Finite-order algebraic automorphisms of affine varieties.

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1. Introduction

Here are a few definitions. Let **F** denote either **R** or **C**. An *algebraic map of affine spaces* $\mathbf{F}^n \to \mathbf{F}^m$ is a map $\mathbf{F}^n \to \mathbf{F}^m$ whose coordinate functions are polynomials with coefficients in **F**. An *algebraic map of affine varieties* $V \to W$, where $V \subset \mathbf{F}^n$, $W \subset \mathbf{F}^m$, is a map $V \to W$ which extends to an algebraic map of affine spaces $\mathbf{F}^n \to \mathbf{F}^m$. The definitions of *algebraic isomorphism* and *algebraic automorphism* are now self-evident. The automorphism group of a variety V is denoted by $\mathrm{Aut}(V)$. An *algebraic action* of a finite group G on a variety V is a homomorphism $\phi: G \to \mathrm{Aut}(V)$. This definition suffices for the results of this paper. For the results cited below for non-finite groups, one needs the action map $G \times V \to V$ defined by ϕ to be algebraic.

Let X be either $\mathbf{R}^n$ or $\mathbf{C}^n$. The following two problems have been popularized by H. Bass and H. Kraft.

Linearity problem

(Kambayashi [7]). Suppose G acts as a group of algebraic automorphisms of X. Show that the action is conjugate to a linear action, i.e. if the action is represented by $\phi: G \to \mathrm{Aut}(X)$, then ϕ is conjugate to a homomorphism of G into the linear subgroup GL_n of $\mathrm{Aut}(X)$.

A summary of some of the known results is given in Bass and Haboush [1] and Kraft [10], and is reproduced here:

For actions on $\mathbf{C}^n$, the problem has been solved in the following cases:

1. G is an algebraic torus acting effectively, and either
   a. $\mathrm{Dim}\, G = n$ or $n - 1$ (Bialynicki-Birula [2], [3]) or
   b. the action is "unmixed" in the sense of Kambayashi and Russell [8] and $\mathrm{Dim}\, \mathbf{C}^n/G \le 2$
2. G is connected and semi-simple, and $n \le 4$ (Jung [6], Van der Kulk: see Kambayashi [7], Draft–Popov [9], Panyushev [15]).

Fixed-point problem

If G acts as a group of algebraic automorphisms of X, then show that there is a fixed point, i.e. show that $X^G \neq \varnothing$.

For actions on $\mathbf{C}^n$, this problem has been solved in the following cases:

1. G is a finite p-group (see [4]).
2. G is a torus (Bialynicki-Birula [2]).
3. $G = \mathrm{SL}_2(\mathbf{C})$ and the action has no 3-dimensional orbits (Panyushev [15]).
4. G is finite and has a normal series $P < H < G$ where P and G/H are of prime power order and H/P is cyclic.

Point 4 is a consequence of Verdier's version of the Lefschetz Fixed-Point Theorem (6.2), the fact that algebraic varieties have cohomology groups (with compact supports) which are finitely generated over $\mathbf{Z}$, and Smith Theory. This result is certainly not well known. See Lemma 3.2 for a proof.

Two of the main results of this paper are that if V is a smooth nonsingular algebraic variety on which a finite group G acts algebraically, then:

1. V has a compactification V^* (see Section 4)
2. V has the G homotopy type of a finite CW complex (see Section 3)

The latter routinely implies that the Lefschetz Fixed-Point Theorem (3.1) holds for algebraic actions of finite groups on nonsingular algebraic varieties, and gives a direct proof of Verdier's theorem in this case. We should emphasize that neither of these results is true in general for smooth actions even under the hypothesis of finite generation of the cohomology of all fixed sets (6.1). In Section 6 we discuss those aspects of smooth transformation groups which have a bearing on the comparison of these points in the smooth and algebraic categories.

Experience from the subject of smooth actions of groups on affine spaces and spheres suggests the following approach to the Linearity and Fixed-Point Problems:

1. Determine the properties of algebraic actions on a variety which depend on the homology of the variety and/or the assumption that the action is algebraic.
2. Then determine the properties which depend upon the specific algebraic structure on the variety.

We have in mind the standard algebraic structure on $\mathbf{F}^n$ when the variety is diffeomorphic to $\mathbf{F}^n$. Theorems 3.1 and 3.2 are examples of 1. Regarding 2: In Section 5 we give an example which shows that if the Linearity Problem has an affirmative answer, then it must depend in an essential way on the standard algebraic structure of $\mathbf{F}^n$. There we construct many examples of an algebraic action of a group G on a variety V which is diffeomorphic to $\mathbf{R}^n$, such that V^G is not contractible. Linear actions have contractible fixed-points sets, so this action is not even smoothly conjugate to a linear action.

An algebraic action on $\mathbf{F}^n$ gives in particular a smooth action on $\mathbf{F}^n$. In order to bring to bear the tools and results from the subject of smooth actions, we need to answer the:

Realization problem

Which smooth actions on $\mathbf{R}^n$ are algebraically realized? (A smooth action on $\mathbf{R}^n$ is *algebraically realized* if it is equivariantly diffeomorphic to an algebraic action on a variety.)

