, h_2, \ldots, g_s, h_s\}$. *Then the multipliers of the following* $6s - 3$ *Möbius transformations determine* $\mathcal{H}$ *uniquely up to conjugation:*

$$g_j,\ h_j,\ g_j \circ h_1, \qquad j = 1, \ldots, s$$
$$h_j \circ h_1,\ g_j \circ g_1,\ h_j \circ g_1, \qquad j = 2, \ldots, s. \quad \square$$

Suppose next that the commutator

$$c_s = h_s \circ g_s^{-1} \circ h_s^{-1} \circ g_s$$

is hyperbolic and that c_s has a given representation in the group generated by the set $\{g_1, h_1, \ldots, g_{s-1}, h_{s-1}\}$. Suppose also that c_s and h_1 have no common fixed points. Let the multipliers of the following $6s - 9$ Möbius transformations be given:

$$g_j,\ h_j,\ g_j \circ h_1, \qquad j = 1, \ldots, s - 1.$$
$$h_j \circ h_1,\ g_j \circ g_1,\ h_j \circ g_1, \qquad j = 2, \ldots, s - 1.$$

Then, by Theorem 2.1, the set $\{g_1, h_1, \ldots, g_{s-1}, h_{s-1}\}$ is determined up to conjugation. Hence $k(c_s)$ is uniquely determined.

Assume that also $k(g_s)$ and $k(h_s)$ are given. Then, by Theorem 1.4, $k(g_s \circ h_s)$ has at most two possible values. Choose one of these. By Theorem 1.1,

$$t_s = (r(g_s), r(h_s), a(h_s), a(g_s))$$

is uniquely determined.

Finally, give $k(h_s \circ h_1)$. Let (g_s', h_s') be another candidate for the pair (g_s, h_s). Since both have the same cross-ratio t_s, there exists a Möbius transformation ψ such that

$$h_s' = \psi \circ h_s \circ \psi^{-1}$$
$$g_s' = \psi \circ g_s \circ \psi^{-1}.$$

Since (g_s, h_s) and (g_s', h_s') have the same commutator c_s, ψ is hyperbolic and has the same axis as c_s. Denote by a and r the common fixed points of c_s and ψ.

Denote $y_1 = r(h_s)$, $y_2 = a(h_s)$. Since $k(h_s \circ h_1) = k(h_s' \circ h_1)$, it follows as in Lemma 2.1 that $y_2\psi(y_1) = y_1\psi(y_2)$. By Lemma 2.2, we have two alternatives. Either $\psi = id$ or

$$\psi(z) = \frac{(ak - r)z - ar(k-1)}{(k-1)z - kr + a},$$
$$k = \frac{r}{a}\frac{(y_1 - a)(y_2 - a)}{(y_1 - r)(y_2 - r)}. \tag{2.2}$$

Hence we have in general two alternatives for the pair (g_s, h_s).

THEOREM 2.2. *Let $\mathcal{H} = \{g_1, h_1, \ldots, g_s, h_s\}$ be a set of hyperbolic Möbius transformations of D onto itself. Suppose that the cyclic order of the fixed points is*

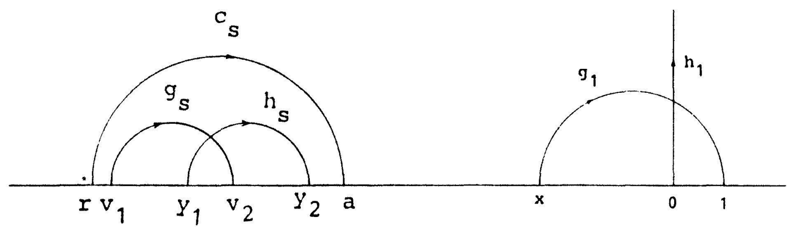

Figure 5.

