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I. INTRODUCTION

A hyperbolic 3-manifold is a Riemannian manifold of constant sectional curvature −1. We will restrict our attention to complete orientable hyperbolic 3-manifolds M; as such, we can think of M as H^3/Γ where Γ is a discrete torsion-free subgroup of $\text{Isom}_+(H^3)$, the orientation-preserving isometries of hyperbolic 3-space. We will generally work in the upper-half-space model H^3 of hyperbolic 3-space, in which case $PGL(2, \mathbf{C})$ acts as orientation-preserving isometries on H^3 by extending the action of $PGL(2, \mathbf{C})$ on the Riemann sphere (boundary of H^3) to H^3. An orbifold is a space locally modelled on $\mathbf{R}^n$ modulo a finite group action. Complete orientable hyperbolic 3-orbifolds Q correspond to discrete subgroups Γ of $PGL(2, \mathbf{C})$. If the discrete group Γ corresponding to M or Q has parabolic elements then M or Q is said to be cusped.

Unless otherwise stated, we will assume all manifolds and orbifolds are orientable. Mostow's theorem implies that a complete, hyperbolic structure on a 3-orbifold of finite volume is unique. Consequently, hyperbolic volume is a topological invariant for orbifolds admitting such structures. Jørgensen and Thurston proved (see [T] section 6.6) that the set of volumes of complete hyperbolic 3-manifolds is well-ordered and of order type ω^ω. In particular, there is a complete hyperbolic 3-manifold of minimum volume V_1 among all complete hyperbolic 3-manifolds, and a cusped hyperbolic 3-manifold of minimum volume V_ω. Further, all volumes of closed manifolds are isolated, while volumes of cusped manifolds are limits from below (thus the notation V_ω).

Modifying the proofs in the Jørgensen–Thurston theory yields similar results for complete hyperbolic 3-orbifolds (this result is folklore, and we will not prove it here). In particular, there is a hyperbolic 3-orbifold of minimum volume V_1', and a cusped hyperbolic 3-orbifold of minimum volume V_c'.

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In [M1] and [M2] it is proved that

$$0.00064 < V_1 \leq \operatorname{vol}(M_{(5,1)}) \approx 0.98$$
$$\sqrt{3}/4 \leq V_\omega \leq \operatorname{vol}(S^3 - \text{figure-eight knot}) = 2V \approx 2.02988$$
$$0.0000013 < V_1' \leq 2 \cdot \operatorname{vol}(\text{o=o—o≡o}) \approx 0.072$$
$$\sqrt{3}/24 \leq V_c' \leq \operatorname{vol}(H^3/PGL_2(\mathcal{O}_3)) = V/12 \approx 0.0846$$

where $M_{(5,1)}$ is the manifold obtained by performing (5, 1) Dehn surgery on the figure-eight knot in the 3-sphere, V is the volume of the ideal regular tetrahedron in H^3, o=o—o≡o denotes the (non-orientable) tetrahedral orbifold with that Coxeter diagram (see [T] theorem (13.5.3)), and $\mathcal{O}_3$ is the ring of integers in $\mathbf{Q}(\sqrt{-3})$.

The left-hand inequalities of all of these estimates can be improved by using sphere-packing arguments. In this paper we prove,

$$0.00082 < V_1 \leq 0.98\ldots$$ [2]
$$V/2 \leq V_\omega \leq 2V$$ [3]
$$0.0000017 < V_1' \leq 0.07177\ldots$$
$$V/12 \leq V_c' \leq V/12$$

From the last set of inequalities we see $V_c' = V/12$, i.e.

THEOREM. *The orbifold $Q_1 = H^3/PGL_2(\mathcal{O}_3)$ has minimum volume among all orientable cusped hyperbolic 3-orbifolds.*

NOTE. Q_1 is the orientable double-cover of the (non-orientable tetrahedral orbifold with Coxeter diagram o—o—o≡o (see [H] section 1). This tetrahedral orbifold has fundamental domain 1/24 of the ideal regular hyperbolic tetrahedron (use the symmetries). In particular, Q_1 has a cusp and its volume is 1/12 the volume of the ideal regular tetrahedron, i.e. $\operatorname{vol}(Q_1) = V/12 \approx 0.0846$.

Remark. The four right-hand inequalities above are simply a list of the lowest volume orbifolds and manifolds of the various types known to date. These volumes are computed by decomposing the orbifold or manifold into hyperbolic

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[2] Jeff Weeks has found a hyperbolic 3-manifold with less volume than $M_{(5,1)}$ (Princeton Univ. Ph.D. thesis, 1985).