In order to treat this problem, we introduce in Section 4 a compactification V^* of a variety V with an algebraic action of a group G. This is a compact smooth manifold with boundary. It supports a smooth G-action and its interior is eqivariantly diffeomorphic to V. When V is diffeomorphic to $\mathbf{R}^n$, V^* is an n-disk (Theorem 4.2). This means that smooth actions on $\mathbf{R}^n$ which are algebraically realized must extend to the n-disk. (Here $\mathbf{R}^n$ is viewed as the interior of the n-disk when considered as a smooth manifold.) Because of this, one should view algebraic actions on varieties diffeomorphic to $\mathbf{F}^n$ as being more closely related to smooth actions on disks than to smooth actions on $\mathbf{F}^n$. We believe that this point of view is helpful in treating the Linearity and Fixed-Point Problems. For example, a cyclic group acting on a disk must have a fixed point, while on $\mathbf{R}^n$ it need not (Theorem 1.1). Indeed, Theorem 4.1 easily implies the Lefschetz Fixed-Point Theorem for cyclic groups acting algebraically. The Linearity Problem for smooth actions of a finite group on $\mathbf{R}^n$ has an affirmative answer if the fixed-point set of each subgroup is contractible (Rothenberg [18]). The corresponding theorem for smooth actions on the closed disk $D(\mathbf{R}^n)$ is false. An algebraic realization of one of the exotic actions on $D(\mathbf{R}^n)$ could lead to a negative response to the Linearity Problem. There are a number of papers giving non-linear smooth actions of finite groups on $\mathbf{R}^n$. See Edmonds and Lee [5] and Bredon [4, p. 55–62].

Before proceeding with details, we make a few remarks about ideas and methods. We establish an essential property of an *algebraic* action on a non-singular variety, namely that there exists an equivariant Morse function *with only finitely many critical points*. This is Corollary 2.6. The reader should bear in mind that this result, like many others in this paper, is false for smooth actions. Here is a brief outline of some of the ideas. Recall that if M is a smooth manifold, then $f: M \to \mathbf{R}$ is a *Morse function* if f has no degenerate critical points, and no critical points on the boundary ∂M of M; and for each $a \in \mathbf{R}$, $f^{-1}(-\infty, a]$ is compact. Let G be a finite group acting on a non-singular variety V. We produce an equivariant polynomial function $f: V \to \mathbf{R}$ whose restriction to $V^{(H)}$ (the set of points of V whose isotropy group is H) is a Morse function for each subgroup H of G. This function may have degenerate critical points, but only finitely many (by construction). By modifying f in an arbitrarily small neighborhood of its critical points, we can produce an equivariant Morse function $f': V \to \mathbf{R}$ which has finitely many critical points, but which is not algebraic. Equivariant Morse Theory implies that V has a compactification (4.1) and has the homotopy type of a finite G-CW complex (2.7). Thus for each $g \in G$ the Lefschetz fixed-point formula

$$L(V, g) = x(V^g)$$

holds, where

$$L(V, g) = \sum (-1)^i \operatorname{trace}(g_* : H_i(V) \to H_i(V))$$

is the Lefschetz number of g acting upon V, and $\chi(V^g)$ is the Euler characteristic of V^g. (Here singular cohomology is used.) This is Theorem 3.1. If V is contractible, then $L(V, g) = 1$ (since g_* is the identity on $H_0(V) \simeq \mathbf{Z}$, the only non-zero homology group), and so $V^g \neq \varnothing$. This argument uses the fact that g acts algebraically to deduce that the Morse function constructed has only finitely many critical points: indeed, the fixed-point conjecture is false for C^∞-actions of cyclic groups.

THEOREM 1.1 [4, p. 61]. *If r is not a prime power, and L is a finite complex then there exists an integer m such that $\mathbf{R}^n$ admits a self-diffeomorphism of period r whose fixed-point set has the homotopy type of L, for all $n \geq m$ (including the case in which $L = \varnothing$).*

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2. Existence of equivariant Morse functions with finitely many critical points

Let X denote $\mathbf{R}^n$ and let $V \subset X$ be a non-singular affine variety with an algebraic action of a finite group G. Let $X(G)$ be the space of functions $G \to X$, made a G-space by the following rule: if $g, g' \in G$ and $f \in X(G)$, then $(g'f)(g) = f((g')^{-1}g)$. There is a natural algebraic map $\theta: V \to X(G)$ which is a closed equivariant immersion such that if $v \in V$, $g \in G$ then $\theta(v)(g) = gv$. As a vector space, $X(G)$ is the direct sum of $|G|$ copies of X. As a G-space, $X(G)$ is n copies of the regular representation of G.

Let V^H denote the subset of V fixed by H. i.e.

$$V^H = \{v \in V \mid hv = v \text{ for all } h \in H\},$$

and let $V^{(H)}$ denote the set of points in X with isotropy group H, i.e.

$$V^{(H)} = \{v \in V \mid G_v = H\}, \text{ where}$$
$$G_v = \{g \in G \mid gv = v\}.$$

If $f: V \to W$ is an equivariant map, let $f^H = f \mid V^H$, $f^{(H)} = f \mid V^{(H)}$.

The results and proofs of this section hold for all finite groups, but for simplicity of notation only we shall suppose G to be abelian. If $H \subset G$, we may then assume that $V^{(H)}$ and V^H are G-spaces, rather than $N(H)$-spaces, where $N(H)$ is the normalizer of H in G.

Let Γ be a real representation of G, given as a group of orthogonal matices, so that the standard norm $\| \ \|$ is G-invariant, and let $P(\Gamma)$ denote the set of all polynomials $\Gamma \to \mathbf{R}$ which are G-equivariant.

THEOREM 2.1 (Hilbert, see [24]). *$P(\Gamma)$ is finitely generated, i.e. there exist $p_1, \ldots, p_m \in P(\Gamma)$ such that $f \in P(\Gamma)$, then there exists a polynomial function $q: \mathbf{R}^m \to \mathbf{R}$ such that $f = q(p_1, \ldots, p_m)$.*

Let $p = (p_1, \ldots, p_m): \Gamma \to \mathbf{R}^m$.

