The linking number of singular maps.

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The linking number of singular maps

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The linking number of a generic smooth map $f: M^{2k} \to R^{3k}$ can be defined not only for $k = 1$ (as we have done in [6]) but also for $k > 1$.

DEFINITION 1. Given a generic (see Definition 2) smooth map $f: M^{2k} \to \mathbb{R}^{3k}$ denote by $\Sigma = \Sigma(f)$ the set of singular points of f and by $f(\Sigma)$ its image. (In these dimensions all the singular points are of type Σ^1 and they constitute a submanifold of dimension $k = 1$) Let us denote by $\Delta = \Delta(f)$ the closure of the set of double points of map f, i.e.

$$\Delta(f) =: \overline{\{x \in M \mid \exists y \in M : y \neq x \text{ and } f(x) = f(y)\}}.$$

$f(\Delta)$ – the image of Δ – is a k dimensional immersed submanifold in R^{3k} with $f(\Sigma)$ as its boundary. Let us denote by v a normal vector field of $f(\Sigma)$ in $f(\Delta)$ directed outward from $f(\Delta)$ (see Figure 1.). The endpoints of these vectors form a $(k-1)$-dimensional manifold $\tilde{\Sigma}$. Denote by $l(f)$ the linking number of $\tilde{\Sigma}$ and $f(M)$. (This linking number does not depend on the choice of v if v consists of short enough vectors.) $l(f)$ is an integer if M^{2k} is oriented and k is even, otherwise it is an element of the group Z_2. $l(f)$ will be called the linking number of the map f.

DEFINITION 2. A smooth map of positive codimension (i.e. dim range > dim domain) is *generic* if

1) its jet sections are transversal to the Boardman manifolds (see [1]);
2) its self-intersections are transversal;
3) the intersections of its singular submanifolds with one another and with the nonsingular part are transversal.

We shall deal with the following

PROBLEM. How can be expressed $l(f)$ in terms of the invariants of the map f?

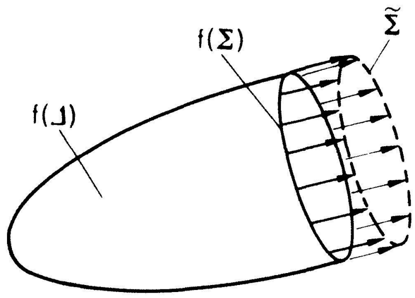

Fig. 1.

This problem is closely related to the following question. What is the homology class realized by the r-tuple points of a (generic) map? For the case of *immersions* the answer was given by Herbert [2]. (See also F. Ronga [4].) When the target is an Euclidean space this answer sounds like this:

The modulo two homology class of the r-tuple points of a codimension k generic immersion into an Euclidean space is dual to the class $\bar{W}_k^{r-1}$ where $\bar{W}_k$ denotes the k^{th} normal Stiefel-Whitney class of the immersion.

Ronga proved that the same formula remains valid for the double points even if the map has singular points.

Our theorem below, which solves the Problem above shows that the homology class of the *triple* points of a singular map $f: M^{2k} \to R^{3k}$ already can not be given by the same formula. (Compare also with [5].)

THEOREM. *Given a generic map* $f: M^{2k} \to R^{3k}$ *the linking number* $l(f)$ *defined as above can be expressed by the formula*:

$$l(f) = \mathscr{D}\bar{W}_k^2 - [\Delta^{(3)}(f)] \qquad (\substack{*\\*})$$

where $[\Delta^{(3)}(f)]$ *is the* 0*-dimensional homology class of* M^{2k} *defined by the triple points, and* $\mathscr{D}$ *is the Poincarè duality operator.*

(In formula ($\substack{*\\*}$) all the entries are integers if k is even and M is oriented, otherwise both sides belong to the group Z_2.)

Remark 1. We shall show that for any manifold M^{2k} there exists a generic singular map $f: M^{2k} \to R^{3k}$ such that $l(f) \neq 0$.