[3] Colin Adams has improved the left-hand inequality for V_ω by a factor of 2 (preprint, 1985).

tetrahedra and then using Lobachevsky's formula to compute the volumes of these tetrahedra (see [T] chapter 7 for the case of ideal hyperbolic tetrahedra, and [La] for the case of non-ideal tetrahedra – actually, these tetrahedra must be further decomposed into "doubly-rectangular" tetrahedra). The decomposition into tetrahedra for tetrahedral orbifolds is trivial. The tetrahedral decomposition of the figure-eight knot complement in the 3-sphere is carried out in [T] pages 3.6 and 3.7. Finally, solving the holonomy equations in section 4.6 of [T] for $(p, q) = (5, 1)$ produces a decomposition of $M_{(5,1)}$ into ideal hyperbolic tetrahedra (off of the surgered geodesic).

II. Sphere-packing

We will be concerned with how densely equal radius balls can be packed without overlapping. In general, the density of S with respect to (finite volume) T is

$$d(S, T) = \frac{\mathrm{vol}\,(S \cap T)}{\mathrm{vol}\,(T)}.$$

We can extend this notion to Euclidean n-space $\mathbf{E}^n$, i.e. $T = \mathbf{E}^n$ and $S =$ (the union of non-overlapping, equal-radius balls), by defining *upper* and *lower densities*

$$d_U = \limsup_{r \to \infty} d(S, B(p, r)) \quad \text{and} \quad d_L = \liminf_{r \to \infty} d(S, B(p, r))$$

where $B(p, r)$ is the radius r ball in $\mathbf{E}^n$ centered at p. If $d_L = d_U$ then we have a notion of *global density* for $\mathbf{E}^n$. The fact that d_L and d_U are independent of the base point p chosen is proven in [FT] pages 161, 162 (see also pg. 261). The argument hinges on the fact that

$$\lim_{r \to \infty} \frac{\mathrm{vol}\,(B(p, r + \varepsilon))}{\mathrm{vol}\,(B(p, r))} = 1.$$

Attempting to use this notion of global density in hyperbolic n-space H^n is problematic because

$$\lim_{r \to \infty} \frac{\mathrm{vol}\,(B(p, r + \varepsilon))}{\mathrm{vol}\,(B(p, r))} = e^{\varepsilon(n-1)}$$

(in H^3, $\text{vol}(B(p, r)) = \pi(\sinh(2r) - 2r)$). We will avoid this problem by dealing with a "local" notion of density. Given a collection $\mathscr{B}$ of equal radius, non-overlapping balls in H^n we define the *local density* of a ball B in $\mathscr{B}$ to be

$$\ell d(B, \mathscr{B}) = \frac{\text{vol}(B \cap D)}{\text{vol}(D)} := d(B, D)$$

where $D = \{p \in H^n : p \text{ is closer to } B \text{ than to any other ball } B' \text{ in } \mathscr{B}\} := D(B, \mathscr{B})$ is the Dirichlet region for B with respect to $\mathscr{B}$. This notion is ideally suited to studying volumes of hyperbolic 3-manifolds $M = H^3/\Gamma$ because, given an embedded ball in M, the collection of all lifts of this ball to H^3 gives a packing $\mathscr{B}$ of H^3 upon which Γ acts transitively, and $D(B, \mathscr{B})$ for any B in $\mathscr{B}$ is a fundamental domain for $M = H^3/\Gamma$ (see [G] Section 2.5). A similar notion holds for orbifolds $Q = H^3/\Gamma$, but we may have to "chop" B and D due to torsion elements in Γ. That is, if Γ_b is the stabilizer of the center b of B, then D/Γ_b is a fundamental domain for $Q = H^3/\Gamma$ (see [Be] Section 9.6). This is not a problem, because $d(B, D) = d(B/\Gamma, D/\Gamma_b)$.