COROLLARY 2.2. *$p: \Gamma \to \mathbf{R}^m$ factors through Γ/G to give an imbedding $\bar{p}: \Gamma/G \to \mathbf{R}^m$.*

Proof. It suffices to show that $\bar{p}$ is injective. Let $x, y \in \Gamma$ be such that $y \neq gx$ for all $g \in G$. In view of Lemma 2.1 it is sufficient to produce an equivariant polynomial $\psi: \Gamma \to \mathbf{R}$ such that $\psi(x) \neq \psi(y)$. The equivariant polynomial

$$\psi(z) = \prod_{g \in G} \|z - gx\|^2$$

has the required properties.

We now apply this to the case in which $\Gamma = X(G)$. There is a commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{\theta} & X(G) & \xrightarrow{p} & \mathbf{R}^m \\ \downarrow & & \downarrow & & \downarrow \\ V/G & \xrightarrow{\bar{\theta}} & X(G)/G & \xrightarrow{\bar{p}} & \mathbf{R}^m \end{array}$$

where θ, $\bar{\theta}$ and $\bar{p}$ are injective, and θ and p are algebraic. Note that $V^{(H)}/G \subset X(G)^{(H)}/G$ are manifolds. Via the maps $\bar{\theta}$ and $\bar{p}$, we shall regard $V^{(H)}/G$ as a submanifold of $\mathbf{R}^m$.

Our aim is to produce an equivariant Morse function $\Phi: V \to \mathbf{R}$ with finitely many critical points. For $z \in \mathbf{R}^m$, let

$$f_z: \mathbf{R}^m \to \mathbf{R}: x \to \|z - x\|^2.$$

If $H \subset G$, let

$$S_H = \{z \in \mathbf{R}^m \mid f_z | V^{(H)}/G \text{ is a Morse function}\}.$$

Each $V^{(H)}/G$ is a manifold, and each S_H is an open dense subset of $\mathbf{R}^m$ (see [12, Theorem 6.6]). Since G has finitely many subgroups, the set $S = \bigcap_{H \subset G} S_H$ is also dense in $\mathbf{R}^m$, and is in particular not empty. Choose $z_0 \in S$, and let $\phi = f_{z_0}$, i.e.

$$\phi: \mathbf{R}^m \to \mathbf{R}: x \to \|z_0 - x\|^2.$$

Let $\Phi: V \to \mathbf{R}$ be the composition $\phi \circ p \circ \theta$. Then for each subgroup H of G, $\Phi \mid V^{(H)}$ is a Morse function. A critical point of Φ which lies in $V^{(H)}$ is a critical point of $\Phi \mid V^{(H)}$, so

$$C(\Phi) \subset \bigcup_{H \subset G} C(\Phi \mid V^{(H)}),$$

where $C(h)$ denotes the critical set of h. Since V^H is a non-singular variety and Φ is a polynomial, $C(\Phi \mid V^H)$ is an algebraic variety by [12, 2.7], and hence has a finite number of topological components by [12, 2.4]. Since $V^{(H)} \subset V^H$ is open, and $\Phi \mid V^{(H)}$ is a Morse function, each point of $C(\Phi \mid V^{(H)})$ is isolated in $V^{(H)}$, so each of these points is a topological component of $C(\Phi \mid V^H)$, so $C(\Phi \mid V^{(H)})$ is finite. We record this in the following lemma:

LEMMA 2.3. *Let V be a non-singular algebraic variety on which a finite group G acts algebraically. Then there exists an equivariant map $\Phi: V \to \mathbf{R}$ which has finitely many critical points; moreover, for each subgroup H of G, $\Phi^{(H)}$ is a Morse function on $V^{(H)}$.*

The function Φ is not a Morse function on V. We shall modify Φ to produce an equivariant function $\Phi': V \to \mathbf{R}$ which is a Morse function (but which is not algebraic), which has finitely many critical points, and which agrees with Φ outside a small equivariant tubular neighborhood of $C(\Phi)$. This modification (Lemma 2.5) uses the following equivariant version of the Morse Lemma (Lemma 2.4). In Section 7 we supply a proof which we feel is more accessible than the outline in [23] (see [23, 4.8] and subsequent remarks).

LEMMA 2.4. *Let M be a compact smooth G-manifold, $N \subset M$ a closed invariant set which contains ∂M, and let $f: M \to \mathbf{R}$ be a G-invariant C^∞-function with no critical points on ∂M and no degenerate critical points on N. Then there exists a G-invariant Morse function $f': M \to \mathbf{R}$ which agrees with f on an open invariant neighborhood of N.*

Proof. See Section 7.

LEMMA 2.5. *Let W be a smooth G-manifold and let $f: W \to \mathbf{R}$ be a G-invariant C^∞-function with no critical points on ∂W. If the critical point set $C(f)$ of f is finite, then there exists a G-invariant Morse function $f': W \to \mathbf{R}$ whose critical set is finite.*

Proof. Let M be a closed G-invariant tubular neighborhood (see [4, p. 306]) of $C(f)$ in W. Since $C(f)$ is finite, M is compact. (In fact M is a finite union of disks.) Let $N = \partial M$, $h = f \mid M: M \to \mathbf{R}$. Apply Lemma 2.4 to M, h and N to produce a G-invariant Morse function $h': M \to \mathbf{R}$ with $h' = h = f$ on ∂M. Since M is compact, $C(h')$ is finite, because non-degenerate critical points are isolated. Define $f': M \to \mathbf{R}$ by

$$f' \mid W - M = f \mid W - M, f' \mid M = h'.$$

Then f' is a Morse function such that $C(f') = C(h')$, a finite set.

COROLLARY 2.6. *Let V be a non-singular affine variety on which the finite group G acts algebraically. Then there exists a G-invariant Morse function $f: V \to \mathbf{R}$ with finitely many critical points.*

Proof. Apply Lemma 2.5 to the function $\Phi: V \to \mathbf{R}$ produced in Lemma 2.3.