Before beginning the proof let me say what I mean by the *tubular neighbourhood* of an embedded or immersed submanifold *with boundary* in an euclidean space: it is the union of small discs D_p^k orthogonal to the tangent space

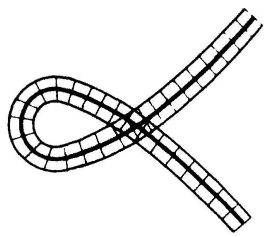

Fig. 2. The middle line denotes the submanifold with boundary.

T_pM for $\forall p \in M$ (k is the codimension of the submanifold). So this tubular neighbourhood is *not* a neighbourhood of the submanifold unless the submanifold is closed. (The points of the boundary of the submanifold may not be interior points of the tubular neighbourhood (see Figure 2.).)

Proof. The set Σ is a $(k-1)$-dimensional submanifold of the k-dimensional manifold Δ. Let us make a cut on Δ along Σ (see Figure 3.). We obtain a manifold Δ' with boundary $\partial\Delta'$. There is a map $\varphi : \Delta' \to \Delta$ which is a 1–1 map on $\Delta' \backslash \partial\Delta'$ and $\varphi|_{\partial\Delta'} : \partial\Delta' \to \Sigma \subset \Delta$ is a double cover. Now first we suppose that f has no triple points. Then we have Σ as the fixed-points set of the natural involution T on Δ, which is defined by the equation $f(Tx) = f(x)\ \forall x \in \Delta$. T defines on Δ' a *free* involution which we also denote by T such that $T \circ \varphi = \varphi \circ T$.

Let us denote by ξ the normal bundle of Δ in M, by ξ' the bundle $\varphi^*\xi$ over Δ' and by $\eta = f_!\xi$ the bundle over $\Delta'/T = f(\Delta)$ defined as follows:

The fibre over a point $[x, Tx] \in \Delta'/T$ $(x \in \Delta')$ is $\xi'_x \oplus \xi'_{Tx}$.

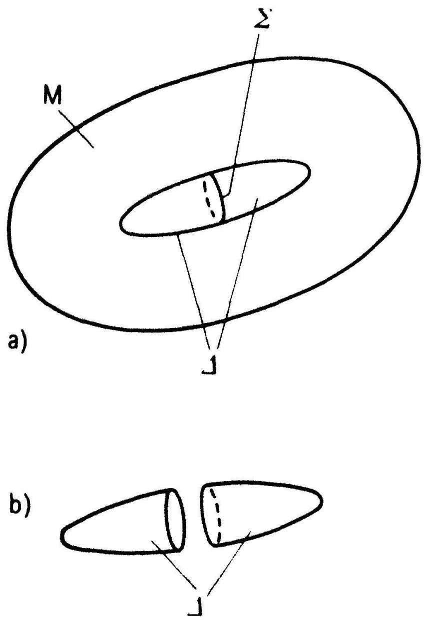

Fig. 3.

Notice that this bundle η splits into a direct sum only locally: in general η is not a direct sum of two k-dimensional bundles. However the subset X of the total space of η consisting of the union of subsets $\xi'_x \oplus 0 \cup 0 \oplus \xi'_{Tx}$ in every fibre of η is well defined. Then the bundle $\eta = f_! \xi$ can be identified with a tubular neighbourhood U of $f(\Delta)$ in R^{3k}. Moreover $f_! \xi$ and U can be identified in such a way that the subset X be identified with $f(M) \cap U$.

Any section s of the bundle ξ naturally defines a section z (which could be denoted by $f_! s$) of the bundle $\eta = f_! \xi$ as follows: $z([x, Tx]) = (s(x), s(Tx)) \in \xi'_x \oplus \xi'_{Tx} = \eta_{[x, Tx]}$. (Notice that over $(\partial \Delta')/T = f(\Sigma)$ the section z belongs to the "diagonal" subbundle of $\xi|_\Sigma \oplus \xi|_\Sigma = \{(y_1, y_2) \,|\, y_1, y_2 \in \xi|_\Sigma\}$ defined by the equation $y_1 = y_2$.)