We can generalize local density to deal with a horoball packing ("horoball" is defined in Section III). The notion of a Dirichlet region $D = D(B, \mathscr{B})$ still makes sense if we define the distance of a point p from a horoball B to be the length of the unique perpendicular geodesic from p to the horosphere boundary of B. The fact that $B \cap D$ and D have infinite volume creates some problems. Thus, we define *local density* $\ell d(B, \mathscr{B})$ in a 2-step procedure: Assume we are in upper-half-space H^3 and that B is centered at the point at infinity. Then, we define

$$d_t = \lim_{c \to \infty} \frac{\text{vol}(B \cap D \cap A(t, c))}{\text{vol}(D \cap A(t, c))}$$

where $A(t, c) = \{(x, y, z) : -c < x < c,\ -c < y < c, \text{ and } z \geq t\}$. This definition is independent of the choice of origin (here the origin is $(0, 0, t)$); the independence-of-origin proof is a re-working of the proof for $\mathbf{E}^n$ mentioned above, using the fact that horoballs have Euclidean structures on their horosphere boundaries and that $\text{vol}(A(t, c)) = c^2/2 \cdot t^2$. Since d_t is an increasing function of t, we can define $\ell d(B, \mathscr{B}) = \lim_{t \to 0} d_t$.

This is the appropriate notion of local density to use in studying hyperbolic 3-manifolds $M = H^3/\Gamma$ with cusps. If we know that a cusped manifold contains an embedded cusp neighborhood, then lifting these cusp neighborhoods to H^3 gives a collection $\mathscr{B}$ of disjoint horoballs B upon which Γ acts transitively; but $D(B, \mathscr{B})$ is no longer a fundamental domain for Γ. To get a fundamental domain F for Γ

we simply take F to be a fundamental domain for the action of Γ_c on $D(B, \mathscr{B})$ where Γ_c is the stabilizer of the center c of B (Γ_c is made up entirely of parabolic transformations). Using the above definition of local density for horoball packings, we have

$$\ell d(B, \mathscr{B}) = \frac{\text{vol}\,(B \cap F)}{\text{vol}\,(F)}.$$

The above holds verbatim for cusped orbifolds $Q = H^3/\Gamma$ except that Γ_c may have elliptic as well as parabolic transformations.

We now state Böröczky's theorem (which applies to constant curvature spaces of arbitrary dimension) in the case of hyperbolic 3-space (See [B] theorems 1 and 4):

THEOREM (Böröczky). *Consider* 4 *spheres of radius r in H^3 each touching all the others. Their centers determine a regular tetrahedron T of edge length* $2r$ *and dihedral angles* 2α *where* $\sec(2\alpha) = 2 + \operatorname{sech}(2r)$. *Let S be the union of the* 4 *balls of radius r bounded by the* 4 *spheres. Then, for any radius r sphere-packing $\mathscr{B}$ in H^3 the local density satisfies*

$$\ell d(B, \mathscr{B}) \le \frac{\text{vol}\,(S \cap T)}{\text{vol}\,(T)} = \frac{(6\alpha - \pi)(\sinh(2r) - 2r)}{\text{vol}\,(T)} := d(r).$$

This result holds for horosphere packings as well, in which case the centers of the horoballs (points of tangency with ∂H^3) determine an ideal regular tetrahedron T, and

$$\ell d(B, \mathscr{B}) \le \frac{\text{vol}\,(S \cap T)}{\text{vol}\,(T)} = \frac{4(\sqrt{3}/8)}{V} = \frac{\sqrt{3}}{2V} \approx 0.853, \quad \textit{where} \quad V = \text{vol}\,(T).$$

Remark. It was shown in [BF] that $d(r)$ is an increasing function of r. The number $d(0) \approx 0.7797$ is the density (with respect to the regular tetrahedron they determine) of 4 mutually touching equal radius balls in $\mathbf{E}^3$. The 4 horoball packing can be extended uniformly to all of H^3. In some sense, this is the densest packing of equal radius spheres in H^3. The densest packing of equal radius spheres in $\mathbf{E}^3$ is not known even though the analog of the above theorem holds for $\mathbf{E}^n$. The difficulty is that the above tetrahedral packing does not extend uniformly to a global packing of $\mathbf{E}^3$ (See [SL] and [R]).

III. Remarks on hyperbolic space

As mentioned in Section 1, we are working in the upper-half-space model for hyperbolic 3-space, $H^3 = \{(x, y, z) : z > 0\}$ with metric $ds^2 = (dx^2 + dy^2 + dz^2)/z^2$ and volume form $dV = dx\,dy\,dz/z^3$; $\partial H^3 = \mathbf{C} \cup \{\infty\}$. The orientation-preserving isometries of hyperbolic 3-space can be identified either with $PGL_2(\mathbf{C}) = GL_2(\mathbf{C})/\mathbf{C}^*$ or $PSL_2(\mathbf{C}) = SL_2(\mathbf{C})/\pm I$ (See [S] pg. 448–449). But note that if $\mathcal{O}_d$ is the ring of integers in $\mathbf{Q}(\sqrt{-d})$ then $PGL_2(\mathcal{O}_d)/PSL_2(\mathcal{O}_d) = \mathbf{Z}/2\mathbf{Z}$ where $PGL_2(\mathcal{O}_d) = GL_2(\mathcal{O}_d)/\{\lambda I : \lambda \in \mathcal{O}_d^*\}$ and $PSL_2(\mathcal{O}_d) = SL_2(\mathcal{O})_d/\pm I$ (See [H] pg. 346). Thus, the use of $PGL_2(\mathcal{O}_d)$, and not $PSL_2(\mathcal{O}_d)$, in the statement of Theorem 1.