Note that Corollary 2.6 produces a *smooth* equivariant Morse function with finitely many critical points. It would be interesting to know whether there is an *algebraic* function with this property.

COROLLARY 2.7. *If G acts algebraically on a non-singular affine variety V, then V has the equivariant homotopy type of a finite G-CW complex.*

Proof. This follows immediately from Corollary 2.6, since V has the homotopy type of a complex which has a cell for each critical point of the equivariant Morse function (see [12, Theorem 3.5]).

3. The Lefschetz Fixed-Point Theorem

We now use the results of Section 2 to prove the Lefschetz Fixed-Point Theorem:

THEOREM 3.1. *Let V be a non-singular affine algebraic variety on which the finite group G acts algebraically. Then the Lefschetz fixed-point formula*

$$L(V, g) = \chi(V^g)$$

holds for each $g \in G$.

Proof. By Corollary 2.7, V has the equivariant homotopy type of a finite G-CW complex, and so the Lefschetz fixed-point formula holds.

As mentioned in the introduction, this leads to the following Fixed-Point Theorem. See Section 6 for a corresponding discussion of smooth actions on disks.

LEMMA 3.2. *Let G be a finite group having a normal series $P < H < G$ where P and G/H are groups of prime power order. If G acts algebraically on an acyclic variety V, then V has a fixed point.*

Proof. We shall treat the case in which V is nonsingular. This condition may be removed by using Verdier's theorem (6.2) in place of 3.1. The order of P is a power of p and the order of G/H is a power of q for some primes p and q. Since P is a p-group, V^P is $\mathbf{Z}_p$-acyclic by Smith Theory. Since V^P is a finite CW-complex, the cohomology groups of V^P are finitely generated, and so V^P is rationally acyclic. The Lefschetz Fixed-Point Theorem (3.1) for the action of the cyclic group H/P on V^P with fixed set V^H implies that $\chi(V^H)=1$. The q-group G/H acts on V^H with fixed set V^G. Thus $\chi(V^G)\equiv\chi(V^H) \pmod p$, so $\chi(V^G)\equiv 1 \pmod p$ and $V^G \neq \emptyset$. (This congruence involving the Euler characteristic of a q-group action is elementary for an equivariant CW-complex.)

4. Algebraic realizations of smooth actions

In this section we look at the connection between the smooth and algebraic cases. We say that a smooth action may be *algebraically realized* if it is equivariantly diffeomorphic to an algebraic action. It is not true that every smooth action may be algebraically realized since there exist smooth fixed-point-free actions of finite cyclic groups on $\mathbf{R}^n$, but every algebraic action of a cyclic group on $\mathbf{R}^n$ must have a fixed point (compare Theorems 1.1 and 3.1). This section gives a necessary condition for a smooth action to be realized algebraically. Throughout this section, G will denote a finite group, so the results of Section 2 apply.

Let V be a smooth G-manifold without boundary. A *compactification* V^* of V is a *compact* smooth G-manifold V^* such that $V^*-\partial V^*$ is equivariantly diffeomorphic to V, where ∂V^* denotes the boundary of V^*. Compactifications do not always exist, but do exist if V is a non-singular affine variety on which G acts algebraically.

THEOREM 4.1. *If a finite group G acts algebraically on a non-singular affine variety V, then V has a compactification V^*.*

Proof. Let $f:V\to\mathbf{R}$ be a G-invariant Morse function with finitely many critical points (Lemma 2.5.). Let $n>0$ be an integer such that $V^n=f^{-1}(-\infty, n]$ contains all the critical points of f in its interior $f^{-1}(-\infty, n)$. Then V^n is a smooth manifold with boundary $\partial V^n=f^{-1}(n)$. If $i\geq n$, then $V^{i,i+1}=f^{-1}[i, i+1]$ contains no critical points of f, and so it is equivariantly diffeomorphic to $f^{-1}(\epsilon)\times[i, i+1]$ for $\epsilon=i$ or $i+1$. In the non-equivariant case, this is one version of the fundamental lemma of Morse theory [12, Theorem 3.1]. The idea is to integrate a vector field X which is defined on V and which satisfies

$$X(gx)=dgX(x),$$

where $x\in V$, $g\in G$ and $dg:TV\to TV$ is the differential of g. In addition, $X \mid V^{i,i+1}$ is to be $\|\mathrm{grad} f\|^{-2}\,\mathrm{grad} f$, where grad is defined with respect to a G-invariant Riemannian metric on V. Since $V^{i,i+1}$ is compact and G-invariant, there is such a vector field which vanishes outside a compact set, and so may be integrated. This produces a one-parameter group of diffeomorphisms $\phi : V \to V$ satisfying

$$(d/dt)\phi_t(x)|_{t=0} = X(x).$$

Note that

$$(d/dt)f(\phi)_t(x)|_{t=0} = 1$$

for all $x \in V^{i,i+1}$. Then $\psi : f^{-1}(i) \times [i, i+1] \to V^{i,i+1}$ defined by $\psi(x, i+t) = \phi_t(x)$, where $x \in f^{-1}(i)$ and $t \in [0, 1]$, is a G-diffeomorphism The reader may easily produce an equivariant diffeomorphism of $f^{-1}(n) \times [n, \infty) \cong \partial V^n \times [0, 1)$ onto

$$\bigcup_{i=n}^{\infty} V^{i,i+1}.$$

Since V is the union of V^n and the $V^{i,i+1}$ for $i > n$,

$$V \cong V^n \cup \partial V^n \times [0, 1) \cong V^n - \partial V^n.$$

The last equivariant isomorphism follows from the equivariant collar neighborhood theorem [4, p. 229]. This shows that $V^* = V^n$ is a compactification of V.