If s is transversal to the zero section and avoids Σ, then $z = f_! s$ will be transversal to $f(M)$. The k-dimensional vector bundle ξ has a nonzero section *over* Σ. (Indeed, the obstructions for the existence of a nonzero section of the bundle $\xi|_\Sigma$ lie in the groups $H^i(\Sigma, \pi_{i-1}(S^{k-1}))$ which are zero for any $i \le \dim \Sigma = k-1$.) Now let us choose an arbitrary nonzero section of ξ over Σ and extend it to a section s of ξ over the whole base space Δ so that

a) s be transversal to the zero section Δ;

b) s should avoid the triple points of f (when they exist at all);

c) the intersection of the image of s with Δ should not contain points corresponding to each other under T.

Now as we explained earlier this section s defines a section $f_! s$ of $f_! \xi$, and $f_! \xi$ can be identified with a tubular neighbourhood U of $f(\Delta)$. Hence the image of $f_! s$ can be considered as a k-dimensional submanifold N_1 in R^{3k} transversal to $f(M)$. The boundary of N_1 is $(f_! s)(f(\Sigma))$ which we denote by $\bar{\bar{\Sigma}}$. It is easy to see that $\bar{\bar{\Sigma}}$ and $\bar{\Sigma}$ (see definition 1) are isotopic submanifolds in $R^{n+k} \setminus f(M)$.

Hence there exists an embedding $h: \bar{\Sigma} \times I \hookrightarrow R^{3k}$ of the cylinder $\bar{\Sigma} \times I$ such that

a) $h(\bar{\Sigma} \times 0) = \bar{\Sigma}$;

b) $h(\bar{\Sigma} \times 1) = \bar{\bar{\Sigma}}$;

c) $h(\bar{\Sigma} \times I) \cap f(M) = \emptyset$.

Now the union $N = h(\bar{\Sigma} \times I) \cup N_1$ is a k dimensional chain with $\bar{\Sigma}$ as its boundary. By definition $l(f)$ equals to the algebraic number of intersection points of N and $f(M)$. Let us compute this number:

1) $h(\bar{\Sigma} \times I)$ does not intersect $f(M)$ at all, hence $N \cap f(M) = N_1 \cap f(M)$.

2) There is a $1-1$ correspondence between the points of intersection of the image of $f_! s$ with $f(M) \cap U = X$ on the one hand and the points of intersection of the image of s with the zero section of ξ on the other hand. (This correspondence leaves unchanged the signs of the intersection points – if these signs can be defined.) Since Δ represents the homology class Poincare dual to $\bar{W}_k(M)$ the

intersection represents the class $\mathscr{D}\bar{W}_k^2$. The proof is finished for the case when f has no triple points.

The scheme of this proof can be summed up by the following chain of equalities:

$$l(f) = N \cap f(M) = N_1 \cap f(M) = f_! s \cap X = s \cap \Delta = \text{the selfintersection of } \Delta \text{ in } M = \mathscr{D}\bar{W}_k^2.$$

Now consider the case when f has triple points. Then first of all the involution T can not be defined on Δ as above. (T is not well defined at the triple points.) In this case we replace Δ by the appropriate subset of $M \times M$, i.e. we consider instead of Δ the set $\tilde{\Delta}$ defined as follows

$$\tilde{\Delta} = \overline{\{(x, y) \mid x \in M, y \in M; x \neq y; f(x) = f(y)\}}$$

where the bar indicates the closure in $M \times M$.

Instead of the involution T on Δ we consider an involution $\tilde{T}$ on $\tilde{\Delta}$ given by the formula $\tilde{T}(x, y) = (y, x)$.