In H^3 a *horoball* B is either:

1) a Euclidean ball in $\{(x, y, z) : z \geq 0\}$ which is tangent to the xy plane, the point of tangency being the *center* of B; or it is

2) a half space of the form $\{(x, y, z) : z \geq a > 0\}$, in which case the center of B is the point at ∞.

Note that the hyperbolic metric on H^3 induces the Euclidean metric $ds^2 = (dx^2 + dy^2)/a^2$ on $\partial B \cap H^3 = \{(x, y, z) : z = a\}$, that is the bounding horosphere of the horoball B is flat. There is no real distinction between horoballs of type 1 and type 2, because there are isometries of H^3 taking either to the other. In particular, all horospheres are flat.

A discrete group Γ is said to have a *cusp* if Γ contains a parabolic element γ. Let the fixed point of γ be $p \in \partial H^3$; then Γ_p, the stabilizer of p, is of importance. Γ_p contains no hyperbolic elements (See [Be] theorem 5.1.2). In the manifold case Γ_p contains only parabolic transformations. In the orbifold case Γ_p may have elliptic elements.

IV. Sphere-packing and volume

It can be proved that short geodesics (length less than approximately 0.107) in complete hyperbolic 3-manifolds have embedded tubular neighborhoods ("solid tubes"), and that the shorter the geodesic the bigger the volume of the solid tube (See [M1]). This solid tube construction can be used to produce a lower bound for the volume of complete hyperbolic 3-manifolds (without cusps). The argument is as follows. A non-cusped hyperbolic 3-manifold $M = H^3/\Gamma$ must have either an embedded ball of radius r or a geodesic of length less than $2r$. If we take $r = 0.053475$ then the embedded ball $B(0.053475)$ contributes at least 0.00064 to the volume of M, while a geodesic of length at most $2r = 0.10695$ has an embedded tubular neighborhood of volume at least 0.00068 (See [M1]). Thus, the volume of a closed hyperbolic 3-manifold must be greater than 0.00064. By choosing a smaller r we get more volume in the solid-tube case, but less in the embedded-ball case; thus the overall volume estimate is lower. The value $r = 0.053475$ was chosen to maximize the overall volume estimate; call this value or r the "trade-off value". (Since cusped hyperbolic 3-manifolds have volume greater than $\sqrt{3}/4$ we have that all complete hyperbolic 3-manifolds have volume at least 0.00064, i.e. $V_1 > 0.00064$ (See [M1]).)

Böröczky's theorem can be used to improve the lower bound of 0.00064. Specifically, Böröczky's theorem yields an improved volume contribution in the embedded-ball case. The argument is as follows. As mentioned in Section 2, the lifts of an embedded ball $B(r)$ to H^3 yield a packing $\mathcal{B}$ of H^3; and a Dirichlet domain $D(B, \mathcal{B})$ for any ball B in the packing is a fundamental domain for Γ. Using Böröczky's theorem, we have $\mathrm{vol}(B(0.053475))/\mathrm{vol}(H^3/\Gamma) = \mathrm{vol}(B(0.053475))/\mathrm{vol}(D(B, \mathcal{B})) \le d(0.053475)$. Thus $\mathrm{vol}(H^3/\Gamma) \ge \mathrm{vol}(B(0.053475))/d(0.053475) > 0.00082$, and we have improved our estimate if an embedded ball of radius 0.053475 sits in M. This technique does not effect the solid-tube contribution; thus, if r is taken as 0.053475 then our lower bound is still 0.00064. However, we can take a smaller value of r and improve our solid-tube volume contribution while only marginally effecting our embedded-ball volume. In particular taking $r = 0.053463$ yields a solid-tube volume greater than 0.00082 while the embedded-ball volume is still greater than 0.00082. Thus, we have that 0.00082 is a lower bound for the volume of complete hyperbolic 3-manifolds; that is $V_1 > 0.00082$.