If G acts smoothly on $\mathbf{R}^n$, then we say that *the action extends to the unit disk* $D(\mathbf{R}^n)$ if the action of G on $\mathbf{R}^n$ viewed as the interior of $D(\mathbf{R}^n)$ extends to a smooth action on $D(\mathbf{R}^n)$.

THEOREM 4.2. *A smooth action of a finite group G on $\mathbf{R}^n$, where $n \geq 5$, may be algebraically realized only if the action extends to $D(\mathbf{R}^n)$.*

Proof. Suppose G acts algebraically on a non-singular variety V, realizing the given action on $\mathbf{R}^n$. Then a compactification V^* of V exists by Theorem 4.1. A theorem of Stallings [19] asserts that for $n \geq 5$, if a compact n-manifold with boundary has interior diffeomorphic to $\mathbf{R}^n$, then the manifold is diffeomorphic to $D(\mathbf{R}^n)$. Thus V^* is diffeomorphic to $D(\mathbf{R}^n)$.

The converse to Theorem 4.2 remains an interesting and essential problem for understanding algebraic actions on $\mathbf{R}^n$. For example, the family of finite groups which act smoothly on a disk without fixed points is infinite. In particular, it contains the alternating group A_5 (see [4]). An algebraic realization of this action would provide a fixed-point free algebraic action of A_5 on a variety diffeomorphic to $\mathbf{R}^n$, and hence a negative answer (modulo identification of algebraic structure) to the Fixed-Point Problem.

5. A non-linear algebraic action

We now give an example of a non-linear algebraic action of the cyclic group of order two on a variety diffeomorphic to Euclidean space.

Let $X = \mathbf{C}^{n+1}$ with coordinates $(z_0, \ldots, z_n)$. Define S^{2n+1} to be the sphere

$$|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1$$

and $\Sigma = \Sigma_q^{2n-1}$ to be the intersection of the Brieskorn variety

$$f(z) = z_0^q + z_1^2 + z_2^2 + \cdots + z_n^2 = 0$$

with S^{2n+1}.

Let G be the cyclic group of order 2 generated by τ, and consider the following representation of G on $\mathbf{C}^{n+1}$:

$$\tau(z_0, z_1, \ldots, z_n) = (z_0, z_1, z_2, -z_3, -z_4, \ldots, -z_n).$$

The variety Σ is invariant, and if

$$p = (0, i/\sqrt{2}, 1/\sqrt{2}, 0, \ldots, 0) \in \mathbf{C}^{n+1},$$

then $p, -p \in \Sigma^G \subset (S^{2n+1})^G$; so in particular the tangent space $T_{-p}S^{2n+1}$ of S^{2n+1} at $-p$ is a $(2n+1)$-dimensional real representation of G denoted by $\mathbf{R}^{2n+1}$. This tangent space is the hyperplane $v \cdot p = -1$, where $v = (v_0, \ldots, v_n) \in \mathbf{C}^{n+1}$, and $\cdot$ denotes the *real* inner product on $\mathbf{C}^{n+1}$.

Let $\phi : \mathbf{R}^{2n+1} \to S^{2n+1} - p$ be the inverse of stereographic projection from p of $S^{2n+1} - p$ onto $T_{-p}S^{2n+1} = \mathbf{R}^{2n+1}$. One finds that

$$\phi(v) = v + [(u(v) - 1) / u(v)]p,$$

where $u(v) = (v^2 + 3)/4$ and v^2 denotes $\|v\|^2$. Moreover, ϕ is an equivariant diffeomorphism.

Let $V = \phi^{-1}(\Sigma - p)$. We claim:

1. V is a non-singular real algebraic variety.
2. G acts algebraically on V.
3. $\pi_1(V^G)$ is cyclic of order q, and if n and q are odd, then V is diffeomorphic to $\mathbf{R}^{2n-1}$.

We verify these assertions. Since $u(v)$ is a real polynomial which never vanishes, V is the zero-set of the real and imaginary parts of the *polynomial*

$$u(v)^q f(\phi(v)), \quad \text{where} \quad v \in \mathbf{R}^{2n+1} \subset \mathbf{C}^{n+1}.$$

This shows that V is an algebraic variety. Since ϕ is a diffeomorphism, and the complex variety $f(z)=0$ intersects S^{2n+1} transversely in Σ [11, p. 17], one readily verifies that V is non-singular.

Since ϕ is G-equivariant and since f and u are G-equivariant polynomials, G acts algebraically on V. Assertion 1 follows from these properties of Σ which may be found in [4. p. 275]: If n and q are odd, $\Sigma = \Sigma_q^{2n-1}$ is homeomorphic to S^{2n-1} while $\Sigma^G = \Sigma_q^3$ is a Lens space whose fundamental group is cyclic of order q. This means that $V = \phi^{-1}(\Sigma - p)$ is diffeomorphic to $\mathbf{R}^{2n-1}$ if n and q are odd, and $V^G \simeq \Sigma_q^3 - p$ has fundamental group which is cyclic of order q.

If n and q are odd, then V is a real algebraic variety diffeomorphic to $\mathbf{R}^{2n-1}$. It supports an algebraic action of G, and if $q \neq 1$ is odd, then $\pi_1(V^G) \neq 0$. Since fixed-point sets of linear actions are contractible, the action on V is not even smoothly conjugate to a linear action on $\mathbf{R}^{2n-1}$.