Instead of the bundle ξ (which was the normal bundle of Δ in M) we define a bundle $\tilde{\xi}$ over $\tilde{\Delta}$ as follows:

At any point (x, y) of $M \times M$ one can consider the tangent space of the ("horizontal") submanifold $M \times y$. These tangent spaces define an n-dimensional vector bundle over $M \times M$, which we shall denote by h.

Since $n < 2k$ for a generic map $f: M^n \to R^{n+k}$ the submanifold $\tilde{\Delta} \subset M \times M$ has no tangent vector lying in a horizontal space. Hence the linear space generated by the tangent space of $\tilde{\Delta}$ and the fibre of h has dimension $2n - k$ at every point $(x, y) \in \tilde{\Delta}$. So these linear spaces form a $(2n - k)$-dimensional vector bundle over $\tilde{\Delta}$, which we denote by ζ. Now we define the bundle $\tilde{\xi}$ by the equality:

$$\zeta \oplus \tilde{\xi} = T(M \times M)\,|_{\tilde{\Delta}}.$$

(The bundle on the right side is the tangent bundle of $M \times M$ restricted over $\tilde{\Delta}$.)

Let π be the natural projection $\tilde{\Delta} \to \tilde{\Delta}/\tilde{T}$ and let $\tilde{\eta}$ be the bundle $\pi_! \tilde{\xi}$ over $\tilde{\Delta}/\tilde{T}$.

In the case when f had no triple points the total space of $\eta (= f_! \xi)$ could be identified with a tubular neighbourhood of $f(\Delta)$.

In the present case when f may have triple points the total space $E(\tilde{\eta})$ of the bundle $\tilde{\eta} = \pi_! \tilde{\xi}$ can be mapped onto a tubular neighbourhood U of $f(\Delta)$ by a map

$\alpha: E(\tilde{\eta}) \to U$ which is

1. One to one off the triple points of f.
2. If x, y, z are triple points with the same image then

$$\alpha([x, y]) = \alpha([x, z]) = \alpha([y, z]) = f(x) \quad (= f(y) = f(z))$$

where $[x, y]$ denotes the point $\pi(x, y) \in \tilde{\Delta}/\tilde{T}$ corresponding to $(x, y) \in \tilde{\Delta}$.

3. α restricted to the zero section coincides with the restriction of the composition $f \circ p$ to $\tilde{\Delta}$. Here $p: M \times M \to M$ denotes the projection onto the first factor.

Now we compute the linking number $l(f)$ of the map f. For this purpose – like in the previous special case – we consider a section s of the bundle ξ which does not vanish over $\Sigma(f)$. This section defines a section z of the bundle $\tilde{\eta} = \pi_! \xi$ (see the case when f has no triple points). Composing this section z with the map α we obtain a map $\alpha \circ z: \tilde{\Delta}/\tilde{T} \to U$. The image of this map will be denoted – again – by N_1.

Let us define the map $h: \tilde{\Sigma} \times I \to R^{3k}$ and the set $N = N_1 \cup h(\tilde{\Sigma} \times I)$ as above.

Now the computation of $l(f)$ goes by a chain of equaties almost identical to the one of the previous case:

$$l(f) \overset{①}{=} N \cap f(M) \overset{②}{=} N_1 \cap f(M) \overset{③}{=} \mathrm{Im}\,(\alpha \circ z) \cap X \overset{④}{=}$$

$$= (\mathrm{Im}\, s) \cap \Delta + \text{algebraic number of triple points} \overset{⑤}{=}$$

$$= \mathscr{D}\bar{W}_k^2 + \text{algebraic number of triple points.}$$

The only difference from the previous case is equality ①. Now we explain why it holds.