given by Fig. 3 and the classes of the pairs (γ, g_1) *and* (γ, h_1) *are fixed for all* $\gamma \in \{g_2, h_2, \ldots, g_s, h_s\}$. *Suppose that the commutator* $c_s = h_s \circ g_s^{-1} \circ h_s^{-1} \circ g_s$ *is hyperbolic and has a given representation in the subgroup generated by* $\{g_1, h_1, \ldots, g_{s-1}, h_{s-1}\}$. *Suppose that* c_s *and* h_1 *have no common fixed points. Let the multipliers of the following* $6s-6$ *Möbius transformations be given:*

$$\begin{aligned}
&g_j,\ h_j, && j = 1, \ldots, s,\\
&g_j \circ h_1, && j = 1, \ldots, s-1,\\
&h_j \circ h_1, && j = 2, \ldots, s,\\
&g_j \circ g_1 \quad \text{and} \quad h_j \circ g_1, && j = 2, \ldots, s-1.
\end{aligned} \tag{2.3}$$

Then the set $\{g_1, h_1, \ldots, g_{s-1}, h_{s-1}\}$ *is uniquely determined up to conjugation. If* $\{g_1, h_1, \ldots, g_{s-1}, h_{s-1}\}$ *is fixed then* (g_s, h_s) *has at most four possible alternatives.* □

The $6s-6$ multipliers in Theorem 2.2 give in fact local parameters for the Teichmüller space of the group generated by $\mathcal{H}$. To obtain a global parametrization for the Teichmüller space, additional multipliers are needed.

We retain the assumptions of Theorem 2.2. We add to the $6s-6$ multipliers (2.3) also $k(g_s \circ h_1)$ and suppose that the cyclic order of the fixed points of c_s is given by Fig. 5. Denote $v_1 = r(g_s)$ and $v_2 = a(g_s)$. Then similarly as for h_s, $v_2\psi(v_1) = v_1\psi(v_2)$. Hence we get a second expression for the number k in (2.2):

$$k = \frac{r}{a}\frac{(v_1 - a)(v_2 - a)}{(v_1 - r)(v_2 - r)}.$$

Hence

$$\frac{v_1 - a}{v_1 - r} \cdot \frac{v_2 - a}{v_2 - r} = \frac{y_1 - a}{y_1 - r} \cdot \frac{y_2 - a}{y_2 - r}$$

or

$$(v_1, y_1, a, r) = (y_2, v_2, a, r).$$

But this is impossible, since $(v_1, y_1, a, r) > 1$ and $(y_2, v_2, a, r) < 1$. Hence $\psi = id$.

The set $\mathcal{H}$ is not uniquely determined up to conjugation by the $6s-5$ multipliers (i.e. the $6s-6$ multipliers of the transformations (2.3) plus $k(g_s \circ h_1)$ since $k(g_s \circ h_s)$ still has in general two possible values. Theorem 1.4 and its Corollary give a clear picture of these two alternatives.

If we finally give $k(g_s \circ h_s)$, then $\mathcal{H}$ is uniquely determined up to conjugation.

COROLLARY. *Suppose that $\mathcal{H}$ satisfies the hypotheses of Theorem 2.2. If the cyclic order of the fixed points of c_s is given by Fig. 5, then the multipliers of the following $6s-4$ Möbius transformations determine $\mathcal{H}$ uniquely up to conjugation:*

$$\begin{array}{ll} g_j, h_j, & j = 1, \ldots, s \\ g_j \circ h_1, & j = 1, \ldots, s \\ h_j \circ h_1, & j = 2, \ldots, s \\ g_j \circ g_1, h_j \circ g_1, & j = 2, \ldots, s-1 \\ g_s \circ h_s. & \end{array}$$

Note that the $6s-3$ multipliers in Theorem 2.1 give global parameters also for a set $\mathcal{H}$ without any relation.

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University of Helsinki
Department of Mathematics
Hallituskatu 15
SF-00100 Helsinki
Finland

University of Joensuu
Department of Mathematics
P.O. Box 111
SF-80101 Joensuu
Finland

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