Finally, it seems worthwhile to explicitly illustrate the defining equations for V for the case in which $n=3$, $q=3$. Let $v_j = a_j + ib_j$, where $a_j, b_j \in \mathbf{R}$ for $j = 0, 1, 2, 3$. The hyperplane $v \cdot p = -1$ given by

$$a_2 + b_1 + \sqrt{2} = 0$$

and V is the set of points in this hyperplane which satisfy

$$(a_0^3 - 3a_0^2b_0^3) + u(v)(a_1^2 + a_2^2 + a_3^2) + u(v)(u(v) - 1)(a_2 - b_1)\sqrt{2} = 0,$$
$$(3a_0^2b_0 - b_0^3) + 2u(v)(a_1b_1 + a_2b_2 + a_3b_3) + u(v)(u(v) - 1)(a_1 + b_2)\sqrt{2} = 0,$$

where

$$u(v) = (v^2 + 3)/4, \quad \text{and} \quad v^2 = \sum_{j=0}^{3} (a_j^2 + b_j^2).$$

6. The influence of the projective class group in comparing smooth and algebraic actions

There are some problems from transformation groups regarding fixed points, existence of compactifications, and the homotopy type of a finite G-CW complex which are significant for smooth actions. In this paper, we have settled these quite simply for algebraic actions. In order to put these two categories in perspective, and to illustrate a unifying theme involving the projective class group K_0, we now discuss these matters.

First we mention Verdier's fixed point theorem, which does not seem to be well known among topologists. Indeed, it was brought to our attention by the referee. Verdier's hypotheses are that X is a locally compact space of finite topological dimension on which a finite group G acts continuously, and which also satisfies:

HYPOTHESIS 6.1. *$H^i(X^H, \mathbf{Z})$ is finitely generated over $\mathbf{Z}$ for every i and every subgroup H of G.*

(Here we are using cohomology with compact supports.)

Define

$$L_c(G, X) = \sum (-1)^i [H^i(X, \mathbf{Q})] \in K_0(\mathbf{Q}(G)),$$

where $[M]$ denotes the class of a $\mathbf{Q}(G)$-module M in $K_0(\mathbf{Q}(G))$. This gives a virtual representation of $\mathbf{Q}(G)$ whose trace defines a function on G. We denote the value of this function at $g \in G$ by $L_c(g, X)$.

THEOREM 6.2 (Verdier [21]).

$$L_c(g, X) = L_c(1, X^g) = \chi_c(X^g).$$

It is interesting to see how 6.1 and the projective class group $K_0(\mathbf{Z}(G))$ are used in the proof of Verdier's theorem. For the proof of 6.2 it suffices to let G be the cyclic group generated by g. By additivity, $L_c(G, X)$ is the sum of the $L_c(G, X^{(H)})$ for $H \subset G$. Thus it must be shown that $L_c(g, X^{(H)})$ is zero unless $H = G$. We may further suppose that $H = 1$ by considering G/H. Thus we may assume that G acts freely on X and satisfies 6.1. Using 6.1 we may add a finite number of G-cells of type $G \times D^i$ to X for $i \leq \dim X$ to get a new G space X' such that $L_c(G, X') = L_c(G, X)$ and $H^i(X') = 0$ for $i \neq n = \dim X$. We claim $H^n(X') = M$ is a projective $\mathbf{Z}(G)$-module. This is proved in Lemma 6.5 below. Assuming this, we have

$$(-1)^n [M \otimes \mathbf{Q}] = L_c(G, X).$$

By a theorem of Swan [20], a $\mathbf{Q}(G)$-module of type $P \otimes \mathbf{Q}$ is free if P is projective. Since trace $(g \mid F)=0$ if F is a free $\mathbf{Q}(G)$-module and $g \neq 1$, $L_c(g, X)=0$.

It should be noted that the hypothesis 6.1 cannot be replaced by finite generation over $\mathbf{Q}$. Indeed, Bredon [4, p. 60] constructs a smooth fixed-point-free action of a cyclic group G on a manifold X diffeomorphic to $\mathbf{R}^n$ such that the fixed set of each Sylow p-subgroup G_p has $H_1(X_p^G)=\mathbf{Z}[1/m_p]$, where p divides m_p. Verdier's theorem shows that this nonfinite generation is typical of smooth fixed-point-free actions of cyclic groups on $\mathbf{R}^n$.

One might ask for a description of the class of groups which act smoothly on a manifold X diffeomorphic to $\mathbf{R}^n$, satisfy 6.1, and which necessarily have a fixed point. It is easy to see (compare 3.2) using 6.2 and Smith Theory that G is in this class if G has a normal series $P<H<G$ where P is a p-group, H/P is cyclic, and G/H is a q-group for primes p and q. On the other hand Oliver [13] has shown that this is exactly the class of groups which act on a disk and necessarily have a fixed point. With regard to fixed points then, smooth actions on disks and smooth actions on $\mathbf{R}^n$ which satisfy 6.1 are the same. Since the n-disk is the compactification of $\mathbf{R}^n$, one might wonder whether 6.1 is the proper assumption on smooth actions to show the existence of a compactification or that the manifold is a finite G-CW complex. Specifically, does a smooth G-manifold X which satisfies 6.1.

1. have the homotopy type of a finite G-CW complex, or
2. have a compactification?

In general the answer to both is 'no'. For example, if G acts freely on X and $\pi_1(X)=0$, then there is an obstruction $\sigma \in \tilde{K}_0(\mathbf{Z}(G))$ to point 1 above. This is due to Wall [22]. Siebenmann, in his Princeton thesis, modified this to produce an obstruction to the existence of X^* which lives in a related projective class group and which depends on the action "at infinity." The main results of this paper show directly that these obstructions vanish.