In a small neighbourhood of a triple point the picture can be visualized as follows (see Figure 4.):

In R^{3k} (which is a neighbourhood V of a triple point) there are three pairwisely orthogonal k-dimensional subspaces R_i^k $i = 1, 2, 3$ through the origin (= the triple point). These subspaces are $f(\Delta) \cap V$. The $2k$-dimensional subspaces generated by any two of these k-dimensional spaces are the three branches of $f(M)$ lying at the triple point under considerations. The submanifold N_1 (or more precisely $N_1 \cap V$) can be visualized (locally) as three k-dimensional spaces $\tilde{R}_i^k$ which are parallel to R_i^k and disjoint from the subset $R_1^k \cup R_2^k \cup R_3^k$ of R^{3k}. Now it is easy to see that in the neighbourhood V exactly three new intersection points of

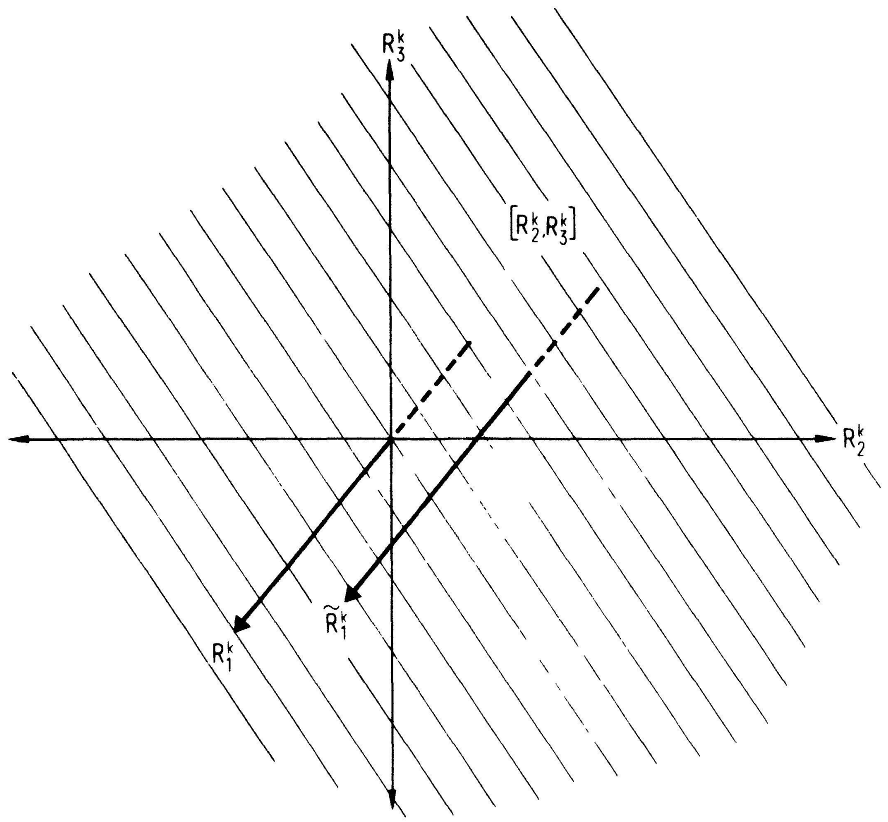

Fig. 4.

N_1 and $f(M)$ arise, namely: $\tilde{R}_1^k \cap \{R_2^k, R_3^k\}$, $\tilde{R}_2^k \cap \{R_1^k, N_3^k\}$, $\tilde{R}_3^k \cap \{R_2^k, R_3^k\}$, where the brackets { , } stand for the generated subspace. These intersection points are all of the same sign as the triple point itself (if this sign makes sense at all, i.e. if k is even and M is oriented). The theorem is proved.

It remains to prove Remark 1.