For orbifolds $Q = H^3/\Gamma$ without cusps the analysis is essentially the same except that the relevant "trade-off" radius is 0.0535 and the volume of the "chopped" solid ball is roughly 0.00000134 (see [M2]). Thus by the density argument $\mathrm{vol}(Q) > 0.0000017$, i.e. $V_1' > 0.0000017$.

In dealing with cusped manifolds $M = H^3/\Gamma$ we do not have to resort to this trading-off argument. In [M1] it is shown that there is a cusp neighborhood C in M of volume at least $\sqrt{3}/4$. This neighborhood yields a horoball packing $\mathcal{B}$ of H^3. Further, given B in $\mathcal{B}$ centered at p we have that a fundamental domain F for the action of Γ_p on $D(B, \mathcal{B})$ is a fundamental domain for Γ. Applying Böröczky's theorem, we have

$$\frac{\mathrm{vol}(C)}{\mathrm{vol}(M)} = \frac{\mathrm{vol}(B \cap F)}{\mathrm{vol}(F)} = d(B, \mathcal{B}) \le \sqrt{3}/2V.$$

Thus, $\mathrm{vol}(M) \ge \mathrm{vol}(C)/(\sqrt{3}/2V) \ge (\sqrt{3}/4)(2V/\sqrt{3}) = V/2$ and $V_\omega \ge V/2 \approx 0.5072$.

This argument works for cusped orbifolds $Q = H^3/\Gamma$ as well, except that the cusp neighborhood C in Q in the worst case only contributes $\sqrt{3}/24$ to the volume of Q (See [M2]). Thus $\text{vol}(Q) \geq (\sqrt{3}/24)(2V/\sqrt{3}) = V/12$, $V_c' \geq V/12 \approx 0.0846$. Since $Q_1 = H^3/PGL_2(\mathcal{O}_3)$ has volume $V/12$ we have (See Section 1):

THEOREM. $Q_1 = H^3/PGL_2(\mathcal{O}_3)$ *has minimum volume among all orientable cusped hyperbolic* 3-*orbifolds.*

Remark. There are cusped orbifolds on which Dehn surgery cannot be performed. Consequently, unlike the manifold case, there are cusped hyperbolic 3-orbifolds whose volumes are isolated—Q_1 is such an orbifold. The question of finding "the least limiting orbifold" remains open.

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BIBLIOGRAPHY

[Be] A. BEARDON, *The Geometry of Discrete Groups,* Springer-Verlag, New York (1983).
[B] K. BÖRÖCZKY, *Packing of Spheres in Spaces of Constant Curvature,* Acta Math. Acad. Sci. Hungar., *32* (1978) 243–261.
[BF] K. BÖRÖCZKY and A. FLORIAN, *Über die dichtest Kugelpackung im hyperbolischen Raum,* Acta Math. Acad. Sci. Hungar. *15* (1964) 237–245. [MR28(3369)].
[FT] L. FEJES TÓTH, *Regular Figures,* the MacMillan Co., New York, (1964).
[G] L. GREENBERG, *Finiteness Theorems for Fuchsian and Kleinian Groups*: "Discrete Groups and Automorphic Functions" (edited by W. J. Harvey), Academic Press, London, 1977.
[H] A. HATCHER, *Hyperbolic Structures of Arithmetic Type on Some Link Complements,* J. London Math. Soc. (2) *27* (1983) 345–355.
[La] B. L. LAPTEV, *The Volume of a Pyramid in a Lobačevskiĭ Space,* Kazan, Gos. Univ. Uč. Zap. *114*, no. 2 (1954) 53–77 (Russian), MR17(777).
[L] J. LEHNER, *A Short Course in Automorphic Functions,* Holt, Rinehart and Winston, New York, 1966.
[M1] R. MEYERHOFF, *A Lower Bound for the Volume of Hyperbolic 3-Manifolds,* preprint.
[M2] R. MEYERHOFF, *A Lower Bound for the Volume of Hyperbolic 3-Orbifolds,* preprint.
[R] C. A. ROGERS, *The Packing of Equal Spheres,* Proc. London Math. Soc. (3) *8* (1958) 609–620.
[S] P. SCOTT, *The Geometries of 3-Manifolds,* Bull. London Math. Soc., *15* (1983) 401–487.
[SL] N. J. A. SLOANE, *The Packing of Spheres,* Scientific Amer. *250* (Jan. 1984) 116–125.
[T] W. THURSTON, *The Geometry and Topology of 3-Manifolds,* Princeton University preprint (1978).

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