Note again that X^* is a disk when X is a G-manifold diffeomorphic to $\mathbf{R}^n$, and in the case of algebraic action, X^* always exists. We are interested in the question of which actions on disks arise in this way from algebraic actions on $\mathbf{R}^n$. An interesting class of smooth actions on disks appears in the papers of Oliver [13], [14]. There he treats an invariant in $\tilde{K}_0(\mathbf{Z}(G))/B_0(G)$ in dealing with the construction of a smooth fixed-point-free action of G on a disk. In [16], Petrie deals with an invariant in $\tilde{K}_0(\mathbf{Z}(G))$ which is relevant to G-maps between smooth G-manifolds. All the projective class group constructions mentioned so far are related geometrically and algebraically, and depend upon showing that a certain $\mathbf{Z}(G)$-module arising from the cohomology of a G-space is projective.

The theme (used in [16]) which unifies these projective class group constructions depends upon a theorem of D. S. Rim:

THEOREM 6.3 (Rim [17]). *Let M be a finitely generated $\mathbf{Z}(G)$-module which is $\mathbf{Z}$ torsion free. Then M is a projective module if and only if for each prime p, $H^i(G_p, M \otimes \mathbf{Z}_p) = 0$ for large i.*

There are two typical cases in which this is applied:

6.4.
1. *$M = H^n(X)$, where X is a locally compact G-space of finite topological dimension which satisfies $H^i(X) = $ for $i \neq n$. (Here cohomology with compact supports is used.)*
2. *$M = H^n(X, A)$, where A and B are finite-dimensional G-CW complexes which satisfy $H^i(X, A) = 0$ for $i \neq n$. (Here singular cohomology is used.)*

The following is an easy application:

LEMMA 6.5. *Suppose* 6.4.1 *holds and G acts freely on X. Then M is projective.*

Proof. The spectral sequence

$$E_2^{i,j} = H^i(G_p, H^i(X, \mathbf{Z}_p)) \Rightarrow H^{i+j}(X/G_p, \mathbf{Z}_p)$$

collapses because $E_2^{i,j} = 0$ for $j \neq n$. Thus

$$H^i(G_p, M \otimes \mathbf{Z}_p) = H^{i+n}(X/G_p, \mathbf{Z}_p).$$

Since X/G_p has finite topological dimension (because X does), $H^i(G_p, M \otimes \mathbf{Z}_p)$ is zero for large i. By Rim's theorem, M is projective.

The hypothesis in this lemma that G acts freely on X can be replaced by

6.6. *For each prime p, $H^*(X^H, \mathbf{Z}_p) = 0$ whenever H is a nontrivial p-group.*

In this case the above spectral sequence collapses to $H^*_\Gamma(X, \mathbf{Z}_p)$ for $\Gamma = G_p$. Here

$$H^*_\Gamma(X, \mathbf{Z}_p) = H^*(E \times_\Gamma X, \mathbf{Z}_p),$$

where E is a contractible space on which Γ acts freely. A localization lemma in the equivalent cohomology theory H^*_Γ relates $H^*_\Gamma(X, \mathbf{Z}_p)$, $H^*_\Gamma(X^H, \mathbf{Z}_p)$, and 6.6, and shows that $H^i(\Gamma, M \otimes \mathbf{Z}_p)$ is zero for large i (see [16, Cor. 3.1]), so again M is projective. The added generality of 6.6 is needed in the papers of Oliver and Petrie cited above.

7. Proof of Lemma 2.4

In this section we prove Lemma 2.4. Lemma 7.1 is a local result which leads to removal of a degenerate critical point which lies in M^G, so long as f^G is a Morse function. The proof of 7.1, with minor alterations, is taken from [23]. Lemmas 7.2 and 7.3 provide the argument required for the inductive step.

LEMMA 7.1 [23, 4.10]. *Let V be a Euclidean space on which G acts linearly. Let $f: V \to \mathbf{R}$ be a C-invariant G^∞-function such that f^G is a Morse function and $0 \in V$ is the only critical point of f in V^G. Then there exists a G-invariant C^∞-function $f': V \to \mathbf{R}$ such that*

1. *f' and f agree outside the unit disk in V.*
2. *f' has only non-degenerate critical points in V^G.*

Proof. First we remark that f' may have degenerate critical points in $V - V^G$. This lemma will be used as the basis of an inductive argument, and these critical points will be made non-degenerate.

Let $\lambda: \mathbf{R} \to \mathbf{R}$ be a monotone decreasing function such that $\lambda[0, 1/2] = 1$, $\lambda[1, \infty) = 0$. Choose coordinates x along V^G, and y in the normal direction. A point $v \in V$ may be written $v = (x, y)$ and its norm squared $\|(x, y)\|^2$ may be written (by abuse of notation) $x^2 + y^2$. Define

$$\begin{aligned} f'(x, y) &= f(x, y) + \epsilon\lambda(\|(x, y)\|^2)\,\|(0, y)\|^2 \\ &= f(x, y) + \epsilon\lambda(x^2 + y^2)y^2, \end{aligned}$$

where ϵ is a constant to be chosen later. Note that condition 1 is satisfied, f and f' agree on V^G, and their differentials also agree on V^G. This means that 0 is the only critical point of f' in V^G because this is the case for f. The Hessian of f' at 0 has the form:

$$H = \begin{bmatrix} A & B \\ C & D + 2\epsilon I \end{bmatrix}$$

where A is the Hessian of f^G at 0. Since f^G is a Morse function, A is non-singular, so $\det H$ is a non-zero polynomial in ϵ, with roots $\epsilon_1, \ldots, \epsilon_k$, say. By choosing $\epsilon \neq \epsilon_i$, condition 2 can be achieved.

In the next two lemmas, we will deal with a smooth G-manifold M and G-invariant smooth functions $M \to \mathbf{R}$. All such functions will have no critical points on ∂M. Let H be a subgroup of G, and let $\bar{U}$ denote the closure of U.