Suppose $f: M^{2k} \to R^{3k}$ is a generic map with a nonempty singular set Σ. Then $f(\Sigma)$ is a $(k-1)$-dimensional closed submanifold of $\mathbb{R}^{3k}$. Take a small $(2k+1)$-dimensional disc D^{2k+1} centered around a point of $f(\Sigma)$ and orthogonal to $f(\Sigma)$. Denote by S^{2k} the boundary of this disc. Form the *joint union* of this sphere S^{2k} with $f(M)$ as follows: joint a point of S^{2k} with a nonsingular simple point of $f(M)$ by a regular curve γ orthogonal at its endpoints to S^{2k} and $f(M)$ respectively. Now move a small $2k$-disc D_P^{2k} so that its center P should travel along γ, furthermore D_P^{2k} be orthogonal to γ and at the endpoints of γ this $2k$-disc lie in S^{2k} and in $f(M)$ respectively. Let T and H be the union of discs $\{D_P^{2k} \mid P \in \gamma\}$ and spheres $\{\partial D_P^{2k} \mid P \in \gamma\}$ respectively. Consider $\tilde{M} = H \cup f(M) \cup S^{2k} \backslash (D_A^{2k} \cup D_B^{2k})$ where A and B are the endpoints of γ (Figure 5.). $\tilde{M}$ is (after an eventual smoothing) diffeomorphic to M. Denote $\tilde{f}: M \to R^{3k}$ a map with image $\tilde{M}$. We claim that $l(\tilde{f}) \neq l(f)$. (Hence at least one of them is nonzero.)

Indeed, let W be a $(2k+1)$-chain in R^{3k} with boundary $\partial W = f(M)$. Then

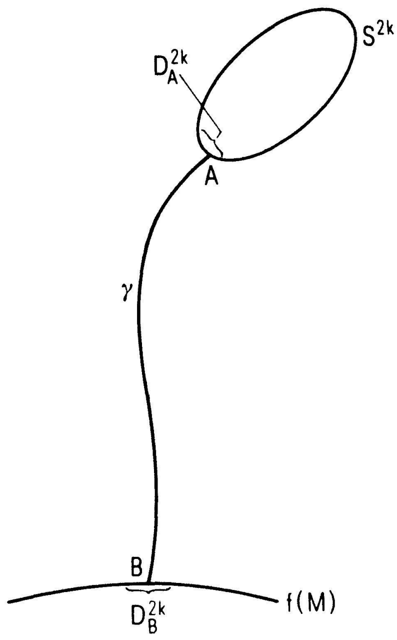

Fig. 5.

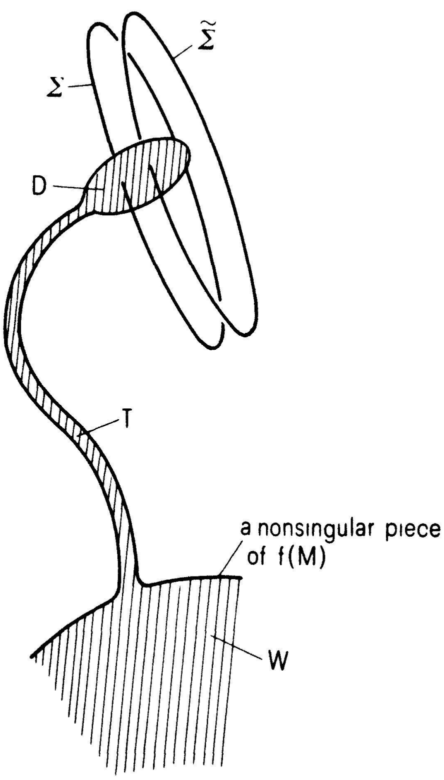

Fig. 6.

$l(f) = \#(W \cap \tilde{\Sigma})$ where $\#(\ )$ denotes the algebraic or mod 2 number of intersection points.

Let $\tilde{W}$ be the union $W \cup T \cup D^{2k+1}$ (see Figure 6.). Then $\partial\tilde{W} = \tilde{f}(M)$ and $l(\tilde{f}) = \#(\tilde{W} \cap \tilde{\Sigma}) = \#(w \cap \tilde{\Sigma}) + \#((T \cup D^{2k+1}) \cap \tilde{\Sigma})$ and $\#((T \cup D^{2k+1}) \cap \tilde{\Sigma}) = \pm 1$. Q.E.D.

Now Remark 1 is proved modulo the following

PROPOSITION. *For any manifold M^{2k} there exists a generic map $f: M^{2k} \to \mathbb{R}^{3k}$ with a nonempty singular set.*

Proof. Notice that it is enough to prove the proposition for the case when M^{2k} is a sphere S^{2k}. Indeed taking the joint union of a singular generic map $S^{2k} \to \mathbb{R}^{3k}$ with an arbitrary generic map $M^{2k} \to \mathbb{R}^{3k}$ we obtain a generic map of the joint union $M^{2k} \# S^{2k}$ (which is diffeomorphic to M^{2k}) into $\mathbb{R}^{3k}$ and this map surely will have singular points.

Now first we define a map $S^{k+1} \to \mathbb{R}^{2k+1}$ which is generic and has nonempty singular set as follows:

Let $\varphi: \mathbb{R}^{k+1} = \{x_1, \ldots, x_{k+1}\} \to R^{2k+1} = \{y_1, \ldots, y_{2k+1}\}-$ dimensional Whitney-umbrella-map i.e. $\varphi = (\varphi_1, \ldots, \varphi_{2k+1})$ and

$$\begin{aligned} \varphi_1(x_1, \ldots, x_{k+1}) &= x_1^2 & \varphi_{k+2}(x_1, \ldots, x_{k+1}) &= x_1x_2 \\ \varphi_2(x_1, \ldots, x_{k+1}) &= x_2 & \varphi_{k+3}(x_1, \ldots, x_{k+1}) &= x_1x_3 \\ &\vdots & &\vdots \\ \varphi_{k+1}(x_1, \ldots, x_{k+1}) &= x_{k+1} & \varphi_{2k+1}(x_1, \ldots, x_{k+1}) &= x_1x_{k+1}. \end{aligned}$$

The map φ has a Σ^1 singular point at the origin $x_1 = x_2 = \cdots = x_{k+1} = 0$ which goes into the origin $y_1 = \cdots = y_{2k+1} = 0$. Now consider the intersection of the image of φ with a disc centered around the point $y_1 = \cdots = y_{2k+1} = 0$. Take two copies of this $2k+1$-disc – denote them by D_1 and D_2 – and attach them to each other along their boundaries using the identity map between the two copies of the boundary. We obtain a $(2k+1)$-dimensional sphere and a subset in it which is the image of a generic singular map

$$\bar{g}: S^{k+1} \to S^{2k+1} \ (\operatorname{Im} \bar{g} \cap D_i = \operatorname{Im} \varphi \cap D_i \quad \text{for} \quad i = 1, 2).$$

Now throwing out an arbitrary point from S^{2k+1} which does not belong to the image of g we obtain a map $g: S^{k+1} \to \mathbb{R}^{2k+1}$. Let $i: \mathbb{D}^{k-1} \hookrightarrow \mathbb{R}^{k-1}$ be the standard inclusion of a disc into the euclidean space. Then $h = g \times i: S^{k+1} \times D^{k-1} \to R^{3k}$ is a generic map with nonempty singular set.

The sphere S^{2k} contains an open subset H such that $\bar{H}$ is diffeomorphic to $S^{k+1} \times D^{k-1}$. Let $\bar{f}: S^{2k} \to R^{3k}$ be an arbitrary smooth map which agrees with h on H. Let K be an open subset which is contained compactly in H (i.e. $\bar{K} \subset H$). The map $\bar{f}$ can be approximated by a generic map $f: S^{2k} \to R^{3k}$ which coincides with h on K. This f has all the required properties (it is generic and $\Sigma(f) \neq \emptyset$). Remark 1 is proved.

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