LEMMA 7.2. *Let $f: M \to \mathbf{R}$ be a G-invariant smooth function which is bounded and has no critical points in a closed invariant set A which contains ∂M and all M^K for $K > H$. Suppose there exist open invariant subsets U and W of M such that $A \subset U \subset \bar{U} \subset W$, $\bar{W} - U$ is compact, and $\bar{W}$ contains no degenerate critical points of f. Then there exist a G-invariant smooth function $f': M \to \mathbf{R}$ such that f and f' agree on an open invariant set which contains A, and f'^H is a Morse function (i.e. all critical points of f'^H are non-degenerate).*

Proof. First we take the case in which $G = \{1\}$, the trivial group. Let $\lambda: M \to \mathbf{R}$ be a non-negative smooth function which is 1 on U and 0 on $M - W$. By [12, Corollary 6.8], there exists a smooth Morse function $h: M \to \mathbf{R}$ which uniformly approximates f and whose ith order derivatives, for $i \leq 3$, uniformly approximate the corresponding derivatives of f on $\bar{W} - U$. Then $f' = \lambda f + (1 - \lambda)h$ is a Morse function with the required properties if h is sufficiently close to f.

We now reduce the general case to this special case. Note that $N = M - \bar{U}$ is a smooth G-manifold. Choose open invariant sets U', U'' such that $\bar{U} \subset U' \subset \bar{U}' \subset U'' \subset \bar{U}'' \subset W$. The group G/H acts freely on N^H, so the orbit space N^H/G is a smooth G-manifold. Apply the case $G = \{1\}$ to the map $f^H/G: N^H/G \to \mathbf{R}$, where the set $(\bar{U}' - \bar{U})^H/G$ plays the role of A, and $(U'' - \bar{U})^H/G$ plays the role of U. (Note that since $\bar{W}$ contains no degenerate critical points of f, $\bar{W}^H$ contains no degenerate critical points of f^H.) This produces a Morse function $h/G: N^H/G \to \mathbf{R}$ which lifts to a G-invariant Morse function $h: N^H \to \mathbf{R}$. Extend h to $h': \bar{U}' \cup N^H \to \mathbf{R}$ by setting $h' = f$ on $\bar{U}'$.

Since $\bar{U}' \cup N^H$ is a closed invariant subset of M on which f and h' are equivariantly homotopic, the equivariant homotopy extension theorem implies that h' may be extended to a smooth function $f': M \to \mathbf{R}$. Then f'^H is a Morse function, and f' and f agree on an open invariant neighborhood of A.

LEMMA 7.3. *Now suppose that M is compact. Let $f: M \to \mathbf{R}$ be a G-invariant C^∞-function which has no degenerate critical points in a closed invariant set A which contains ∂M and M^K for $K > H$. Then there exists a G-invariant C^∞-function $f': M \to \mathbf{R}$ such that f and f' agree on an open invariant neighborhood of A and f' has no degenerate critical points in $A \cup M^H$.*

Proof. Apply Lemma 7.2 to produce a G-invariant function $h: M \to \mathbf{R}$ which agrees with f on ∂M and on an open invariant neighborhood of A, and such that h^H is a Morse function. (Since M is compact, the open sets U and W with $\bar{W} - U$ compact required in Lemma 7.2 do exist.) Thus the critical points of h^H are isolated. Let x be a critical point of h with isotropy group H, and let V be the representation of H on the tangent space T_xM of M at x. With respect to a

G-invariant Riemannian metric on M, the exponential map of T_xM to M gives rise to an H-equivariant imbedding of V onto an open neighborhood of x in M such that $0 \in V$ corresponds to $x \in M$. Identify V with its image in M. Then the orbit $G(V)$ of V is a G-invariant neighborhood of the orbit $G(x)$ of x. Since x is an isolated critical point of h^H, we may suppose that V^H contains no other critical points of h^H, and hence no other critical points of h. Now apply Lemma 7.1 to the function $h \mid V : V \to \mathbf{R}$ with G of the lemma replaced by H to produce an H-invariant C^∞-function $h' : V \to \mathbf{R}$ such that

1. h' and h agree outside the unit disk of V.
2. h' has only non-degenerate critical points in V^H.

Extend h' to a G-invariant C^∞-function $f' : M \to \mathbf{R}$ by

$$f'(gv) = gh'(v) \quad \text{for} \quad g \in G,\ v \in V,$$

$$f' \mid M - G(V) = f \mid M - G(V).$$

Since M is compact, and f^H is a Morse function, f has finitely many critical points in M^H, and this process may be repeated for each critical point of f in M^H. The result is a function f' which agrees with f outside a neighborhood of the critical points of f^H in M^H and which has no degenerate critical points in a neighborhood of $A \cup M^H$.

Proof of Lemma 2.4

Let S be a set of subgroups of G with the property that if $H \in S$ and K is a subgroup containing H, then $K \in S$. Let H be a maximal subgroup of G which is not in S. Let A_S be the union of N (see the statement of Lemma 2.4) and the M^K, for $K \in S$. Suppose that $f_S : M \to \mathbf{R}$ is a G-invariant C^∞-function with no degenerate critical points in A_S and suppose that f and f_S agree on an open neighborhood of N. Apply Lemma 7.3 to produce a G-invariant function $f' : M \to \mathbf{R}$ such that f' and f_S agree on an open invariant neighborhood of A_S and f' has no degenerate critical points in $A_S \cup M^H$. Let $S' = S \cup \{H\}$ and $f_{S'} = f'$. Then $f_{S'}$ satisfies the same conditions with respect to S' as f_S does with respect to S. Repeat this procedure until S is the set of all subgroups of G. Then $f_S = f'$ has the properties asserted in Lemma 2.4.

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